Local Times for Vector Functions: Energy Integrals and Local Growth Rates

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Abstract. Let $F: E \to R^m$ ($E \subset R^n$ compact) have a local time $\alpha(x,dt)$, $x \in R^m$, and let $I(\phi)$ denote the integral of $\phi(s-t)$ against $\alpha(x,ds)\alpha(x,dt)dx$; here ϕ is a "potential kernel" on R^n, so that $I(\phi)$ is an (averaged) "energy integral" for the distribution $\alpha(x,dt)$ of mass on $\{t \in E: F(t)=x\}$. We show that if $\phi \in L^1(dt)$ and $\phi = \sup_n \phi_n$ for a sequence $\{\phi_n\}$ of Fourier transforms of positive L^1-functions, then $I(\phi)$ is well approximated by certain functionals of the increments of F. We then draw the following conclusions about the local growth and fluctuations of F: if F is continuous, $I(\phi)<\infty$ and ϕ is radial, then (i) $\text{ap}\overline{\lim}_{s \to t} ||F(s)-F(t)||/V(||s-t||) = \infty$ a.e. (dt) on E, where $V(\varepsilon) \equiv (\varepsilon^n \phi(\varepsilon))^{1/m}$, and (ii)

$$\left[\int_0^\varepsilon \phi(r) r^{n-1} dr\right]^{\frac{1}{m}} = o\Big(\sup_{\substack{s,t \in E \\ ||s-t|| \le \varepsilon}} ||F(s)-F(t)||\Big) \quad \text{as} \quad \varepsilon \downarrow 0.$$

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<u>Introduction</u>. By and large, the study of local times has been confined to probabilistic settings, either as in Markov processes where the potential theoretic and stochastic analysis are fused, or as in [2], [3], [6] [7] and [8], where "real variable" results may be separately developed, but with an eye toward applications, especially to sample function analysis. (See the discussion in §0 of [8] and the references therein.) Here we deal exclusively with non-random functions. More specifically, we intend to further develop the observation of S. M. Berman that, loosely, "the more regular the local time, the more irregular the function," by amplifying several earlier results of ours and J. Horowitz, such as: if a function has a local time, any "approximate local modulus" grows at least linearly, and grows faster than linearly if the local time is continuous in its "time" parameter. (See the comments after the corollaries.)

Let $\mathcal{B}_k$ denote the Borel sets in R^k (Euclidean k-space), $\lambda_k(dt)$ be Lebesgue measure in R^k (just dt for integration), and let $c_k = \lambda_k\{B_k(o,1)\}$, $B_k(t,\varepsilon)$ being the open ball in R^k centered at t and of radius ε. Further, let $E\in\mathcal{B}_n$ be bounded and $F: E\to R^m$ Borel measurable. The *occupation measure* of F is $\mu(B) = \lambda_n\{F^{-1}(B)\}$, $B\in\mathcal{B}_m$. If $\mu \ll \lambda_m$ (i.e. $\lambda_m(B) = 0 \Rightarrow \mu(B) = 0$, $B\in\mathcal{B}_m$), then for each $A\in\mathcal{B}_n$, the measure $\lambda_n\{t\in E\cap A: F(t)\in dx\}$ is also dominated by $\lambda_m(dx)$, and we may select versions $\alpha(x,A)$ of the Radon-Nikodym derivatives such that (i) $\alpha(\cdot,A)$ is $\mathcal{B}_m$-measurable for each $A\in\mathcal{B}_n$ and (ii) $\alpha(x,\cdot)$ is a finite measure on $\mathcal{B}_n$ $\forall x$. We call this family $\alpha(x,dt)$ of measures the *local time* of F because it represents the "time spent" by F in the state x during dt.

By definition, for any $B\in\mathcal{B}_m$, $A\in\mathcal{B}_n$,

(1) $$\lambda_m\{F^{-1}(B)\cap A\} = \int_B \alpha(x,A)dx ,$$

which extends to

(2) $$\int_E H(t,F(t))dt = \int_{R^m}\int_E H(t,x)\alpha(x,dt)dx$$

for any non-negative, Borel measurable H on $R^n \times R^m$. It follows that $\alpha(x,M_x^c) = 0$ for λ_m -- a.e. x, where $M_x = \{t\in E: F(t)=x\}$.

