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INVARIANCE PRINCIPLES FOR THE COUPON COLLECTOR'S PROBLEM: A MARTINGALE APPROACH*

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ABSTRACT

For the coupon collector's problem, invariance principles for the partial sequence of bonus sums after n coupons as well as for the waiting times to obtain the bonus sum t are studied through a construction of a triangular array of martingales related to these sequences and verifying the invariance principles for these martingales. This approach provides simpler and neater proofs than given in Rosen (1969, 1970) and strengthens his convergence of finite dimensional results to those of weak invariance principles.

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Key Words and Phrases: Bonus Sums, coupon collector's situation, finite-dimensional distributions, Gaussian functions, invariance principles, martingales, tightness, waiting times.

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1. INTRODUCTION

Consider a sequence $\{\Omega_N, N \geq 1\}$ of *Coupon collector's situations*

$$\Omega_N = \{(a_N(1), p_N(1)), \ldots, (a_N(N), p_N(N))\}, \quad N \geq 1, \tag{1.1}$$

where $a_N(s)$ and $p_N(s)$ (>0) are real numbers and $\sum_{s=1}^{N} p_N(s) = 1$. Consider also a (double) sequence $\{I_{Nk}, k \geq 1\}$ of (row-wise) independent and identically distributed random variables (i.i.d.r.v.), where for each N,

$$P\{I_{Nk} = s\} = p_N(s), \quad 1 \leq s \leq N. \tag{1.2}$$

Let then

$$Y_{Nk} = \begin{cases} a_N(I_{Nk}), & \text{if } I_{Nk} \notin (I_{N1}, \ldots, I_{Nk-1}), \\ 0, & \text{otherwise, for } k \geq 1; \end{cases} \tag{1.3}$$

$$Z_{Nn} = \sum_{i=1}^{n} Y_{Nk}, \quad k \geq 1 \text{ and } Z_{N0} = Y_{N0} = 0. \tag{1.4}$$

Then, Z_{Nn} is termed the *bonus sum after* n *coupons* in the collector's situation Ω_N. If the $a_N(s)$ are all non-negative, Z_{Nn} is non-decreasing in $n(\geq 0)$, and, for every $t \geq 0$, let

$$U_N(t) = \max\{k: Z_{Nk} \leq t\}. \tag{1.5}$$

Then, $U_N(t)$ is termed the *waiting time to obtain the bonus sum* t *in the coupon collector's situtation* Ω_N.

For an arbitrary positive integer b and $\{n_{1N} < \ldots < n_{bN}\}$ satisfying $0 < \liminf_{N\to\infty} N^{-1} n_{1N} \leq \limsup_{N\to\infty} N^{-1} n_{bN} < \infty$, under certain regularity conditions, Rosen (1969) has established (by very elaborate analysis) the asymptotic (multi)normality of (the standardized form of) $\{Z_{Nn_{1N}}, \ldots, Z_{Nn_{bN}}\}$; in a follow up ([6]), he has studied parallel results for $\{U_{Nt}\}$. The object of the present investigation is to propose and formulate an alternative approach to this problem based on the weak convergence of a suitably constructed martingale sequence associated with the Z_{Nn}. This provides a simpler, shorter and neater proof of the aforesaid normality and also an invariance principle for the partial sequence $\{Z_{Nk}, k \le n\}$ as well as for the corresponding sequence of waiting times. The basic regularity conditions are outlined in Section 2. Asymptotic normality of Z_{Nn} is established in Section 3 with the aid of some recently developed martingale central limit theorems, and the multi-normality extension is presented in Section 4. Section 5 is devoted to the (weak) invariance principle for $\{Z_{Nk}, k \le n\}$ and allied results for the sequence $\{U_{Nt}, t \ge 0\}$ are studied in Section 6. A few remarks are made in the concluding section.

2. PRELIMINARY NOTIONS

Note that by (1.2)-(1.4), for every $N(\ge 1)$,

(2.1) $$EY_{Nk} = \sum_{s=1}^{N} a_N(s)p_N(s)[1-p_N(s)]^{k-1}, \quad k \ge 1, \quad EY_{N0} = 0 ;$$

(2.2) $$\phi^*_{Nn} = EZ_{Nn} = \sum_{s=1}^{N} a_N(s)\{1[1-p_N(s)]^n\}, \quad n \ge 1, \quad EZ_{N0} = 0 .$$

Let us denote by

(2.3) $$\phi_{Nn} = \sum_{s=1}^{N} a_N(s)[1-e^{-np_N(s)}], \quad n \ge 0,$$

$$(2.4) \qquad d_{Nn}^2 = \sum_{s=1}^N a_N^2(s) e^{-np_N(s)} (1-e^{-np_N(s)}) - n\left(\sum_{s=1}^N a_N(s) p_N(s) e^{-np_N(s)}\right)^2 , \quad n \geq 0 ;$$

$$(2.5) \qquad A_{Nr} = N^{-1} \sum_{s=1}^N |a_N(s)|^r , \quad \text{for} \quad r = 1,2,3,4 .$$

We assume that

$$(2.6) \qquad \sup_N \left\{ \max_{1 \leq s \leq N} Np_N(s) \right\} \leq M_1 < \infty ;$$

$$(2.7) \qquad \lim_{N \to \infty} \left\{ \max_{1 \leq s \leq N} |a_N(s)| / N^{\frac{1}{2}} A_{N2}^{\frac{1}{2}} \right\} = 0 ;$$

$$(2.8) \qquad \liminf_{N \to \infty} \left[\left(\sum_{s=1}^N a_N^2(s) p_N(s) \right) / A_{N2} \right] \geq M_2 > 0 .$$

Note that $xe^{-x} \leq e^{-1}$, $\forall\ x > 0$ and for $0 \leq x \leq 1$, $0 \leq e^{-nx} - (1-x)^n \leq nx^2 e^{-nx}$. Hence, from (2.2) and (2.3), we have

$$|\phi_{Nn}^* - \phi_{Nn}| = |\sum_{s=1}^N a_N(s)(s)[e^{-np_N(s)} - \{1-p_N(s)\}^n]|$$

$$(2.9) \qquad \leq \sum_{s=1}^N |a_N(s)| p_N(s) \{np_N(s) e^{-np_N(s)}\} \leq e^{-1} M_1 A_{N1} , \ \forall\ n \geq 0,\ N \geq 1 .$$

