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On the Exponential Boundedness of Stopping Times of Invariant SPRT's

by
Holger Rootzén[1]
Department of Mathematical Statistics
University of Lund
and
Department of Statistics
University of North Carolina at Chapel Hill
and
Gordon Simons[2]
Department of Statistics
University of North Carolina at Chapel Hill

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Short Title: Exponential Boundedness of Stopping Times

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Holger Rootzén[1] University of Lund and University of North Carolina
Gordon Simons[2] University of North Carolina

Abstract: It is shown, under conditions which include invariant sequential probability ratio tests, that the stopping time is always exponentially bounded when the null or alternative hypothesis holds, except in a trivial instance.

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This paper is primarily concerned with showing that, for invariant sequential probability ratio tests, the stopping time is exponentially bounded under the null and alternative hypotheses, with the exception of trivial situations. This work is the by-product of a largely unsuccessful attempt to obtain general results under non-model distributions. Since we view the lack of these general results as a major deficiency in the theory of sequential analysis, we shall suggest what we believe are reasonable conjectures and shall discuss some of the past literature on the subject.

Let $(\Omega,\mathcal{B})$ be a measurable space and $\{\mathcal{B}_n, n\geq 1\}$ be a nondecreasing sequence of sub-σ-fields of $\mathcal{B}$. A stopping time N adapted to $\{\mathcal{B}_n, n\geq 1\}$ [3] is said to be *exponentially bounded* for a family of probability measures $\mathcal{R}$ on $(\Omega,\mathcal{B})$ if for each probability measure $R \in \mathcal{R}$ there exist constants $c > 0$ and $\rho < 1$ such that $R(N>n) < c\rho^n$, $n = 1,2,\ldots$. Such a condition implies that N is finite a.s. $(\mathcal{R})$ and that N has a moment generating function in some neighborhood of the origin for each $R \in \mathcal{R}$.

Let P and Q be elements in $\mathcal{R}$ and let L_n denote the P - Q likelihood ratio for $\mathcal{B}_n$.[4] Sequential probability ratio tests are defined in terms of a stopping variable N of the general form

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[3] Here, N is said to be adapted to $\{\mathcal{B}_n, n\geq 1\}$ if $[N = 1,2,\ldots \text{ or } \infty] = \Omega$ and $[N=n] \in \mathcal{B}_n$ for $n = 1,2,\ldots$.

[4] For a precise definition of this concept see Eisenberg, Ghosh and Simons (1976). Essentially, L_n is the Radon-Nikodym derivative dQ/dP relative to $\mathcal{B}_n$.

(1) N = the first $n \geq 1$ such that $L_n \notin (A,B)$

$= \infty$ if no such n exists,

where $0 < A < B < \infty$. At this level of generality, it is easy to construct examples for which N is exponentially bounded for $\{P,Q\}$ and others for which it is not. All that is known, in general, is derived from the following set of inequalities, described by Eisenberg, Ghosh and Simons (1976):

$$AP(N>n) \leq Q(N>n) \leq BP(N>n), \quad n = 1,2,\ldots .$$

It follows immediately that

(a) $P(N<\infty) = 1$ iff $Q(N<\infty) = 1$,

(b) for each $r > 0$, $\int N^r dP < \infty$ iff $\int N^r dQ < \infty$, and

(c) N is exponentially bounded for $\{P\}$ iff it is exponentially bounded for $\{Q\}$.

Now suppose $X_1, X_2, \ldots$ is a sequence of random elements on $(\Omega, \mathcal{B})$ (to the statistician, "potential data") and $\mathcal{B}_n = \sigma(X_1,\ldots,X_n)$ (the σ-field generated by $X_1,\ldots,X_n$). If $X_1, X_2, \ldots$ is an i.i.d. sequence under P and Q, then $\{\log L_n, n\geq 1\}$ is a random walk under both P and Q. It follows that N is exponentially bounded for $\{P,Q\}$ unless $L_1 = 1$ a.s. (P,Q). This is a well-known result due to Stein (1946). We shall now describe a similar result for invariant sequential probability ratio tests.

Consider the following mathematical structure:

(i) $\mathcal{P}$ and $\mathcal{Q}$ are two disjoint families of probability measures on $(\Omega,\mathcal{B})$.

(ii) $X_1,X_2,\ldots$ is an i.i.d. sequence of random elements for each probability measure $R \in \mathcal{P} \cup \mathcal{Q}$.