Consider the measures $I_x(dsdt) = \alpha(x,ds)\alpha(x,dt)$ and $I(dsdt) = \int I_x(dsdt)dx$ on $\mathcal{B}_n \otimes \mathcal{B}_n$; the latter is the measure $H(ds,dt)$ which figures in [2]. For $0 \le k(s,t)$ Borel measurable, we write $I_x(k)$ and $I(k)$ for the corresponding integrals:

(3) $$I(k) = \int_{R^m} I_x(k)dx = \int_{R^m}\int_E\int_E k(s,t)\alpha(x,dt)\alpha(x,ds)dx .$$

Let ϕ be a "potential kernel" on R^n , i.e. ϕ is positive and continuous on $R^n/\{0\}$ and $\phi(0)=\infty$, and recall that $A\in\mathcal{B}_n$ is said to have "positive ϕ-capacity" if there exists a non-zero, finite measure $\gamma(ds)$ concentrated on A such that $\phi(s-t) \in L^1(\gamma(ds)\gamma(dt))$. Various authors (see e.g. [1], [2], [9], [10], [11]) have considered the capacity of the "level sets" M_x for Gaussian (and other) stochastic processes, and $\alpha(x,dt)$ has been the natural measure to use; typically, one obtains probabilistic conditions for $\int I(k)dP < \infty$ for some kernel $k(s,t) = \phi(s-t)$. See also the "concluding remark."

To approximate $I(k)$ using the *increments* of F, we define, for each $\varepsilon>0$,

(4) $$T_\varepsilon(k) = \frac{1}{c_m\varepsilon^m}\int_E\int_E k(s,t)\xi_\varepsilon(F(s)-F(t))dsdt ,$$

where $\xi_\varepsilon(u) = 1_{B_m(0,\varepsilon)}(u)$, and consider the following question: for which functions $k(s,t)$ does $T_\varepsilon(k)\to I(k)$ as $\varepsilon\downarrow 0$? To see this may fail, notice first that $I \perp \lambda_{2n}$ because, with $G = \{(s,t)\in E\times E\colon F(s)=F(t)\}$, we have $G\in\mathcal{B}_{2n}$, $I(G^c) = 0$, whereas $\mu \ll \lambda_m$ implies that $\nu \equiv \mu^*\text{-}\mu \ll \lambda_m$ and hence $\lambda_{2n}(G) = \nu(\{0\})=0$. Consequently, with $k(s,t) = 1_G(s,t)$, we have $T_\varepsilon(k)\equiv 0$ but

$$I(k) = \int_{R^m} \alpha^2(x,E)dx > 0 \ .$$

(Other examples are easily constructed.)

Naturally we would like to have $T_\varepsilon(\phi)\to I(\phi)$ for a wide class of potential kernels, for example for the Riesz potentials $\phi_\alpha(t) = C(n,\alpha)||t||^{\alpha-n}$, $0<\alpha<n$ (here $I(\phi)$, $T_\varepsilon(\phi)$ stand for $I(k)$, $T_\varepsilon(k)$, $k(s,t) = \phi(s-t)$). Suppose ϕ is a kernel of "positive type," i.e. $\phi\in L^1(dt)$ and has a positive Fourier transform $\hat{\phi}$. Let $\hat{\alpha}_x$ be the Fourier transform of the measure $\alpha(x,dt)$, $x\in R^m$. Then

$$I(\phi) = \int I_x(\phi)dx = \int_{R^m} dx \int_{R^n} \hat{\phi}(\lambda)|\hat{\alpha}_x(\lambda)|^2 d\lambda \quad ([9, \text{p. } 141])$$

$$= \int_{R^n} d\lambda\hat{\phi}(\lambda) \int_{R^m} |\hat{\alpha}_x(\lambda)|^2 dx$$

$$= \int_{R^n} \hat{\phi}(\lambda) I(e^{i\lambda\circ(t-s)})d\lambda \ .$$

As will be seen in the course of the proof of the Theorem, $I(k) = \lim_\varepsilon T_\varepsilon(k)$ for any *continuous* k, and hence

$$I(\phi) = \int_{R^n} \hat{\phi}(\lambda) \lim_\varepsilon T_\varepsilon(e^{i\lambda\circ(t-s)})d\lambda \ ,$$

which, proceeding "formally,"

$$= \lim_\varepsilon \int_{R^n} \hat{\phi}(\lambda)T_\varepsilon(e^{i\lambda\circ(t-s)})d\lambda = \lim_\varepsilon T_\varepsilon\left(\hat{\hat{\phi}}(t-s)\right) = \lim_\varepsilon T_\varepsilon(\phi) \ .$$

Among other problems, however, $\hat{\phi}$ *has no transform* because ess sup $\phi = \infty$ implies $\hat{\phi} \notin L^1(dt)$; but this does suggest how to proceed.

Main result. Let F^∞ denote the class of functions $\phi \in L^1(\lambda_n)$, $\phi(0)=\infty$, such that $\phi(t) = \sup_n \phi_n(t)$ where $0 \le \phi_1 \le \phi_2 \dots$, $\phi_n \in L^1(dt)\ \forall n$, and each ϕ_n is the Fourier transform of some Borel measurable $0 \le f \in L^1(dt)$. For $n=1$, F^∞ contains any even function in $L^1(dt)$ which is convex, continuous, and decreasing on $(0,\infty)$ (approximate ϕ by convex, continuous functions and use Polya's theorem). For $n>1$, F^∞ also contains the usual kernels.