In fact, if the $a_N(s)$ are all non-negative then $\phi_{Nn} \geq \phi_{Nn}$. Also, noting that $e^{-x}(1-e^{-x}) \leq x$, $\forall\ x \geq 0$, we obtain from (2.4) that

$$(2.10) \qquad d_{Nn}^2 \leq \sum_{s=1}^N a_N^2(s) e^{-np_N(s)} [1-e^{-np_N(s)}]$$

$$\leq n \sum_{s=1}^N a_N^2(s) p_N(s) \leq nM_1 A_{N2} = 0(nA_{N2}) , \ \forall\ n \geq 1,\ N \geq 1 .$$

Further, using the fact that for $0 < x \leq 1$, $(1-e^{-nx}) = (1-e^{-x}) \sum_{k=0}^{n-1} e^{-kx} > x(1-\frac{1}{2}x) \sum_{k=0}^{n-1} e^{-kx}$, we obtain that for $N > \frac{1}{2} M_1$,

$$d_{Nn}^2 \ge (1-\tfrac{1}{2}M_1/N)\sum_{k=0}^{n-1}\left\{\sum_{s=1}^{n} a_N^2(s)p_N(s)e^{-(n+k)p_N(s)} - \left[\sum_{s=1}^{N} a_N(s)p_N(s)e^{-np_N(s)}\right]^2\right\}$$

$$\text{(2.11)} \qquad \ge (1-\tfrac{1}{2}N^{-1}M_1)\sum_{k=0}^{n-1}\left\{\sum_{s=1}^{N} a_N^2(s)p_N(s)e^{-(n+k)p_N(s)}[1-e^{-(n-k)p_N(s)}]\right\}$$

$$\ge (1-\tfrac{1}{2}N^{-1}M_1)e^{-(2n-1)M_1/2N}[(n-1)/2]\sum_{s=1}^{N} a_N^2(s)p_N(s)[1-e^{-\frac{1}{2}(n+1)p_N(s)}] \ .$$

Now, by (2.6), (2.7) and (2.8), $A_{N2}^{-1}\sum_{\{s:p_N(s)>\varepsilon/N\}} a_N^2(s)p_N(s) \ge A_{N2}^{-1}\sum_{s=1}^{N} a_N^2(s)p_N(s)-\varepsilon$ $\forall\, \varepsilon>0$, and noting that for $p_N(s)>\varepsilon/N$ and $n/N>\eta>0$, $1-e^{-\frac{1}{2}(n+1)p_N(s)} \ge c(\varepsilon,\eta)>0$, we obtain from (2.11) that if

$$\text{(2.12)} \qquad 0 < \liminf_{N\to\infty} N^{-1}n \le \limsup_{N\to\infty} N^{-1}n < \infty \ ,$$

then $\liminf_{N\to\infty} (d_{Nn}^2/nA_{N2}) > 0$. Thus, we have

$$\text{(2.13)} \qquad 0 < \liminf_{N\to\infty} (d_{Nn}^2/nA_{N2}) \le \limsup_{N\to\infty} (d_{Nn}^2/nA_{N2}) < \infty \ ,$$

when (2.6)-(2.8) and (2.12) hold.

We are primarily concerned here with the limiting behavior of the partial sequence $d_{Nn}^{-1}(Z_{Nk}-\phi_{Nk};\ k\le n)$. Since $d_{Nn}^{-1}a_N(s)$, $1\le s\le N$ remain invariant under any scalar multiplication, we may without any loss of generality set

$$\text{(2.14)} \qquad A_{N2} = N^{-1}\sum_{s=1}^{N} a_N^2(s) \sim 1 \ , \quad \text{for every } N \ .$$

Then, by (2.9), (2.13) and (2.14), we have $d_{Nn}^{-1}\left\{\max_{1\le k\le n}|\phi_{Nk}^*-\phi_{Nk}|\right\} \to 0$, so that, we may equivalently consider the partial sequence $d_{Nn}^{-1}(Z_{Nk}-\phi_{Nk};\ k\le n)$. In the remaining of this section, we consider a basic lemma, to be used repeatedly in subsequent sections. Let $Q_{Nk}=p_N(I_{Nk})$, $k\ge 1$ and let $g_{Nu}(Y_{Nk},Q_{Nk})$, $u=1,\dots,p(\ge 2)$ be such that

$$\text{(2.15)} \qquad \max_{1\le u\le p}\left\{\max_{1\le s\le N}|g_{Nu}(a_N(s),p_N(s))|\right\} \le M_{N,3} \ , \quad \sup_N M_{N,3} < \infty \ ,$$

$$(2.16)\qquad \max_{1\le u\le p}\left\{\sum_{s=1}^{N}|g_{Nu}(a_N(s),p_N(s))|\right\}\le M_{N,4},\ \sup_N N^{-1}M_{N,4}<\infty .$$

Note that (2.15) and (2.16) insure that

$$(2.17)\qquad \max_{1\le u\le u'\le p}\left\{\sum_{s=1}^{N}|g_{Nu_1}(a_N(s),p_N(s)g_{Nu_2}(a_N(s),p_N(s))|\right\}$$

$$\le M_{N,5}\quad \text{where}\quad M_{N,5}\le M_{N,3}M_{N,4}\ .$$

Lemma 2.1. Under (2.6), (2.15) and (2.16), for every $0=\nu_0<\nu_1<\ldots<\nu_p\le n$,

$$(2.18)\qquad E\prod_{u=1}^{p}g_{Nu}(Y_{N\nu_u},Q_{N\nu_u})=\prod_{u=1}^{p}Eg_{Nu}(Y_{N\nu_u},Q_{N\nu_u})$$

$$+\,0(N^{-1}M_{N,4}^{p-1}[M_{N,3}\vee N^{-1}M_{N,4}^{2}]),$$

$$(2.19)\quad \mathrm{Cov}[g_{N1}(Y_{N\nu_1},Q_{N\nu_1}),g_{N2}(Y_{N\nu_2},Q_{N\nu_2})]=0(N^{-2}[M_{N,5}\vee N^{-1}M_{N,4}]),$$

$$(2.20)\qquad V[g_{N1}(Y_{N\nu_1},Q_{N\nu_1})]=0([N^{-1}M_{N,5}]\wedge[(N^{-1}M_{N,4})^{2}])\ .$$