(iii) For each $n \geq 1$, $\mathcal{B}_n = \sigma(T_n)$ where T_n is a statistic which is a symmetric function of $X_1,\ldots,X_n$.

(iv) For each $n \geq 1$, T_k is $\mathcal{B}_n$-measurable for $k = 1,2,\ldots,n$. (I.e., $\mathcal{B}_1 \subset \mathcal{B}_2 \subset \ldots$.)

(v) For each $n \geq 1$, the probability space $(\Omega,\mathcal{B}_n,R)$ is the same for every $R \in \mathcal{P}$, and is the same for every $R \in \mathcal{Q}$.

This is the structure that one encounters when invariant sequential probability ratio tests are being considered. If one wishes to test the (composite) hypothesis that the true probability measure R belongs to $\mathcal{P}$ against the (composite) alternative that it belongs to $\mathcal{Q}$, the following sequential probability ratio test suggests itself: Let $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$ be chosen in an arbitrary manner, and let L_n be the $P - Q$ likelihood ratio for $\mathcal{B}_n$ $(n \geq 1)$. Because of property (v), each possible pair (P,Q) leads to the same sequence of likelihood ratios $L_1,L_2,\ldots$. Let N be the stopping variable described in (1) and choose $\mathcal{P}$ or $\mathcal{Q}$ according as $N < \infty$ and $L_N \leq A$ or $N < \infty$ and $L_N \geq B$, respectively. (If $N = \infty$, no decision is made.) We have the following theorem:

<u>THEOREM</u> *Either* $L_n = 1$ *a.s.* $(\mathcal{P} \cup \mathcal{Q})$ *for* $n \geq 1$, *or* N *is exponentially bounded for* $\mathcal{P} \cup \mathcal{Q}$.

PROOF In view of (c) and (v), N is exponentially bounded for $\mathcal{P} \cup \mathcal{Q}$ iff it is exponentially bounded for $\{P\}$. Likewise, $L_n = 1$ a.s. $(\mathcal{P} \cup \mathcal{Q})$ for $n \geq 1$ iff $P(L_n = 1) = 1$ for $n \geq 1$. Suppose the latter is false, i.e., that for some k, $P(L_k = 1) < 1$. For simplicity, we shall assume that k can be taken to be unity. If a larger value of k is required, the proof we shall give that N is exponentially bounded for $\{P\}$ can be easily modified. Express T_1 as $t(X_1)$ and L_1 as $\ell(X_1)$ and observe that $EL_1 \leq 1$ and (since $P(L_1=1) < 1$) that $E \log L_1 < 0$. Set $\mathcal{B}'_n = \sigma(t(X_1), \ldots ,t(X_n))$ and observe that $L'_n = \Pi_{i=1}^{n} \ell(X_i)$ is the P - Q likelihood ratio for $\mathcal{B}'_n$ in view of condition (ii) $(n \geq 1)$. Moreover, conditions (iii) and (iv) imply that $\mathcal{B}'_n \subset \mathcal{B}_n$ and, hence, $E^{\mathcal{B}'_n} L_n \leq L'_n$ [5] for $n \geq 1$. Thus, for $\rho < 1$, to be chosen later,

$$P^{\mathcal{B}'_n}(L_n > A,\ L'_n \leq \rho^n) \leq A^{-1} I_{[L'_n \leq \rho^n]}\ E^{\mathcal{B}'_n} L_n \leq A^{-1}\rho^n$$

and, hence,

$$\begin{aligned} P(N>n) &\leq P(L_n>A) = EP^{\mathcal{B}'_n}(L_n>A) \\ &\leq A^{-1}\rho^n + P(L'_n > \rho^n). \end{aligned}$$

Thus it suffices to show, for properly chosen $\rho < 1$ and $\rho' < 1$, that

$$P(L'_n > \rho^n) \leq (\rho')^n .$$

Since $EL_1 \leq 1$, $E(L_1)^t \leq 1$ for $0 < t \leq 1$. In fact, if ρ is chosen so that $E \log L_1 < \log \rho < 0$, then there exists a small positive t_0 such that

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[5] Equality holds if P and Q are equivalent probability measures on $\mathcal{B}_n$.