Writing just $\alpha(x)$ for $\alpha(x,E)$ and with $A_\delta = \{(s,t)\in E\times E\colon ||s-t|| \le \delta\}$,

Theorem. Suppose $\alpha \in L^2(dx)$ and $\phi \in F^\infty$. Then

(5) $$I(\phi) = \lim_\varepsilon T_\varepsilon(\phi) = \sup_\varepsilon T_\varepsilon(\phi) \le \infty .$$

Moreover, if $I(\phi)<\infty$,

(6) $$\lim_{\delta\downarrow 0} \sup_\varepsilon T_\varepsilon(\phi\cdot 1_{A_\delta}) = 0 .$$

Corollaries 1 and 2 about the local growth of F depend on (6), which, in turn, rests on (5). The proof of the Theorem will be split into several steps.

Step 1^0: $I(k) = \lim_\varepsilon T_\varepsilon(k)$ for k continuous. For any bounded, complex-valued g on R^n,

$$\int_{R^m}\int_E |g(s)|\alpha(x,ds)dx = \int_E |g(s)|ds < \infty$$

and hence for λ^m -- a.e. y,

$$\int_E g(s)\alpha(y,ds) = \lim_\varepsilon \frac{1}{c_m\varepsilon^m}\int_{B_m(y,\varepsilon)} dx \int_E g(s)\alpha(x,ds) .$$

Using (2) and the fact that the exceptional set of y's has μ-measure 0, we find that, for λ_n -- a.e. t,

$$\int_E g(s)\alpha\big(F(t),ds\big) = \lim_{\varepsilon} \frac{1}{c_m\varepsilon^m}\int_E g(s)\xi_\varepsilon\big(F(s)-F(t)\big)ds\ .$$

Consequently, for any f like g,

$$\begin{aligned} I\big(f(t)g(s)\big) &= \int_{R^m} dx \int_E f(t)\{\int_E g(s)\alpha(x,ds)\}\alpha(x,dt) \\ &= \int_E f(t)\int_E g(s)\alpha\big(F(t),ds\big)dt \quad \text{by (2)} \\ &= \int_E \lim_{\varepsilon}\frac{1}{c_m\varepsilon^m}\int_E f(t)g(s)\xi_\varepsilon\big(F(s)-F(t)\big)dsdt\ , \end{aligned}$$

which will $\quad = \lim_{\varepsilon} T_\varepsilon\big(f(t)g(s)\big)$ if we can show

$$\sup_{\varepsilon}\frac{1}{c_m\varepsilon^m}\left|\int_E f(t)g(s)\xi_\varepsilon\big(F(s)-F(t)\big)ds\right| \in L^1(dt) \text{ on } E\ .$$

Let $Q_\alpha(x)$ be the Hardy-Littlewood maximal function for $\alpha(x)$:

$$Q_\alpha(x) = \sup_{\varepsilon}\frac{1}{c_m\varepsilon^m}\int_{B_m(x,r)}\alpha(y)dy, \qquad x\in R^m\ .$$

If f and g are bounded by k,

$$\begin{aligned} \int_E \sup_{\varepsilon}\frac{1}{c_m\varepsilon^m}\left|\int_E f(t)g(s)\xi_\varepsilon\big(F(s)-F(t)\big)ds\right|dt &\le k\int_E \sup_{\varepsilon}\frac{1}{c_m\varepsilon^m}\int_E \xi_\varepsilon\big(F(s)-F(t)\big)ds\ dt \\ &= k\int_E Q_\alpha\big(F(t)\big)dt = k\int_{R^m} Q_\alpha(x)\alpha(x)dx\ , \end{aligned}$$

which is finite because $\alpha\in L^2(dx)$ implies $Q_\alpha\in L^2(dx)$.

Taking $f\!=\!g\!\equiv\!1$, we find that $T_\varepsilon(1)\to\int\alpha^2$. Consider the probability measures

$$W_\varepsilon(B) = \frac{T_\varepsilon(1_B)}{T_\varepsilon(1)}\ ,\quad W(B) = \frac{I(B)}{I(R^{2n})}\ ,\quad B\in\mathcal{B}_{2n}\ .$$

Choosing $f(t) = e^{i\lambda_1 \cdot t}$, $g(t) = e^{i\lambda_2 \cdot t}$, $\lambda_1, \lambda_2 \in R^n$, we have,

$$\int_{R^n}\int_{R^n} e^{i(\lambda_1,\lambda_2)\cdot(s,t)}\, W(dsdt) = \lim_{\varepsilon} \int_{R^n}\int_{R^n} e^{i(\lambda_1,\lambda_2)\cdot(s,t)}\, W_\varepsilon(dsdt) .$$

In other words, the measures W_ε converge weakly to W; since these measures are supported on $\overline{E\times E}$, which is compact, we obtain 1^0 .