Proof. We shall only prove (2.18); the others follow by similar arguments. Note that

$$E\prod_{u=1}^{p}g_{Nu}(Y_{N\nu_u},Q_{N\nu_u})=\sum_{1\le s_1\neq\ldots\neq s_p\le N}\prod_{u=1}^{p}\left\{g_{Nu}(a_N(s_u),p_N(s_u))\left[1-\sum_{k=u}^{p}p_N(s_k)\right]^{\nu_u-\nu_{u-1}-1}\right\}$$

$$(2.21)\quad =\sum_{1\le s_1\neq\ldots\neq s_p\le N}\prod_{u=1}^{p}\left\{g_{Nu}(a_N(s_u),p_N(s_u))p_N(s_u)e^{-\nu_u p_N(s_u)}[1+0(N^{-1})]\right\}(\text{by }(2.6))$$

$$=\sum_{1\le s_1\neq\ldots\neq s_p\le N}\prod_{u=1}^{p}\left\{g_{Nu}(a_N(s_u)p_N(s_u))p_N(s_u)e^{-\nu_u p_N(s_u)}\right\}+0(N^{-p-1}M_{N,4}^{p}),$$

by (2.6) and (2.16). Similarly, for each $u(=1,\ldots,p)$, (2.22) $Eg_{Nu}(Y_{N\nu_u},Q_{N\nu_u})=\sum_{s=1}^{N} g_{Nu}(a_N(s),p_N(s))p_N(s)e^{-\nu_u p_N(s)}+0(N^{-2}M_{N,4})$,

where, by (2.6) and (2.16), the first term on the right hand side of (2.22) is $0(N^{-1}M_{N,4})$. The product of the p factors of the first erm in (2.22) involves N^p terms where as (2.21) involves $N^{[p]}$ terms; by (2.15) and (2.16), the contribution of these $N^p - N^{[p]}$ terms is $0(M_{N,3}\cdot M_{N,4}^{p-1}\cdot N^{-p})$. Hence, the proof follows from (2.21)-(2.22). For $p=2$, $N^p - N^{[p]} = N$ and $\sum_{s=1}^{N} g_{N1}(a_N(s),p_N(s))g_{N2}(a_N(s),p_N(s))p_N^2(s)e^{-(\nu_1+\nu_2)p_N(s)} = 0(N^{-2})$. $\left(\sum_{s=1}^{N} |g_{N1}(a_N(s),p_N(s))g_{N2}(a_N(s),p_N(s))|\right) = 0(N^{-2}\cdot M_{N,5})$, so that (2.19) follows on parallel lines. Q.E.D.

3. ASYMPTOTIC NORMALITY OF $\{d_{Nn}^{-1}(Z_{Nn}-\phi_{Nn})\}$

The main theorem of this section if the following.

Theorem 3.1. *Under* (2.6)-(2.8) *and* (2.12),

(3.1) $$L(d_{Nn}^{-1}(Z_{Nn}-\phi_{Nn})) \to N(0,1) .$$

Proof. Unlike the approaches of Baum and Billingley (1965) and Rosen (1969, 1970), our proof rests on a construction of a (triangular array of) martingales related to $\{Z_{Nn}\}$. Let B_{Nk} be the sigma-field generated by $\{Y_{Nj},\ j\le k\}$, $k\ge 1$, and let B_{N0} be the trivial signma-field. Then, for every N, B_{Nk} is non-decreasing. For every N, $n(\ge 1)$, we define

(3.2) $$X_{Nk}^{(n)} = Y_{Nk}(1+Q_{Nk})^{k-1}e^{-nQ_{Nk}},\quad Q_{Nk}=p_N(I_{Nk}),\quad k\ge 1;\quad X_{N0}^{(n)}=0,$$

(3.3) $$\xi_{Nk}^{(n)} = \sum_{s=1}^{N} a_N(s)p_N(s)e^{-np_N(s)}[1+p_N(s)]^{k-1},\quad k\ge 1,\quad \xi_{N0}^{(n)}=0,$$

and consider the sequence

$$(3.4)\qquad \tilde{X}_{Nk}^{(n)} = X_{Nk}^{(n)} - E(X_{Nk}^{(n)} \mid \mathcal{B}_{Nk-1})$$

$$= (X_{Nk}^{(n)} - \xi_{Nk}^{(n)}) + \sum_{\nu=0}^{k-1} X_{N\nu}^{(n)} Q_{N\nu}(1+Q_{N\nu})^{k-\nu},\ k \ge 1;\ \tilde{X}_{N0}^{(n)} = 0 .$$

Then, on denoting by

$$(3.5)\qquad \tilde{S}_{Nk}^{(n)} = \sum_{j=0}^{k} \tilde{X}_{Nj}^{(n)},\ k \ge 0 \quad \text{and} \quad \tilde{\xi}_{Nk}^{(n)} = \sum_{j=0}^{k} \xi_{Nk}^{(n)},\ k \ge 0 ,$$

we obtain from (3.2)-(3.5) that

$$(3.6)\qquad \tilde{\xi}_{Nk}^{(n)} = \sum_{s=1}^{N} a_N(s) e^{-np_N(s)} \{[1+p_N(s)]^k - 1\},\ k \ge 0 ;$$

$$(3.7)\qquad \tilde{S}_{Nk}^{(n)} = \sum_{i=1}^{k} Y_{Ni} e^{-nQ_{Ni}} [(1+Q_{Ni})^k - Q_{Ni}(1+Q_{Ni})^{i-1}],\ k \ge 0 ,$$

$$(3.8)\qquad E(\tilde{S}_{Nk}^{(n)} \mid \mathcal{B}_{Nk-1}) = \tilde{S}_{Nk-1}^{(n)},\ \forall\ k \ge 1 ,$$

so that for every N, $n(\geq 1)$, $\{\tilde{S}_{Nk}^{(n)}, \mathcal{B}_{Nk};\ k \ge 0\}$ is a martingale. From (2.6) and (3.6), it readily follows that

$$(3.9)\qquad |\tilde{\xi}_{Nn}^{(n)} - \phi_{Nn}| = 0(1) ,\quad \text{for every } N,n .$$

Also, note that $\sum_{i=1}^{n} |Y_{Ni} Q_{Ni} e^{-nQ_{Ni}} (1+Q_{Ni})^{i-1}| \le \sum_{i=1}^{n} |Y_{Ni}| Q_{Ni} \le \sum_{s=1}^{N} |a_N(s)| p_N(s)$ $\le M_1^{\frac{1}{2}} A_{N2}^{\frac{1}{2}} \sim M_1^{\frac{1}{2}}$, by (2.6) and (2.14), while for $x \in (0,1)$, $1 \ge e^{-nx}(1+x)^n \ge 1-nx^2$, so that under (2.6) and (2.12), we have from (3.7) that $|\tilde{S}_{Nn}^{(n)} - Z_{Nn} + \phi_{Nn}|$ is bounded, with probability one. Thus, by (2.13), (2.14) and the above, we conclude that for every $\varepsilon > 0$, there exists an $N_0(\varepsilon)$, such that under (2.6)-(2.8) and (2.12),

$$(3.10)\qquad P\{d_{Nn}^{-1} |\tilde{S}_{Nn}^{(n)} - Z_{Nn} + \phi_{Nn}| > \varepsilon\} = 0 ,\ \forall\ N \ge N_0(\varepsilon) .$$

