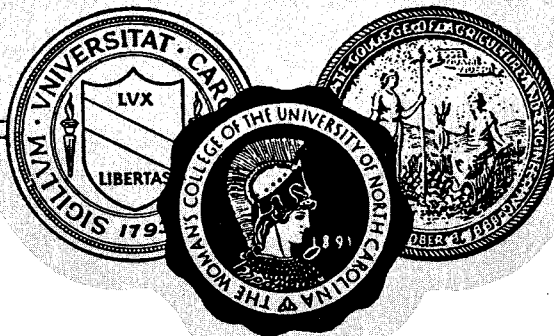


THE INSTITUTE OF STATISTICS

THE CONSOLIDATED UNIVERSITY
OF NORTH CAROLINA



SINGULARITIES IN THE DISTRIBUTION OF THE
INCREMENTS OF A SMOOTH FUNCTION

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§1. By the "distribution of the increments" of a Borel function $F: [0,1] \rightarrow \mathbb{R}$,
I mean the measure

$$\lambda(B) = \int_0^1 \int_0^1 1_B(F(s)-F(t)) ds dt ,$$

B a Borel set in \mathbb{R} . λ is the convolution of the "occupation measure" $\mu(B) = m\{F^{-1}(B)\}$ with $\mu(-B)$; here m is Lebesgue measure. When $\mu \ll m$, write $\alpha(x)$ for the Radon-Nikodym derivative $\frac{d\mu}{dm}(x)$ (the "local time" of F at x). Of course $\mu \ll m$ implies $\lambda \ll m$ and

$$(1) \quad \Lambda(x) \equiv \frac{d\lambda}{dm}(x) = \int_{-\infty}^{\infty} \alpha(y)\alpha(x+y)dy .$$

Although this paper treats only smooth F 's (at least C^1), the relevant background consists of two general results from [3]. Throughout, ψ will denote a nonnegative.

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Borel measurable function. Define

$$I(\psi; F) = \int_0^1 \int_0^1 \psi(F(s) - F(t)) ds dt = \int \psi d\lambda \leq \infty.$$

Then (a) if ψ is even, decreasing on $(0, \infty)$, and nonintegrable on $(0, 1)$, then $I(\psi; F) = \infty$ for any F ; (b) $\mu \ll m$ with $\alpha \in L^2$ if and only if $I(\psi; F) < \infty \forall \psi \in L^1$.

Now for F differentiable a.e., $\mu \ll m$ if and only if $D_0 \equiv \{t: F'(t) = 0\}$ has Lebesgue measure 0. Suppose $F \in C^1$ (i.e. has a continuous derivative, with the usual conventions about the endpoints) and $D_0 \neq \emptyset$, $m(D_0) = 0$. Then, as the Theorem states, $\lim_{x \rightarrow 0} \Lambda(x) = \infty$. (The additional assumptions made on F below are not needed for this.) Hence $I(\psi; F) = \infty$ for some $\psi \in L^1$ (and so $\alpha \notin L^2$) because $\lambda \ll m$ implies

$$(2) \quad I(\psi; F) = \int \Lambda(x) \psi(x) dx.$$

So, the question arises: for which ψ 's - in particular, which monotone ones - is $I(\psi; F) < \infty$? This depends on the nature of the singular points of Λ .

Assume now $D_0 \neq \emptyset$, F is C^2 and that $F''(t) \neq 0$ for all $t \in D_0$. Then D_0 is finite, say $D_0 = \{a_i\}_{i=1}^N$, $0 \leq a_1 < a_2 < \dots < a_N \leq 1$. Let $A_i = F(a_i)$ and let $\{B_i\}_{i=1}^L$ denote the (distinct) elements of $\{A_i - A_j\}$ for which there exist $t_1, t_2 \in D_0$ with $F''(t_1)F''(t_2) > 0$ and $F(t_1) - F(t_2) = A_i - A_j$. $\{B_i\}$ is symmetric about 0 and contains 0. For the version of $\Lambda(x)$ given by (1):

THEOREM. $\Lambda(x)$ is continuous on $\mathbb{R} \setminus \{B_i\}_1^L$ and

$$(3a) \quad 0 < \lim_{x \rightarrow B_i} \frac{\Lambda(x)}{-\log|x - B_i|} \leq \overline{\lim}_{x \rightarrow B_i} \frac{\Lambda(x)}{-\log|x - B_i|} < \infty \quad 1 \leq i \leq L.$$