Let $\phi_n(t) \uparrow \phi(t)$ with $\{\phi_n\}$ as described above.

<u>Step 2^0</u>: <u>$I(\phi) \le \underline{\lim}_{\varepsilon}\, T_\varepsilon(\phi)$.</u> This is easy:

$$I(\phi) = \lim_n I(\phi_n) \quad \text{(by the monotone convergence theorem)}$$

$$= \lim_n \lim_\varepsilon T_\varepsilon(\phi_n) \quad (\text{by } 1^0)$$

$$\le \underline{\lim}_{\varepsilon}\, T_\varepsilon(\phi), \text{ since } \phi_n \le \phi \ \forall n .$$

Next, we introduce the following finite measures on $\mathcal{B}_m$:

$$\Lambda_j(B) = \int_E \int_E \phi_j(s-t) 1_B\big(F(s)-F(t)\big) dsdt, \quad j=1,2,\dots ,$$

$$\Lambda(B) = \int_E \int_E \phi(s-t) 1_B\big(F(s)-F(t)\big) dsdt ,$$

and recall that

$$\nu(B) = \mu^*\text{-}\mu(B) = \int_E \int_E 1_B\big(F(s)-F(t)\big) dsdt .$$

Each of these is absolutely continuous with respect to λ_m . Let

$$\psi_j(y) = \frac{d\Lambda_j}{d\lambda_m}(y) , \quad \psi(y) = \frac{d\Lambda}{d\lambda_m}(y) , \quad \beta(y) = \frac{d\nu}{d\lambda_m}(y)$$

and let $\hat{\psi}_j$, $\hat{\psi}$, and $\hat{\beta}$ be the corresponding Fourier transforms. For example, then,

(6) $$\hat{\psi}_j(\lambda) = \int_{R^m} e^{i\lambda\circ x}\,\psi_j(x)dx = \int_E\int_E e^{i\lambda\circ F(s)}e^{-i\lambda\circ F(t)}\,\phi_j(s-t)dsdt\ ,$$

$$\hat{\beta}(y) = \left|\int_{R^m} e^{i\lambda\circ x}\,\alpha(x)dx\right|^2 \geq 0\ .$$

<u>Step 3^0:</u> <u>$I(\phi) \geq (2\pi)^{-m}\int_{R^m}\hat{\psi}(y)dy$.</u> To start, $\hat{\psi}_j(\lambda) \geq 0\ \forall\lambda$ and $j\geq 1$; this follows from (6) because the ϕ_j's are positive definite. By the dominated convergence theorem, for any $\lambda\in R^m$,

$$\hat{\psi}(\lambda) = \int_E\int_E \phi(s-t)\ e^{i\lambda\circ F(s)}\ e^{-i\lambda\circ F(t)}dsdt = \lim_j \hat{\psi}_j(\lambda) \geq 0\ .$$

Next, since $\alpha \in L^2(dx)$, the Parseval relation yields

$$\int_{R^m}\left|\int_{R^m} e^{i\lambda\circ x}\,\alpha(x)dx\right|^2 d\lambda < \infty\ .$$

That is, $\hat{\beta} \in L^1(d\lambda)$. It follows that β has a bounded (and continuous) version, and recalling that each ϕ_j is bounded, we have, for any $B\in\mathcal{B}^m$,

$$\int_B \Psi_j(x)dx = \Lambda_j(B) \leq ||\phi_j||_\infty\ \nu(B) \leq ||\phi_j||_\infty\ ||\beta||_\infty\ \lambda_m(B)\ .$$

Thus, each ψ_j has a *bounded* version and a *non-negative* Fourier transform. According to [4, p. 66], then, $\hat{\psi}_j \in L^1(d\lambda)$ and the inversion formula holds for λ_m -- a.e. y:

(7) $$\psi_j(y) = (2\pi)^{-m}\int_{R^m} e^{-i\lambda\circ y}\,\hat{\psi}_j(\lambda)d\lambda\ .$$

We now assume β and ψ_j are bounded and continuous; in particular, (7) holds $\forall y$. The continuity of ϕ_j gives

$$\begin{aligned} I(\phi_j) &= \lim_{\varepsilon} T_{\varepsilon}(\phi_j) \\ &= \lim_{\varepsilon} \frac{1}{c_m \varepsilon^m} \int_{B_m(0,\varepsilon)} \psi_j(y)dy \\ &= \psi_j(0) \\ &= (2\pi)^{-m} \int_{R^m} \hat{\psi}_j(\lambda)d\lambda. \end{aligned}$$