Consequently, it suffices to show that under (2.6)-(2.8) and (2.12),

$$(3.11) \qquad L(d_{Nn}^{-1}\tilde{S}_{Nn}^{(n)}) \to N(0,1) \ .$$

Now, for the martingale-difference array $\{d_{Nn}^{-1}\tilde{X}_{Nk}^{(n)};\ k \le n\}$, by (3.4),

$$(3.12) \qquad |\tilde{X}_{Nk}^{(n)}| \le |Y_{Nk}| + |\xi_{Nk}^{(n)}| + \sum_{\nu=1}^{k-1} |Y_{N\nu}| Q_{N\nu} \, , \quad 1 \le k \le n \, ,$$

where $|\xi_{Nk}^{(n)}| \le \sum_{s=1}^{N} |a_N(s)| p_N(s) \le M_1^{\frac{1}{2}} A_{N2}^{\frac{1}{2}} \sim M_1$. Also, $|Y_{Nk}| \le \max_{1\le s\le N} |a_N(s)| = O(N^{\frac{1}{2}}) = O(d_{Nn})$, by (2.7), (2.13) and (2.14). Finally, $\left(\sum_{\nu=1}^{k-1} |Y_{N\nu}| Q_{N\nu}\right) \le \sum_{s=1}^{N} |a_N(s)| p_N(s) \le M_1^{\frac{1}{2}} A_{N2}^{\frac{1}{2}} \sim M_1^{\frac{1}{2}}$, $\forall\ k \ge 1$. Hence, for every $\varepsilon > 0$, there exists an $N_0(\varepsilon)$, such that

$$(3.13) \qquad P\left\{\max_{1\le k\le n} d_{Nn}^{-1} |\tilde{X}_{Nk}^{(n)}| > \varepsilon\right\} = 0 \, , \quad \forall\ N \ge N_0(\varepsilon) \ ;$$

the above equation also insures that

$$(3.14) \qquad \max_{1\le k\le n} d_{Nn}^{-1} |\tilde{X}_{Nk}^{(n)}| \ \text{ is uniformly bounded in } L_2 \text{ norm} \ .$$

Further, by (3.4)-(3.8), $E([\tilde{S}_{Nn}^{(n)}]^2) = \sum_{k=1}^{n} E([\tilde{X}_{Nk}^{(n)}]^2)$, and some routine steps leads us to

$$(3.15) \qquad d_{Nn}^{-2} E([\tilde{S}_{Nn}^{(n)}]^2) \to 1 \quad \text{as} \quad N \to \infty$$

Thus, by virtue of (3.13)-(3.15) and (3.8), we are in a position to use the recently developed central limit theoresm by Dvoretzky (1972), Scott (1973) and McLeish (1974), and to prove (3.11), it suffices to show that

$$(3.16) \qquad d_{Nn}^{-2}\left\{\sum_{k=1}^{N} [\ell_{Nk}^{(n)} - E(\ell_{Nk}^{(n)})]\right\} \overset{p}{\to} 0 \, ,$$

where

$$(3.17)\qquad \ell_{Nk}^{(n)} = E([\tilde{X}_{Nk}^{(n)}]^2|\mathcal{B}_{Nk-1}) = V(\tilde{X}_{Nk}^{(n)}|\mathcal{B}_{Nk-1})$$
$$= \sum_{s=1}^{N} a_N^2(s)p_N(s)e^{-2np_N(s)}[1+p_N(s)]^{2(k-1)}$$
$$- \sum_{\nu=1}^{k-1} Y_{N\nu}^2 Q_{N\nu} e^{-2nQ_{N\nu}}(1+Q_{N\nu})^{2(k-1)}$$
$$- (\xi_{Nk}^{(n)} - \sum_{\nu=1}^{k-1} Y_{N\nu}Q_{N\nu}e^{-nQ_{N\nu}}(1+Q_{N\nu})^{k-1})^2, \quad k \geq 1 .$$

By steps similar to those after (3.12), it follows that the $\ell_{Nk}^{(n)}$ are all bounded with probability 1, while, by (2.13)-(2.14), $d_{Nn}^{-2} = 0(n^{-1})$. Hence, to prove (3.16), it suffices to show that

$$(3.18)\qquad \max_{1\leq k<q\leq n}|\mathrm{Cov}(\ell_{Nk}^{(n)},\ell_{Nq}^{(n)})| \to 0 \quad \text{as} \quad n \to \infty .$$

Defining $g_{Nu}(Y_{N\nu},Q_{N\nu})$ as $Y_{N\nu}^2 Q_{N\nu}e^{-2nQ_{N\nu}}(1+Q_{N\nu})^{2(k-1)}$ (or $Y_{N\nu}Q_{N\nu}e^{-nQ_{N\nu}}(1+Q_{N\nu})^{k-1}$), it readily follows that (2.15)-(2.16) hold with $M_{N,3} = 0(1)$ (or $0(N^{-\frac{1}{2}})$ and $M_{N,4} \sim M_1$ (or $M_1^{\frac{1}{2}}$), and hence, (3.18) can be proved directly by repeated use of (2.18)-(2.20) for the individual terms in the expansion of $\ell_{Nk}^{(n)}\ell_{Nq}^{(n)}$ by (3.17). Q.E.D.

We may remark that intuitively one may attempt to work with the alternative construction: $\tilde{Y}_{Nk} = Y_{Nk} - E(Y_{Nk}|\mathcal{B}_{Nk-1})$, $k \geq 1$, $\tilde{Y}_{N0} = 0$. Then, one would have

$$(3.19)\qquad \tilde{Z}_{Nn} = \sum_{k=1}^{n}\tilde{Y}_{Nk} = \sum_{k=1}^{n}\{Y_{Nk} - \sum_{s=1}^{N}a_N(s)p_N(s) + \sum_{\nu=1}^{k-1}Y_{N\nu}Q_{N\nu}\}$$
$$= (Z_{Nn}-\phi_{Nn}^*) - \sum_{k=1}^{n}(n-k)[Y_{N\nu}Q_{N\nu} - EY_{N\nu}Q_{N\nu}] .$$

Whereas the asymptotic normality of $d_{Nn}^{-1}\tilde{Z}_{Nn}$ may be proved along the same lines as in $\tilde{S}_{Nn}^{(n)}$, the second term on the right hand side of (3.19) is not generally $o_p(n^{\frac{1}{2}})$, so that this particular construction may not be very helpful for the desired normality of $d_{Nn}^{-1}(Z_{Nn}-\phi_{Nn})$.