Consequently, for $\psi \in L^1$

$$(3b) \quad I(\psi; F) < \infty \iff \psi \in L^1 \left\{ \sum_{i=1}^L |\log|x - B_i|| dx \right\}.$$

In particular, if ψ is even and decreasing on $(0, \infty)$, then

$$I(\psi;F) < \infty \iff \int_0^1 \psi(x) \log 1/x \, dx < \infty .$$

§2. The fact that the singularities of Λ occur among the points $\{\Lambda_i - A_j\}$ is fairly obvious. Indeed for F as above ([4])

$$(4) \quad \alpha(x) = \sum_{s \in F^{-1}(\{x\})} |F'(s)|^{-1} .$$

(Since $F(D_0)$ has measure zero, it doesn't matter how α is defined there.) Clearly α is well-behaved off $\{A_i\}$, and, in turn, Λ off $\{\Lambda_i - A_j\}$. (Actually, (4) is valid for any F such that F' exists a.e., although " $s \in F^{-1}(\{x\})$ " must be replaced by " $s \in F^{-1}(\{x\}) \cap \{F' \text{ exists, finite}\}$ " and neither $F(\{|F'| = \infty\})$ nor $F(\{F' \text{ doesn't exist}\})$ need have measure 0.)

The Co-Area Theorem [2], applied to the Lipschitz function $s, t \rightarrow F(s) - F(t)$, leads to this expression for Λ :

$$(5) \quad \Lambda(x) = \iint_{U_x} [(F'(s))^2 + (F'(t))^2]^{-1/2} H(dsdt) ;$$

here $U_x = \{(s,t) : F(s) - F(t) = x\}$ and H is one-dimensional Hausdorff measure in \mathbb{R}^2 . This shows clearly where Λ might explode. Nonetheless, I will not refer again to (5), but instead work with the version of Λ given by (1) with α as in (4).

That the singularities of Λ are logarithmic is perhaps not as evident, and emerged in a curious way. To get an idea of when $I(\psi;F)$ is finite, $\psi \in L^1$, choose a convenient random function $X(t, \omega)$, $0 \leq t \leq 1$, $\omega \in \Omega$, with smooth trajectories and compute the expected value $E\{I(\psi;X(\cdot, \omega))\}$ of the random variable $\omega \rightarrow I(\psi;X(\cdot, \omega))$. For instance, let $X(t, \omega)$ be Gaussian, mean 0, $\sigma^2(s, t) = E(X(s) - X(t))^2$. Then

$$E\{I(\psi; X(\cdot, \omega))\} = \int_0^1 \int_0^1 \int_{-\infty}^{\infty} \psi(x) [2\pi\sigma(s,t)]^{-1} \exp\left\{-\frac{x^2}{2\sigma^2(s,t)}\right\} dx ds dt .$$

For simplicity, and to insure the differentiability of the sample functions, suppose there are constants $0 < C_1 < C_2 < \infty \ni C_1|s-t| < \sigma(s,t) < C_2|s-t| \quad \forall s,t$. (For example, $X(t, \omega)$ is stationary, $r(t) \equiv EX_t X_0 \neq r(0)$, $t \neq 0$, and $-r''(0) < \infty$.) A straightforward computation yields (for ψ even):

$$E\{I(\psi; X(\cdot, \omega))\} < \infty \iff \int_0^{\infty} e^{-y^2} \frac{1}{y} \int_0^y \psi(x) dx dy < \infty ;$$

equivalently, $M_{\psi}(x) \equiv \frac{1}{x} \int_0^x \psi(u) du$ is integrable around the origin, say over $[0,1]$.

If ψ is decreasing on $(0, \infty)$, then $M_{\psi}(x)$ is the usual maximal function:

$$M_{\psi}(x) = \sup_{0 < u < x < v < 1} \frac{1}{v-u} \int_u^v \psi(y) dy, \quad 0 < x < 1 ;$$

hence $M_{\psi} \in L^1[0,1]$ if and only if $\psi \in L \log L$, i.e.

$$\int_0^1 \psi(x) \log^+ \psi(x) dx < \infty .$$

Whether or not ψ is monotone, Fubini's theorem shows

$$\int_0^1 M_{\psi}(x) dx = \int_0^1 \psi(x) \log \frac{1}{x} dx .$$

Consequently, $I(\psi; X(\cdot, \omega)) < \infty$ a.s. for any $0 \leq \psi \in L^1$ with $\psi(x) \log \frac{1}{x} \in L^1[0,1]$, and likewise for any stochastic process which satisfies several mild conditions concerning the distribution of its derivative $X'(s, \omega)$. This is a "stochastic version" of the real-variable theorem above: only the "fixed" singularity of Λ at 0 is picked up; the others - at $\{B_i\} \setminus 0$ - depend on the specific function and will generally occur at any fixed point x_0 with probability 0.