Consequently,

$$\begin{aligned} I(\phi) &= \lim_{j} I(\phi_j) \\ &= \lim_{j} (2\pi)^{-m} \int_{R^m} \hat{\psi}_j(\lambda)d\lambda \\ &\geq (2\pi)^{-m} \int_{R^m} \underline{\lim_{j}}\, \hat{\psi}_j(\lambda)d\lambda \qquad \text{(by Fatou's lemma, since } \hat{\psi}_j \geq 0 \text{)} \\ &= (2\pi)^{-m} \int_{R^m} \hat{\psi}(\lambda)d\lambda \ . \end{aligned}$$

<u>Step 4^0: $I(\phi) \geq \sup_{\varepsilon} T_{\varepsilon}(\phi)$.</u> Assume $I(\phi)<\infty$. From 3^0, $\hat{\psi} \in L^1(d\lambda)$ and we have already seen that $\hat{\psi}\geq 0$. As above, then, we can and do assume

$$\psi(y) = (2\pi)^{-m} \int_{R^m} e^{-i\lambda\circ y}\, \hat{\psi}(\lambda)d\lambda \quad \forall y \ .$$

In particular, $\psi(y) \leq \psi(0)$ $\forall y$, so that

$$\begin{aligned} \sup_{\varepsilon} T_{\varepsilon}(\phi) &= \sup_{\varepsilon} \frac{1}{c_m \varepsilon^m} \int_{B_m(0,\varepsilon)} \psi(y)dy \\ &= \psi(0) \\ &= (2\pi)^{-m} \int_{R^m} \hat{\psi}(\lambda)d\lambda \leq I(\phi) \ . \end{aligned}$$

Combining 2^0 and 4^0 we have (5). As for (6), we'll assume, for notational ease, that $n=1$ and $E\subset[0,1]$; the proof for arbitrary n and bounded E is essentially the same. Writing A_k instead of $A_{1/k}$, $k=2,3,\ldots$,

$$A_k \subset B_k \equiv \bigcup_{j=0}^{k-2} (D_j^k \times D_j^k), \quad D_j^k = E \cap [\tfrac{j}{k}, \tfrac{j+2}{k}], \quad j=0,\ldots,k-2 .$$

In what follows, $I^{j,k}$, $\psi^{j,k}$, etc. refer to the quantities I, ψ, etc. but defined relative to D_j^k instead of E. Now,

$$\begin{aligned}
\sup_\varepsilon T_\varepsilon(\phi\circ 1_{A_k}) &\le \sup_\varepsilon T_\varepsilon(\phi\circ 1_{B_k}) \\
&= \sup_\varepsilon \sum_{j=0}^{k-2} T_\varepsilon^{j,k}(\phi) \\
&\le \sum_{j=0}^{k-2} \sup_\varepsilon T_\varepsilon^{j,k}(\phi) \\
&= \sum_{j=0}^{k-2} I^{j,k}(\phi), \quad \text{by (5).}
\end{aligned}$$

And

$$\begin{aligned}
I^{j,k}(\phi) &= \iint \phi(s-t) I^{j,k}(dsdt) \\
&= \int_{D_j^k}\int_{D_j^k} \phi(s-t) I(dsdt)
\end{aligned}$$

because, for any $E'\subset E$, $\alpha(x, dt\cap E')$ in the local time of F restricted to E'. Furthermore $\phi(0)=\infty$ and $I(\phi)<\infty$ together imply $\alpha(x,\{t\})=0\ \forall t\in E$, for λ_m -- a.e. y, which, in turn, implies that $I(dsdt)$ has no atoms. As a result,

$$I^{j,k}(\phi) = \int_{\frac{j}{k}}^{\frac{j+1}{k}}\int_{\frac{j}{k}}^{\frac{j+1}{k}} \phi dI + \int_{\frac{j}{k}}^{\frac{j+1}{k}}\int_{\frac{j+1}{k}}^{\frac{j+2}{k}} \phi dI + \int_{\frac{j+1}{k}}^{\frac{j+2}{k}}\int_{\frac{j}{k}}^{\frac{j+1}{k}} \phi dI + \int_{\frac{j+1}{k}}^{\frac{j+2}{k}}\int_{\frac{j+1}{k}}^{\frac{j+2}{k}} \phi dI,$$

from which it follows that

$$\sum_{j=0}^{k-2} I^{j,k}(\phi) \le 2 \int_{B_k}\!\!\int \phi dI$$

by re-arranging some terms in the summation just above. Finally, putting $D = \{(t,t): T\epsilon E\}$,