4. CONVERGENCE OF FINITE DIMENSIONAL DISTRIBUTIONS

For an arbitrary positive integer b, consider a sequence of positive integers $\{n_{1N} < \dots < n_{bN}\}$ where

$$0 < \liminf_{N\to\infty} N^{-1} n_{1N} < \limsup_{N\to\infty} N^{-1} n_{bN} < \infty . \tag{4.1}$$

We are basically interested in the asymptotic multinormality of

$$N^{-\frac{1}{2}}[Z_{Nn_{1N}} - \phi_{Nn_{1N}}, \dots, Z_{Nn_{bn}} - \phi_{Nn_{bN}}] , \tag{4.2}$$

when (2.6)-(2.8) and (4.1) hold, and we impose (2.14) without any loss of generality. Let us define $\underset{\sim}{\Sigma}_N = ((\sigma_{kq,N}))_{k,q=1,\dots,b}$ by

$$\sigma_{kq,N} = \begin{cases} N^{-1} d^2_{Nn_{kN}} , k=q(=1,\dots,b) \\ N^{-1}\sum_{s=1}^{N} a_N^2(s) e^{-n_{qN}p_N(s)} (1-e^{-n_{kN}p_N(s)}) \\ \quad -N^{-1} n_{kN}\left(\sum_{s=1}^{N} a_N(s)p_N(s)e^{-n_{kN}p_N(s)}\right)\left(\sum_{s=1}^{N} a_N(s)p_N(s)e^{-n_{qN}p_N(s)}\right), \\ \quad 1 \le k < q \le b , \\ \sigma_{qk,N} , \quad 1 \le q < k \le b . \end{cases} \tag{4.3}$$

Recalling that a vector $\underset{\sim}{L}_N$ is $\sim N_b(\underset{\sim}{\mu}_N, \underset{\sim}{\Sigma}_N)$ if for every $\underset{\sim}{\lambda} \neq \underset{\sim}{0}$, $\underset{\sim}{\lambda}'(\underset{\sim}{L}_N - \underset{\sim}{\mu}_N)$ is $\sim N_1(0, \underset{\sim}{\lambda}'\underset{\sim}{\Sigma}_N\underset{\sim}{\lambda})$, we have the following.

<u>Theorem 4.1</u>. *Under* (2.6)-(2.8) *and* (4.1)

$$L(N^{-\frac{1}{2}}[Z_{Nn_{jN}} - \phi_{Nn_{jN}}, \ 1 \le j \le b]) \to N_b(\underset{\sim}{0}, \underset{\sim}{\Sigma}_N) . \tag{4.4}$$

Proof. Define $\tilde{X}_{Nk}^{(n)}$, $\tilde{S}_{Nk}^{(n)}$, $k \geq 0$, as in Section 3. Let then

$$(4.5) \qquad \hat{X}_{Nk}^{(n)} = \begin{cases} \tilde{X}_{Nk}^{(k)}, & 0 \leq k \leq n, \\ 0, & k > n; \end{cases} \qquad \hat{S}_{Nk}^{(n)} = \sum_{\nu=0}^{k} \hat{X}_{N}^{(n)}, \quad k \geq 0 .$$

Note that $\hat{S}_{Nk}^{(n)} = \tilde{S}_{Nk}^{(n)}$ for $k \leq n$ and $\hat{S}_{Nk}^{(n)} = \tilde{S}_{Nn}^{(n)}$ for $k > n$. Thus, proceeding as in (3.9)-(3.10), under (4.1), it suffices to show that $N^{-\frac{1}{2}}(\hat{S}_{Nn_{jN}}^{(n_{jN})}, 1 \leq j \leq b)$ is $\sim \mathcal{N}_b(\underset{\sim}{0}, \underset{\sim}{\Sigma}_N)$. For this purpose, consider an arbitrary linear compound (where $\underset{\sim}{\lambda} \neq 0$)

$$(4.6) \qquad \sum_{j=1}^{b} \lambda_j N^{-\frac{1}{2}} \hat{S}_{Nn_{jN}}^{(n_{jN})} = N^{-\frac{1}{2}} \sum_{k=1}^{n_{bN}} \sum_{j=1}^{b} \lambda_j \hat{X}_{Nk}^{(n_{jN})}) = N^{-\frac{1}{2}} \sum_{k=1}^{n_{bN}} \hat{X}_{Nk}^{*}, \quad \text{say} ,$$

where $X_{Nk}^{*} = \sum_{j=1}^{b} \lambda_j \hat{X}_{Nk}^{(n_{jN})}$, $k \geq 1$. Note that by (3.2), (3.4) and (4.5), $E(X_{Nk}^{*} | \mathcal{B}_{Nk-1}) = 0$, $\forall\ k \geq 1$, and, we may virtually repeat the proof of Theorem 3.1 [viz., (3.12)-(3.18)] with $\tilde{X}_{Nk}^{(n)}$ being replaced by $\hat{X}_{Nk}^{*}$; the details are omitted. Q.E.D.

The prescribed martingale approach provides a solution, considerably shorter and neater than the one given in Rosen (1969).

5. AN INVARIANCE PRINCIPLE FOR THE BONUS SUM PROCESS

For an arbitrary $T (0 < T < \infty)$, let $J = [0,T]$, and for $0 \leq x \leq y \leq T$, define

$$\gamma_N(x,y) = N^{-1} \sum_{s=1}^{N} a_N^2(s) e^{-Nyp_N(s)} [1 - e^{-Nxp_N(s)}]$$

$$(5.1) \qquad - x \left\{ \sum_{s=1}^{N} a_N(s) p_N(s) e^{-Nxp_N(s)} \right\} \left\{ \sum_{s=1}^{N} a_N(s) p_N(s) e^{-Nyp_N(s)} \right\} ,$$

and let $\gamma_N(x,y) = \gamma_N(y,x)$ for $0 \le y \le x \le T$. Proceeding as in (2.10)-(2.11), it can be shown that $\gamma_N(x,y)$ is (uniformly in N) a continuous function of $(x,y) \in J^2$. Let us then define

$$\gamma_N^0(x,y) = \gamma_N(x,x) + \gamma_N(y,y) - 2\gamma_N(x,y)\ , \quad \forall\ (x,y) \in J^2 . \tag{5.2}$$

It follows from (5.1) and (5.2) that $\gamma_N^0(x,y) \ge 0$, $\forall\ (x,y) \in J^2$, and, moreover, by using (2.6)-(2.8) and (2.14), it follows by some standard steps that there exists a positive constant $K(<\infty)$, depending only on T and the M_i in (2.6)-(2.8), such that

$$\sup_N \gamma_N^0(x,y) \le K|y-x|\ , \quad \forall (x,y) \in J^2 . \tag{5.3}$$

Let $\{\zeta_N = [\zeta_N(x),\ x \in J]\}$ be a sequence of (independent) Gaussian functions on J, where $E\gamma_N(x) = 0$ and $E\gamma_N(x)\gamma_N(y) = \gamma_N(x,y)$, $\forall\ (x,y) \in J^2$. Then $E[\zeta_N(y) - \zeta_N(x)]^4 = 2(E[\zeta_N(y) - \zeta_N(x)]^2)^2 = 2[\gamma_N^0(x,y)]^2 \le 2K^2(y-x)^2$, $\forall (x,y) \in J^2$, so that by the Kolmogorov existence theorem, for every N, ζ_N belongs to the space $C[J]$, with probability 1.