Rounding out the picture, it follows from a theorem of Bulinskaya

[1] that the hypotheses of the theorem are valid for almost every sample function of a stochastic process $X(t, \omega)$ for which: (i) $X(\cdot, \omega)$ is C^2 a.s.,

(ii) for each $0 \leq t \leq 1$, $X'(t, \omega)$ has a density $p_t(x)$ which is bounded in t and x .

Condition (ii) guarantees that $\{t: X'(t, \omega) = X''(t, \omega) = 0\}$ is empty a.s. Our earlier statement " $I(\psi; X(\cdot, \omega)) < \infty$ a.s. for any $\psi \downarrow$ in $L \log L$ " can then be strengthened to " $I(\psi; X(\cdot, \omega)) < \infty$ for all $\psi \downarrow$ in $L \log L$, a.s.," i.e. the exceptional ω -set no longer depends on the particular ψ .

§3. Here is the proof of the theorem, which uses little else than ordinary calculus. Recall that Λ is the version of $d\lambda/dm$ given by

$$\Lambda(x) = \int_{-\infty}^{\infty} \sum_{s \in F^{-1}(\{x+y\})} |F'(s)|^{-1} \sum_{s \in F^{-1}(\{y\})} |F'(s)|^{-1} dy, \quad -\infty < x < \infty.$$

(i) Λ is continuous off $\{B_i\}_1^L$. I will show that Λ is continuous on $\{A_i - A_j\}_{i,j=1}^N \setminus \{B_i\}_1^L$; the proof of continuity at $x \in \mathbb{R} \setminus \{A_i - A_j\}$ goes about the same, except is easier.

Let $A_0 = \inf_s F(s)$, $A_{N+1} = \sup_s F(s)$, and $v(x) = \text{Card}\{s: F(s)=x\} \leq 1 + \text{Card}\{D_0\} < \infty$. First, notice that α is continuous off $\{A_i\}_0^{N+1}$ because F' and F^{-1} (defined piecewise) are continuous, and because $v(x+\epsilon) = v(x)$ for all small ϵ if $x \notin \{A_i\}_0^{N+1}$.

Now fix $A_i - A_j \notin \{B_k\}_1^L$, $1 \leq i, j \leq N$, and let (k_ℓ, r_ℓ) , $\ell = 1, \dots, q$, be those pairs of integers among $\{1, 2, \dots, N\}$ for which $A_{k_\ell} - A_{r_\ell} = A_i - A_j$. Assume $F''(a_i) < 0 < F''(a_j)$; then $F''(a_{k_\ell}) < 0$ (resp. $F''(a_{r_\ell}) > 0$) for each $1 \leq k \leq N$ with $A_k = A_i$ (resp. $A_k = A_j$). It follows that $\alpha(A_j^-) = \lim_{\epsilon \downarrow 0} \alpha(A_j - \epsilon)$ and $\alpha(A_i^+) = \lim_{\epsilon \downarrow 0} \alpha(A_i + \epsilon)$ exist, finite. The same argument applies to each A_{k_ℓ}, A_{r_ℓ} and yields:

$$(*) \quad \alpha(A_{k_\ell}^+) < \infty, \quad \alpha(A_{r_\ell}^-) < \infty, \quad 1 \leq \ell \leq q.$$

Since $\Lambda(x)$ is an even function and $\{A_i - A_j\}_{i,j=1}^N \setminus \{B_i\}_1^L$ is symmetric about

0, it will be enough to check that Λ is right-continuous at $A_i - A_j$. Set $K(x,y) = \alpha(y)\alpha(y+x+A_i-A_j)$ and let

$$T_\delta = \bigcup_{k=1}^N (A_k - \delta, A_k + \delta), \quad W_\delta = \bigcup_{\ell=0}^q (A_{r_\ell} - \delta, A_{r_\ell} + \delta);$$

also, let $\eta > 0$ be the distance from $A_i - A_j$ to $\{A_n - A_m\}$, $(n,m) \neq (k_\ell, r_\ell)$.