$$\overline{\lim_k} \sup_\varepsilon T_\varepsilon(\phi \circ 1_{A_k}) \le 2 \overline{\lim_k} \int_{B_k}\int \phi(s-t)I(dsdt)$$

$$= 2 \int_D\int \phi(s-t)I(dsdt) \quad (\text{since } D = \cap_k B_k \text{ and } I(\phi)<\infty)$$

$$= 0,$$

because $I(D) = \int_{R^m} \sum_{t\epsilon E} \alpha^2(x,\{t\})dx = 0.$ QED

Remark. As the proof shows, $\lim_n \lim_\varepsilon T_\varepsilon(\phi_n) = I(\phi) = \lim_\varepsilon \lim_n T_\varepsilon(\phi_n)$. Suppose we can select ϕ_n's which converge *uniformly* to ϕ away from $t=0$, as, for example, when $n=1$ and ϕ is convex, etc. as described before. Then, in fact, $T_\varepsilon(\phi_n) \to I(\phi_n)$ as $\varepsilon \downarrow 0$ *uniformly* in n if $I(\phi)<\infty$. Briefly, here's why: write

$$|T_\varepsilon(\phi_n)-I(\phi_n)| \le |T_\varepsilon(\phi_n)-T_\varepsilon(\phi)| + |T_\varepsilon(\phi)-I(\phi)| + |I(\phi)-I(\phi_n)|;$$

given $\zeta>0$, the first righthand term is $\le \zeta(1+\int\alpha^2)$ $\forall\varepsilon>0$ and large n, using (6) and the uniform convergence of the ϕ_n's; the second term is $\le \zeta$ for all small ε by (5); and of course, the last term vanishes as $n\to\infty$.

Applications. Throughout this section and the next we assume $\alpha(x,dt)$ exists and $\alpha \in L^2(dx)$. Let $\zeta: (0,\infty) \to (0,\infty)$ be decreasing; $\zeta(0^+)=\infty$, and suppose (i) $t \to \zeta(||t||)$, $t\epsilon R^n$, belongs to F^∞ and (ii) $V(t) \equiv (t^n\zeta(t))^{1/m}$ is increasing for $t>0$, with $V(0^+)=0$. (For example, $\zeta(t) = t^{-\gamma}$, $0<\gamma<n$.)

Corollary 1. Suppose $I_x(\zeta) < \infty$ for μ — a.e. $x\epsilon R^m$. Then

(8) $$\operatorname{ap}\overline{\lim_{s\to t}}\frac{||F(s)-F(t)||}{V(||s-t||)} = \infty \quad \lambda_n - \text{a.e. on } E.$$

Proof. We must show that for λ_n – a.e. $t\in E$:

(9) $$\lim_{\varepsilon\downarrow 0}\,[\lambda_n\{B_n(t,\varepsilon)\}]^{-1}\lambda_n\{s\in E\cap B_n(t,\varepsilon):\ ||F(s)-F(t)||\le kV(||s-t||)\} < 1$$

$\forall k>0$. Denote the ratio in (9) by $\tau_t(k,\varepsilon)$ and define $H_r = \{x\in B_m(0,r):\ I_x(\zeta)\le r\}$ and $E_r = F^{-1}(H_r)$, $r=1,2,\ldots$ Obviously, $H_1\subset H_2\subset\ldots$, $\cup H_r = \{x:\ I_x(\zeta)<\infty\} \equiv H$, and

$$\lambda_n(E_r) = \mu(H_r) \uparrow \mu(H) = \mu(R^m) = \lambda_n(E)\ .$$

Restricting F to E_r,

$$\int_{R^m}\int_{E_r}\int_{E_r}\zeta(||s-t||)\alpha(x,ds)\alpha(x,dt)dx \le c_m\, r^{m+1} < \infty\ ,$$

and therefore we may even assume $I(\zeta)<\infty$ in proving (8). Now,

$$\begin{aligned}\tau_t(k,\varepsilon) &\le \frac{1}{c_n\varepsilon^n}\,\lambda_n\{s\in E\cap B_n(t,\varepsilon):\ ||F(s)-F(t)||\le kV(\varepsilon)\}\\ &= \frac{\zeta(\varepsilon)}{c_n\big(V(\varepsilon)\big)^m}\,\lambda_n\{s\in E\cap B_n(t,\varepsilon):\ ||F(s)-F(t)||\le kV(\varepsilon)\}\\ &\le \frac{1}{c_n\big(V(\varepsilon)\big)^m}\int_{E\cap B_n(t,\varepsilon)}\zeta(||s-t||)\ \xi_{kV(\varepsilon)}\big(F(s)-F(t)\big)ds \quad (\text{since } \zeta\downarrow).\end{aligned}$$