Now, for every N, consider the sample process $W_N = \{W_N(x),\ x \in J\}$, where

$$W_N(x) = N^{-\frac{1}{2}}[Z_{N[Nx]} - \phi_{N[Nx]})\ , \quad x \in J\ , \tag{5.4}$$

$[s]$ being the largest integer $\le s$. Then, W_N belongs to the space $D[J]$, endowed with the Skorokhod J_1-topology.

<u>Theorem 5.1</u>. *Under* (2.6)-(2.8) *and* (2.14), $\{W_N\}$ *and* $\{\zeta_N\}$ *are convergent-equivalent in law, in the* J_1- *topology on* $D[J]$.

Proof. We need to show that (a) the finite dimensional distributions of $\{W_N\}$ are convergent equivalent to the corresponding ones of $\{\zeta_N\}$, and (b) that $\{W_N\}$ is *tight*. Now, (a) follows readily from Theorem 4.1. Also, by (1.4) and (5.5), $W_N(0) = 0$, with probability 1, $\forall$ N. Hence, to prove (b), it suffices to show that for every $\varepsilon > 0$ and $\eta > 0$, there exists a δ: $0 < \delta < T$ and an interger N_0, such that for every $x \in J$ and $N \geq N_0$,

$$(5.5) \qquad P\{\sup[|W_N(y) - W_N(x)|: x \geq y \geq (x-\delta) \vee 0] > \varepsilon\} < \eta\delta/T .$$

Suppose that in (5.4), we replace $Z_{N[N]} - \phi_{N[Nx]}$ by $\tilde{S}^{[Nx]}_{N[Nx]}$, $x \in J$, and denote the resulting process by $\tilde{W}_N$. Then, proceeding as in (3.9)-(3.10), it follows that for every $\varepsilon' > 0$,

$$(5.6) \qquad \limsup_{N\to\infty} P\left\{\sup_{x\in J}|W_N(x) - \tilde{W}_N(x)| > \varepsilon\right\} = 0 .$$

Hence, it suffices to prove (5.5) with $\tilde{W}_N$ replacing W_N. Towards this note that

$$(5.7) \qquad N^{-\frac{1}{2}}(\tilde{S}^{(n)}_{Nn} - \tilde{S}^{(k)}_{Nk}) = N^{-\frac{1}{2}}(\tilde{S}^{(n)}_{Nn} - \tilde{S}^{(n)}_{Nk}) + N^{-\frac{1}{2}}(\tilde{S}^{(n)}_{Nk} - \tilde{S}^{(k)}_{Nk}) , \quad \forall\ k \geq 0 .$$

Since $\{S^{(n)}_{Nk}, \mathcal{B}_{Nk}; k \geq 1\}$ is a martingale, by (3.12), (3.14) and (3.19) [insuring that $\left(\sum_{k=1}^{[Nx]}[\ell^{(n)}_{Nk} - E\ell^{(n)}_{Nk}]\right)/d^2_{N[Nx]} \xrightarrow{p} 1$, $\forall\ x \in J$], we are in a position to use Theorem 2 of Scott (1973) and this, along with (2.13) and (2.14), implies that for every $\varepsilon > 0$ and $\eta > 0$, there exist a δ: $0 < \delta < M$ and an N_0, such that for $N \geq N_0$, $n = [Nx]$, $x \in J$

$$(5.8) \qquad P\left\{\max_{n-\delta N\leq k\leq n} N^{-\frac{1}{2}}|S^{(n)}_{Nn} - S^{(n)}_{Nk}| > \tfrac{1}{2}\varepsilon\right\} < \frac{1}{2T}\eta\delta .$$

Also, if we choose $\delta(>0)$ so small that $\delta M_1 < 1$, then for $[n-k] \leq \delta N$,

$(n-k)\left\{\max_{1\le s\le N} p_N(s)\right\} \le \delta M_1 < 1$, by (2.6). Hence, for $n \ge k \ge (n-\delta N) \wedge 0$, we obtain from (3.7) that

$$(5.9) \quad N^{-\frac{1}{2}}(\tilde{S}_{Nk}^{(k)} - \tilde{S}_{Nk}^{(n)}) = \sum_{i=1}^{k} [g_{n,k,i}(Y_{Ni},Q_{Ni}) - Eg_{n,k,i}(Y_{Ni},Q_{Ni})] ,$$

where

$$(5.10) \quad g_{n,k,i}(a,b) = N^{-\frac{1}{2}} a(1-e^{-(n-k)b})[e^{-kb}(1+b)^k - be^{-kb}(1+b)^{i-1}] ,$$

$$1 \le i \le k \le n \le Nt .$$

Note that by (2.6)-(2.7), for every $n \le Nt$, (2.15)-(2.17) hold with $M_{N,3} = 0(N^{-1}(n-k)N^{-\frac{1}{2}} \max_{1\le s\le N}|a_N(s)|) = [0(1)][(n-k)/N]$, $M_{n,4} = N^{-\frac{1}{2}}(n-k)\cdot 0(1)$ and $M_{N,5} = N^{-2}(n-k)^2\cdot 0(1)$. Hence, by (2.19)-(2.10) and (5.9)-(5.10), we have

$$(5.11) \qquad E\{[N^{-\frac{1}{2}}(\tilde{S}_{Nk}^{(k)} - \tilde{S}_{Nk}^{(n)})]^2\} \le M^*[N^{-1}(n-k)]^2 ,$$

$$\forall\ k\colon NT \ge n \ge k \ge (n- N) \vee 0 ,$$

where $M^*(<\infty)$ does not depend on δ. By (5.11) and Theorem 12.2 of Billingsley (1968, p. 94), we conclude that for every $n \le TN$, $t < \infty$,