Then

$$\begin{aligned} \Lambda(x+A_i-A_j) &= \int_{W_\delta^c \cap T_\delta} K(x,y) dy + \int_{W_\delta^c \cap T_\delta^c} K(x,y) dy + \sum_{\ell \in \Gamma} \int_{A_{r_\ell} - \delta}^{A_{r_\ell} + \delta} K(x,y) dy \\ &\equiv P_1(x) + P_2(x) + \sum_{\ell \in \Gamma} P_{3,\ell}(x) \end{aligned}$$

where the A_{r_ℓ} , $\ell \in \Gamma \subseteq \{1, \dots, q\}$, are distinct and δ is small enough that the intervals $(A_{r_\ell} - \delta, A_{r_\ell} + \delta)$, $\ell \in \Gamma$, are disjoint.

If $0 \leq x < \delta/2$ and $\delta < \eta/2$, $y \in W_\delta^c \cap T_\delta \Rightarrow y + A_i - A_j \in T_\delta^c \Rightarrow y + x + A_i - A_j \in T_{\delta/2}^c$. In particular, $\sup_{y \in W_\delta^c \cap T_\delta} \alpha(y+x+A_i-A_j) < \infty$ for such x 's. Consequently, recalling that $\alpha \in L^1$ and α is continuous a.e., $P_1(x) \rightarrow P_1(0) < \infty$ as $x \downarrow 0$ (dominated convergence theorem). Similarly, $P_2(x) < \infty \quad \forall x \geq 0$ and

$$|P_2(x) - P_2(0)| \leq \sup_{y \in T_\delta^c} \alpha(y) \int_0^\infty |\alpha(x+y) - \alpha(y)| dy \rightarrow 0 \text{ as } x \downarrow 0.$$

Finally,

$$P_{3,\ell}(x) = \int_{A_{r_\ell} - \delta}^{A_{r_\ell} + \delta} K(x,y) dy + \int_{A_{k_\ell}}^{A_{k_\ell} + \delta} \alpha(y - A_i + A_j) \alpha(y+x) dy$$

which converges to $P_{3,\ell}(0) < \infty$ as $x \downarrow 0$ by using (*) and arguing as above with P_1 and P_2 .

Next, $F' \circ F^{-1}$ satisfies upper and lower Hölder conditions of order 1/2 at each A_i , $1 \leq i \leq N$. For convenience, assume $0 < a_0 < a_N < 1$; the other cases only need some additional notation. For each $a_i \in D_0$ and $s \in [0, 1]$ there are numbers $\xi_s, \bar{\xi}_s$ between s and a_i with $F'(s) = F''(\xi_s)(s-a_i)$ and $F(s) - A_i = \frac{1}{2}F''(\bar{\xi}_s)(s-a_i)^2$. It follows that there are constants $0 < C_1, C_2, C_3, C_4 < \infty$ and a $\tilde{\delta}_0 > 0$ such that for each $1 \leq i \leq N$ and $\delta \leq \tilde{\delta}_0$,

$$(6a) \quad C_2 |s-a_i| \leq |F'(s)| \leq C_1 |s-a_i|, \quad s \in (a_i - \delta, a_i + \delta)$$

$$(6b) \quad C_4 |s-a_i|^2 \leq |F(s) - A_i| \leq C_3 |s-a_i|^2, \quad s \in (a_i - \delta, a_i + \delta).$$

Let \hat{F}_i denote the inverse of F on $J_i \equiv [a_i, a_{i+1}]$, $1 \leq i \leq N-1$. From (6b) and the continuity of the \hat{F}_i 's, there is a $\delta_0 > 0$ such that, for each $1 \leq i \leq N-1$, $\delta \leq \delta_0$,

$$(7) \quad \frac{1}{C_3} |y-A_i|^{1/2} \leq |\hat{F}_i(y) - a_i| \leq \frac{1}{C_4} |y-A_i|^{1/2}, \quad y \in (A_i - \delta, A_i + \delta) \cap F(J_i)$$

$$\frac{1}{C_3} |y-A_{i+1}|^{1/2} \leq |\hat{F}_i(y) - a_{i+1}| \leq \frac{1}{C_4} |y-A_{i+1}|^{1/2}, \quad y \in (A_{i+1} - \delta, A_{i+1} + \delta) \cap F(J_i).$$

Let $D(i, \delta) = (A_i, A_i + \delta)$ if $F''(a_i) > 0$, $= (A_i - \delta, A_i)$ if $F''(a_i) < 0$, $1 \leq i \leq N$.