$$\rho' \equiv E\, e^{t_0(\log L_1 - \log \rho)} < 1.$$

Then

$$P(L_n' > \rho^n) = P\left(\prod_{i=1}^{n} \ell(X_i) > \rho^n\right)$$

$$\le \rho^{-nt_0}\, E\left[\prod_{i=1}^{n} \ell(X_i)\right]^{t_0}$$

$$= \rho^{-nt_0}\, E^n(L_1)^{t_0} = (\rho')^n\ .$$

□

We should point out that it is always possible that $L_n = 1$ a.s. $(P \cup Q)$ for $n \ge 1$. (This is the case, for instance, if $T_n \equiv 1$ for $n \ge 1$.) However, this possibility does not detract from the strength of the theorem. For if $L_n = 1$ a.s. $(P \cup Q)$ for $n \ge 1$, then, for each $n \ge 1$, the probability spaces $(\Omega, \mathcal{B}_n, R)$, $R \in P \cup Q$, are all the same and there is no information available for distinguishing between the two families P and Q. In practice, one would never consider proposing such a test.

A close examination of our proof reveals that conditions (ii) and (iii) are stronger than necessary. We shall not try to state the weakest possible conditions, but simply make the observation that the argument works because the σ-fields $\mathcal{B}_n$, $n \ge 1$, grow at a sufficiently fast rate under conditions (ii) and (iii). Note also that the proof shows exponential boundedness under P for the "single boundary" stopping time

$$N^* = \text{the first } n \ge 1 \text{ such that } L_n \ge A,$$
$$= \infty \text{ if no such } n \text{ exists.}$$

It is reasonable to ask whether our theorem has an extension to probability measure $R \notin P \cup Q$ under which $X_1, X_2, \ldots$ is an i.i.d.

sequence. It is possible to extend it to other probability measures within a monotone likelihood ratio class (cf., A. Ifram (1965)) but this amounts to a very restrictive improvement. R. Wijsman (1970) has found an interesting example for which $R(N<\infty) = 1$ but N is not exponentially bounded under R. This would seem to put to rest any hope of finding a general extension of our theorem. However, it must be remembered that a P - Q likelihood ratio is uniquely defined only up to a P and Q equivalence, and, in Wijsman's example, the probability measure R is *orthogonal* to the probability measures in $\mathcal{P} \cup \mathcal{Q}$ on the σ-fields $\mathcal{B}_n$. Indeed there exist other versions of the likelihood ratios L_n, appropriate to his example (albeit less natural from a topological viewpoint), for which N *is* exponentially bounded under R. Recently, Wijsman (1976a) has been working with some examples for which R is dominated (on each $\mathcal{B}_n$) by the probability measures in $\mathcal{P} \cup \mathcal{Q}$. (In such a case, the likelihood ratios L_n *are* uniquely specified up to an R-equivalence.) For these, again, $R(N<\infty) = 1$ but N is not exponentially bounded under R. We believe it is significant that the probability measures R, in his examples, are *not* equivalent to the probability measures in $\mathcal{P} \cup \mathcal{Q}$ (on each $\mathcal{B}_n$). For one thing, there exist examples, within the context of conditions (i) - (v), for which R is dominated by but is not equivalent to the probability measures in $\mathcal{P} \cup \mathcal{Q}$, with the property that $R(N=\infty) = 1$, and yet the condition " $R(L_n=1) = 1$ for $n \geq 1$ " does not hold. However, on the basis of our studies, we are prepared to conjecture that $R(N<\infty) = 1$, whenever R and the probability measures in $\mathcal{P} \cup \mathcal{Q}$ are equivalent on $\sigma(X_1)$ and the probability spaces $(\Omega,\mathcal{B}_n,P)$ and $(\Omega,\mathcal{B}_n,Q)$ are distinct for some $n \geq 1$, $P \in \mathcal{P}$, $Q \in \mathcal{Q}$. Indeed we suspect that these conditions guarantee that N is exponentially bounded under R. Here, we are assuming the structure imposed by conditions (i) - (v).

We do not want to leave the reader with the impression that there are no results of a general nature concerning $R \notin \mathcal{P} \cup \mathcal{Q}$. R. Berk (1970) has obtained results for what he calls *parametric* sequential probability ratio tests and R. Wijsman (1976b) has recently obtained a theorem concerned with exponential boundedness. Their results exploit the fact that, in many useful examples of invariant sequential probability ratio tests, the log-likelihood ratios $\{\log L_n, n \geq 1\}$ become a random walk asymptotically (under R) as $n \to \infty$. This is not always the case. (There are practical examples to the contrary.) Therefore, it appears that some substantially new approaches are called for.

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