$$(5.12) \qquad P\left\{\max_{n-\delta N\le k\le n} N^{-}|\tilde{S}_{Nk}^{(k)} - \tilde{S}_{Nk}^{(n)}| > \tfrac{1}{2}\varepsilon\right\} \le K^*\varepsilon^{-2}\delta^2 , \quad K^* < \infty ,$$

and K^* does not depend on ε and δ. For every $\varepsilon > 0$, $\eta > 0$ and $T < \infty$, we choose $\delta(>0)$ so small that $\delta < \eta\varepsilon^2/2K^*T$, so that the right hand side of (5.12) is $\le \frac{1}{2}\eta\delta/T$. From (5.8) and (5.12), we obtain that

$$(5.13) \quad P\left\{\max_{n-\delta N\le k\le n} N^{-} |\tilde{S}_{Nn}^{(n)} - \tilde{S}_{Nk}^{(k)}| > \varepsilon\right\} \le \eta\delta/T, \ \forall\ NT \ge n \ge k \ge (n- N) \vee 0 ,$$

and this completes the proof of (5.5) (for $\tilde{W}_N$). Q.E.D.

6. INVARIANCE PRINCIPLE FOR THE WAITING TIME PROCESS

We define the *waiting times* $U_N(t)$, $t \geq 0$ as in (1.5). Note that here all the $a_N(s)$ are assumed to be non-negative, so that Z_{Nk} is $\nearrow$ in $k(\geq 0)$. Here also we make the assumptions (2.6)-(2.8) as well as the convention (2.14) i.e., $N^{-1}A_{N2} \sim 1$, so that $0 \leq A_{N1} \leq A_{N2}^{\frac{1}{2}} \sim 1$. We assume that

(6.1) $$\liminf_{N\to\infty} A_{N1} \geq A_1 > 0 .$$

Note that by (2.14) and (2.7), $\max_{1\leq s\leq N} N^{-\frac{1}{2}}|a_N(s)| = 0(1)$, so that $N^{-\frac{1}{2}}|Y_{Nk}|$, $k \geq 1$ are all uniformly asymptotically (as $N \to \infty$) negligible. Further, by definition $Z_{Nn} \leq NA_{N1}$ $\forall\, n \geq 1$, so that in (1.5), we are only interested in the domain $t \leq NA_1$. Let us define ϕ_{Nn}, $n \geq 0$ as in (2.3) and introduce the sequence $\alpha_N = \{\alpha_N(x) \colon 0 \leq x \leq NA\}$ by letting

(6.2) $$\phi_{N\alpha_N(x)} = x , \quad 0 \leq x < NA_1 .$$

Note that by (2.3) and (6.2), $\alpha_N(x)$ is non-decreasing and

(6.3) $$\alpha_N'(x) = \frac{d}{dx}\,\alpha_N(x) = \left[\sum_{s=1}^{N} a_N(s)p_N(s)e^{-\alpha_N(x)p_N(s)}\right]^{-1} ,$$

(6.4) $$\alpha_N''(x) = \frac{d}{dx}\,\alpha_N'(x) = [\alpha_N'(x)]^2\left[\sum_{s=1}^{N} a_N(s)p_N^2(s)e^{-\alpha_N(x)p_N(s)}\right] .$$

Thus, by steps similar to those in (2.10)-(2.11), it can be shown that under (2.6)-(2.8), (2.14) and (6.1), for $x = Nu$, $0 < u < A_1$, $\alpha_N'(Nu)$ and $\alpha_N''(Nu)$ are continuous functions of u and, further,

(6.5) $$\alpha_N'(Nu) = 0_e(1) \quad \text{and} \quad \alpha_N''(Nu) = 0_e(N^{-1}); \quad 0_e = \text{exact order} .$$

Let us also define $\gamma_N(x,y)$ as in (5.1) and set

$$(6.6)\qquad \beta_N^2(Nu) = [\gamma_N(\alpha_N(Nu),\ \alpha_N(Nu))][\alpha_N'(Nu)]^2\ ,\quad 0 < u < A_1\ .$$

Then, note that by (6.5), for every $u \in (0,A_1)$ and fixed $\nu(|\nu| \le K)$, as $N \to \infty$,

$$(6.7)\qquad N^{-\frac{1}{2}}[\alpha_N(Nu + N^{\frac{1}{2}}\nu) - \alpha_N(Nu)]$$

$$= \nu\alpha_N'(Nu) + \tfrac{1}{2}\,\nu^2 N^{\frac{1}{2}}\alpha_N''(Nu + \theta N^{\frac{1}{2}}\nu) \qquad (0 < \theta < 1)$$

$$= \nu\alpha_N'(Nu) + 0(N^{-\frac{1}{2}})\ .$$

Further, by (6.2) and (1.5), for every ν and $u \in (0,A_1)$,

$$(6.8)\quad P\{U_N(Nu) > \alpha_N(Nu+N^{\frac{1}{2}}\nu)\} = P\{Z_{N\alpha_N(Nu+N^{\frac{1}{2}}\nu)} \le Nu\}$$

$$= P\{N^{-\frac{1}{2}}[Z_{N\alpha_N(Nu+N^{\frac{1}{2}}\nu)} - \phi_{N\alpha_N(Nu+N^{\frac{1}{2}}\nu)}] \le N^{-\frac{1}{2}}[Nu - \phi_{N\alpha_N(Nu+N^{\frac{1}{2}}\nu)}]\}$$

$$= P\{W_N(N^{-1}\alpha_N(Nu+N^{\frac{1}{2}}\nu)) \le -\nu\}\ ,\quad \text{by (6.2)},$$

where $W_N(\cdot)$ is defined by (5.4), and by (6.7), $N^{-1}[\alpha_N(Nu+N^{\frac{1}{2}}\nu) - \alpha_N(Nu)] \to 0$ as $N \to \infty$, $\forall\ u \in (0,A_1)$. Hence, by Theorem 5.1 [viz., (5.5)], the right hand side of (6.8) is convergent equivalent to

$$(6.9)\qquad P\{W_N(N^{-1}\alpha_N(Nu)) \le -\nu\}\ ,\quad \text{for every finite } \nu\ .$$

On the other hand, for every finite ν and $u \in (0,A_1)$, by (6.7) and (6.8),

$$P\{N^{-\frac{1}{2}}[U_N(Nu) - \alpha_N(Nu)]/\beta_N(Nu) > \nu\}$$

$$(6.10)\quad = P\{U_N(Nu) > \alpha_N(Nu) + N^{\frac{1}{2}}\beta_N(Nu)\nu\}$$

$$= P\{U_N(Nu) > \alpha_N(Nu+N^{\frac{1}{2}}\beta_N(Nu)\nu/\alpha_N'(Nu) + 0(1))\}$$

$$\sim P\{W_N(N^{-1}\alpha_N(Nu+N^{\frac{1}{2}}\nu\beta_N(Nu)/\alpha_N'(Nu) + 0(1)) \le -\nu\beta_N(Nu)/\alpha_N'(Nu) + 0(N^{-\frac{1}{2}})\}$$