Hence, for any $\delta>0$ and $k=1,2,\ldots$,

(10) $$\int_E \tau_t(k,\varepsilon)dt \le \frac{c_m k^m}{c_n}\, T_{kV(\varepsilon)}(\zeta\circ 1_{A_\delta})\ \ \forall\varepsilon\le\delta\ .$$

Recalling that $V(0^+)=0$, (10) leads to

$$\begin{aligned}\overline{\lim_{\varepsilon}}\int_E \tau_t(k,\varepsilon)dt &\le \lim_{\delta}\ \overline{\lim_{\varepsilon}}\ \frac{c_m k^m}{c_n}\, T_\varepsilon(\zeta\circ 1_{A_\delta})\\ &= 0 \ \text{ by (6).}\end{aligned}$$

Finally, Fatou's lemma shows that $\underline{\lim}_{\varepsilon} \tau_t(k,\varepsilon) = 0$ for λ_n — a.e. $t\in E$, $\forall k$, and hence $\underline{\lim}_{\varepsilon} \tau_t(k,\varepsilon) = 0\ \forall k$, for λ_n — a.e. $t\in E$. QED

We actually found that

(11) $$\lim_{\varepsilon} \int_E \tau_t(k,\varepsilon)dt = 0\ \forall k\ .$$

If $\lim_{\varepsilon} \tau_t(k,\varepsilon)$ *exists*, (11) then implies $\lim_{\varepsilon} \tau_t(k,\varepsilon) = 0\ \forall k$, a.e. on E, i.e.

(12) $$\operatorname{ap}\lim_{s\to t} \frac{||F(s)-F(t)||}{V(||s-t||)} = \infty \quad \lambda_n \text{ — a.e. on } E\ .$$

We don't know, however, whether (12) is true assuming only $I_x(\zeta) < \infty$ μ — a.e.

The (limiting) case ζ = constant should correspond to assuming only that $\alpha(x,dt)$ is a continuous measure $\forall x$, and it does: in [6] for $n=m=1$, then in [7] for arbitrary n,m, we showed that (12) holds with $V(\varepsilon) = \varepsilon^{n/m}$. More generally, in fact, suppose $\alpha(x,dt)$ has a k-dimensional "marginal distribution" dominated by λ_k, $0<k<n$, with the remaining n-k-dimensional marginal distribution continuous; that is, suppose

$$\alpha(x,\ B\times A) = \int_B g(x,s,A)\lambda_k(ds),\quad B\in\mathcal{B}_k\ ,\quad A\in\mathcal{B}_{n-k}$$

where $g(x,s,\circ)$ is a continuous measure on $\mathcal{B}_{n+k}$ $\forall\ x\in R^m$, $s\in E$. (The case $k=0$ corresponds to $\alpha(x,dt)$ continuous.) Then (12) holds with $V(\varepsilon) = \varepsilon^{(n-k)/m}$. (See [7, Lemma 3].)

For our second application, assume E is compact, F is continuous, ζ is as above, and let $\omega(\varepsilon)$ be the modulus of F on E:

$$\omega(\varepsilon) = \sup_{\substack{s,t\in E\\ ||s-t||\le\varepsilon}} ||F(s)-F(t)||,\quad \varepsilon>0.$$

Also, define

$$L(\varepsilon) = \left[\int_0^\varepsilon \zeta(r)r^{n-1}dr\right]^{\frac{1}{m}}, \quad \varepsilon\geq 0;$$

$\zeta(||\cdot||) \in F^\infty$ implies $L(\varepsilon) < \infty\ \forall\varepsilon$.

Corollary 2. Suppose $\mu\{x: I_x(\zeta)<\infty\} > 0$. Then

$$\lim_{\varepsilon\downarrow 0} \frac{\omega(\varepsilon)}{L(\varepsilon)} = \infty .$$

Proof. Choose $A \subset \{x: I_x(\zeta)<\infty\}$ such that $A\in\mathcal{B}_m$, $\mu(A)>0$, and

$$\int_A I_x(\zeta)dx < \infty .$$

Let $E' = F^{-1}(A)$. Restricting F to E', (6) holds with E replaced by E', i.e.

$$(13)\quad \lim_\delta \lim_\varepsilon \frac{1}{c_m\varepsilon^m} \int_{E'}\int_{E'} 1_{B_n(t,\delta)}(s)\ \zeta(||s-t||)\xi_\varepsilon(F(s)-F(t))dsdt = 0.$$