Now, by (5.1) and (6.4), (6.6), $\beta_N(Nu)/\alpha_N'(Nu) = \gamma_N^{\frac{1}{2}}(\alpha_N(Nu), \alpha_N(Nu))$, and by (2.10) and (5.1), it is $0(1)$. Hence, by the convergence equivalence of (6.8)-(6.9), the right hand side of (6.10) is convergent equivalent to

$$(6.11) \qquad P\{W_N(N^{-1}\alpha_N(uN))/\gamma_N^{\frac{1}{2}}(\alpha_N(Nu), \alpha_N(Nu)) \le -\nu\}$$
$$\to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\nu}\exp(-\tfrac{1}{2}t^2)dt = \frac{1}{\sqrt{2\pi}}\int_{\nu}^{\infty}\exp(-\tfrac{1}{2}t^2)dt ,$$

by Theorem 5.1. Hence, we conclude that

$$(6.12) \qquad L(N^{-\frac{1}{2}}[U_N(Nu)-\alpha_N(Nu)]/\beta_N(Nu)) \to N(0,1) , \quad \forall\ u \in (0,A_1) .$$

Let $J^* = [0,T^*]$ for some $0 < T^* < A_1$ and consider a sequence of stochastic processes $W_N^* = \{W_N^*(u),\ u \in J^*\}$, $N \ge 1$, where

$$(6.13) \qquad W_N^*(u) = N^{-\frac{1}{2}}\{U_N(Nu)-\alpha_N(Nu)\}/\alpha_N'(Nu) , \quad 0 \le u \le T^* .$$

Let us define $\gamma_N(.,)$ as in (5.1) and $\alpha_N(.)$ as in (6.2), and let

$$(6.14) \qquad \gamma_N^*(u,\nu) = \gamma_N(\tfrac{1}{N}\alpha_N(Nu) , \tfrac{1}{N}\alpha_N(N\nu)) , \quad u,\nu \in J^{*2}.$$

Since $\alpha_N(x)$ is a monotonic transformation of x, defining $\{\zeta_N\}$ as in after (5.3), by transformation of the time-parameter, we obtain a sequence $\{\zeta_N^*\}$ of Gaussian functions where $\zeta_N^* = \{\zeta_N^*(u),\ u \in J^*\}$ with $E\zeta_N^*(u) = 0$ and $E\zeta_N^*(u)\zeta_N^*(\nu) = \gamma_N^*(u,\nu)$, $\forall\ u,\nu \in J^*$.

Theorem 6.1. *Under* (2.6)-(2.8), (2.14) *and* (6.1), *for every* $T^* < A_1$, $\{W_N^*\}$ *and* $\{\zeta_N^*\}$ *are convergent equivalent in law.*

Proof. For every given $m(\ge 1)$ and $0 \le u_1 < \ldots < u_m \le T^*$, virtually repeating the steps (6.8)-(6.11), but working with the vector case, it follows that

$$(6.15)\qquad [W_N^*(u_1),\ldots,W_N^*(u_m)] \overset{\mathcal{D}}{\sim} [\zeta_N^*(u_1),\ldots,\zeta_N^*(u_m)] .$$

Also, recall that by (1.5)

$$(6.16)\qquad Z_{NU_N(t)} \le t < Z_{NU_N(t)} + Y_{NU_N(t)+1} , \quad \forall\, t \ge 0 ,$$

where by (2.7) and (2.14), $\sup_{k\ge1}|Y_{Nk}| \le \max_{1\le s\le N}|a_N(s)| = O(N^{\frac{1}{2}})$. Hence,

$$(6.17)\qquad N^{\frac{1}{2}}\left\{\sup_{u\in J^*}|N^{-1}Z_{NU_N(Nu)}-u|\right\} \overset{p}{\to} 0 \quad \text{as} \quad N \to \infty .$$

On the other hand, by Theorem 5.1 and (6.2),

$$(6.18)\qquad \sup_{u\in J^*}|N^{-1}Z_{N\alpha_N(u)} - u| \overset{p}{\to} 0 , \quad \text{as} \quad N \to \infty .$$

By (6.5), (6.17) and (6.18), it follows by some standard steps that

$$(6.19)\qquad \sup_{u\in J^*}|N^{-1}U_N(Nu) - N^{-1}\alpha_N(Nu)| \overset{p}{\to} 0 .$$

Having proved this, we may proceed along the lines of the proof of Theorem 17.3 of Billingsley (1968, p. 149) and show that the weak convergent equivalence of $\{W_N\}$ and $\{\zeta_N\}$ in Theorem 5.1 implies the same for $\{W_N^*\}$ and $\{\zeta_N^*\}$. Q.E.D.

7. SOME CONCLUDING REMARKS

Parallel to (6.12), Rosen (1970) has obtained the asymptotic normality along with the asymptotic equivalence of $\alpha_N(Nu)$ and $EU_N(Nu)$ as well as of $\beta_N^2(Nu)$ and $\mathrm{Var}[U_N(Nu)]$. However, his conditions (2.7)-(2.8) are somewhat more restrictive than ours and our Theorem 5.1 provides us with Theorem

6.1 under no entra conditions — Rosen's approach presumably encouters considerable difficulties in this respect. Secondly, in his Theorems 2 and 3, Rosen (1969) has studied the convergence of f.d.d.'s of $\{W_N\}$ to those of $\{\zeta_N\}$ under additional conditions [viz., his (5.4), (5.10)] which insures that $\gamma_N(x,y) \to \gamma(x,y)$ as $N \to \infty$, $\forall\ (x,y) \in J^2$. Under his setup, using his Lemma 3.17, one can also prove the tightness of $\{W_N\}$ (by using Theorem 12.2 of Billingsley (1968)) provided one assumes (in our notations) that for some $r > 2$, $A_{Nr}/A_{N2}^{r/2} = O(1)$, $\forall\ N \geq N_0$. In our approach, these additional conditions do not appear to be necessary. Moreover, contrasted with his Hilbert space approach, the present one appears to be more elementary too. Finally, through the pioneering work of Rosen (1972a,b), the coupon collector's problem occupies a prominent place in the development of the asymptotic theory of finite population sampling (without replacement) with varying probabilities. It is hoped that the results of the present paper will lead to more work in this potential area.

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