But the L.H.S. of (13) is

$$\geq \lim_\delta \overline{\lim_\varepsilon} \frac{1}{c_m\varepsilon^m} \int_{E'}\int_{E'} 1_{B_n(t,\delta)}(s)\ \zeta(||s-t||)1_{[0,\varepsilon]}(\omega(||s-t||))dsdt$$

$$= \lim_\delta \overline{\lim_\varepsilon} \frac{1}{c_m\varepsilon^m} \int_{B_n(0,\delta)} \lambda_n\{E'\cap E'-s\}\zeta(||s||)1_{[0,\varepsilon]}(\omega(||s||))ds$$

by making the change of variables $s-t\to s$ and then reversing the order of integration. Notice that $\lambda_n\{E'\cap E'-s\} \to \lambda_n(E')$ as $s\to 0$ and that $\lambda_n(E') = \mu(A) > 0$. As a result,

$$0 = \lim_\delta \overline{\lim_\varepsilon}\ \varepsilon^{-m} \int_{B_n(0,\delta)} \zeta(||s||)1_{[0,\varepsilon]}(\omega(||s||))ds .$$

Changing to polar coordinates,

$$0 = \lim_{\delta} \overline{\lim_{\varepsilon}}\, \varepsilon^{-m}\left[L\left(\min(\delta,\hat{\omega}(\varepsilon))\right)\right]^m, \hat{\omega}(\varepsilon) \equiv \inf\{t>0\colon \omega(t)>\varepsilon\},$$

$$= \overline{\lim_{\varepsilon}}\, \varepsilon^{-m}\left[L(\hat{\omega}(\varepsilon))\right]^m ,$$

since $\omega(0^+)=0$, which completes the proof.

(Note: Here, the "limiting case" $\zeta \equiv$ constant should be compared to Theorems B, B' of [3].)

Concluding Remark. Suppose F has a differentiable, global modulus W (i.e. $\omega \le W$) for which $\zeta(t) \equiv m(W(t))^{m-1}W'(t)t^{1-n}$ is as above, i.e. $\zeta(||\circ||)\epsilon F^\infty$ and $\zeta(\circ)$ decreases on $(0,\infty)$. Then it follows from Corollary 2 that $I_x(\zeta) = \infty$ for μ – a.e. x. But this, together with the special role of $\alpha(x,dt)$ among the measures concentrated on the M_x's, leads one to believe that, under "suitable conditions," one actually has $\mathrm{Cap}_\zeta M_x = 0$ μ – a.e., i.e.

$$\int_{M_x}\int_{M_x} \zeta(||s-t||)\gamma(ds)\gamma(dt) = \infty$$

for *every* non-trivial probability measure carried by M_x .

Generalizing a result of Kahane, Adler [1] proves that if $F\colon [0,1]^n \to R^m$ is Lipschitz β (i.e. $\omega(\varepsilon) \le$ const. ε^β) and $n-\beta m \ge 0$, then $\dim M_x \le n-\beta m$ for for λ_m – a.e. x, where "dim" stands for Hansdorff dimension. Suppose that $\lambda_m \ll \mu$, i.e. $\alpha(x)>0$ λ_m – a.e. In view of the remarks above, Adler's result might then follow by choosing $W(\varepsilon) =$ const. $\times\, \varepsilon^\beta$ (giving $\zeta(t) =$ const. $\times\, t^{-(n-\beta m)}$) and the fact that $\dim M_x$ is the supremum of the numbers δ for which M_x has positive capacity for $||t||^{-\delta}$.

References

1. R. J. Adler, Hausdorff dimension and Gaussian fields, *Ann. Probability* (to appear).

2. S. M. Berman, Gaussian processes with stationary increments: local times and sample function properties, *Ann. Math. Stat.* **41**, 1260-1272 (1970).

3. ________, Gaussian sample functions: uniform dimension and Hölder conditions nowhere, *Nagoya Math. J.* **46**, 63-86 (1972).

4. S. Bochner, *Lectures on Fourier Integrals*, Ann. of Math Studies, No. 42, Princeton Univ. Press (1959).

5. H. Federer, *Geometric Measure Theory*, Berlin: Springer-Verlag (1969).

6. D. Geman, A note on the continuity of local times, *Proc. Amer. Math. Soc.* **57**, 321-326 (1976).

7. ________, On the approximate local growth of multidimensional random fields, *Z. Wahrscheinlichkietstheorie v. Geb.* (to appear).

8. ________ and J. Horowitz, Local times for real and random functions, *Duke Math. J.*, Dec. 1976 (to appear).

9. J. P. Kahane, *Some Random Series of Functions*, D. C. Heath, Lexington, MA (1968).

10. M. B. Marcus, Capacity of level sets of certain stochastic processes, *Z. Wahrscheinlichkeitstheorie v. Geb.* **34**, 279-284 (1976).

11. S. Orey, Gaussian sample functions and the Hausdorff dimension of level crossings, *Z. Wahrscheinlichkeitstheorie v. Geb.* **15**, 249-256 (1970).