Combining (6a) and (7), and reducing δ_0 if necessary, there are constants $0 < C_5, C_6 < \infty$ such that for each $1 \leq i \leq N-1$, $\delta \leq \delta_0$,

$$(8) \quad C_5 |y-A_i|^{1/2} \leq |F'(\hat{F}_i(y))| \leq C_6 |y-A_i|^{1/2}, \quad y \in D(i, \delta)$$

and likewise (in case $a_N = 1$) with $A_i, D(i, \delta)$ replaced by $A_{i+1}, D(i+1, \delta)$.

We can assume that for each i, j and each small δ , either $D(i, \delta) = D(j, \delta)$ or $D(i, \delta) \cap D(j, \delta) = \emptyset$. Defining $J_0 = [0, a_1]$, $J_N = [a_N, 1]$ and the corresponding inverses \hat{F}_0, \hat{F}_N , it is clear that (8) extends to $F' \circ \hat{F}_0$ and $F' \circ \hat{F}_N$ at the

appropriate places. (By the way, both inequalities in (8) depend on $F'' \neq 0$ on D_0 .)

(ii) $\lim_{x \rightarrow B_i} \Lambda(x) / -\log|x - B_i| > 0, 1 \leq i \leq L.$ Suppose $B_i = A_\ell - A_k, 1 \leq \ell, k \leq N,$

and $F''(a_k) < 0, F''(a_\ell) < 0;$ the other case, namely $F''(a_\ell), F''(a_k) > 0$ is the same..

$$\begin{aligned} \Lambda(x+B_i) &= \int_{-\infty}^{\infty} \alpha(y+x+A_\ell)\alpha(y+A_k)dy \\ &\geq \int_{-\epsilon}^{-|x|} \alpha(y+x+A_\ell)\alpha(y+A_k)dy, \quad |x| < \epsilon. \end{aligned}$$

Now for ϵ small, the conditions $|x| < \epsilon$ and $-\epsilon < y < -|x|$ together imply that $y+x+A_\ell \in D(\ell, \delta_0)$ and $y+A_k \in D(k, \delta_0)$. Consequently,

$$\begin{aligned} \Lambda(x+B_i) &\geq C_5^2 \int_{-\epsilon}^{-|x|} |y+x|^{-1/2} |y|^{-1/2} dy \\ &= C_5^2 \log \left| \frac{2\sqrt{\epsilon^2 - \epsilon x} + 2\epsilon - x}{2\sqrt{x^2 - |x|x} + 2|x| - x} \right| \\ &\geq C \log \frac{1}{|x|}, \end{aligned}$$

for all small x , for some $C > 0$.

(iii) $\Lambda(x) \leq \text{const.} \times [1 + \sum_1^L |\log|x - B_i||] \forall x.$ (This is equivalent to the "lim" part of (3a).) Evidently,

$$\alpha(y) = \sum_{i=0}^{N+1} 1_{F(J_i)}(y) |F' \circ \hat{F}_i(y)|^{-1}.$$

Off T_δ , α is bounded. Let $y \in T_\delta$, say $A_i - \delta < y < A_i + \delta, y \in F[0,1]$. Keeping (8) in mind and that non-identical $D(j, \delta)$'s are disjoint:

$$\begin{aligned} \alpha(y) &= \sum_{j: A_i = A_j} 1_{F(J_j)}(y) |F' \circ \hat{F}_j(y)|^{-1} + \sum_{j: A_i \neq A_j} 1_{F(J_j)}(y) |F' \circ \hat{F}_j(y)|^{-1} \\ &\leq v(y) C_5 |y - A_i|^{-1/2} + v(y) \sup_{s \in H_\delta} |F'(s)|^{-1}, \quad H_\delta = F^{-1} \left[\bigcap_{i=1}^N (A_i - \delta, A_i + \delta)^c \right] \\ &\leq \text{const.} \times [1 + \sum_{i=1}^N |y - A_i|^{-1/2}]. \end{aligned}$$

Let $V = F[0,1]$ and $U = \bigcup_{i=1}^N V - A_i$, which is bounded.

$$\begin{aligned} \Lambda(x) &= \int_V \alpha(y)\alpha(x+y)dy \\ &\leq \text{const.} \times [1 + 2 \sum_{i=1}^N \int_V |y-A_i|^{-1/2} dy + \sum_{i,j=1}^N \int_V |y-A_i|^{-1/2} |y+x-A_j|^{-1/2} dy] \\ &\leq \text{const.} \times [1 + \sum_{i,j=1}^N \int_U |y+A_j-A_i|^{-1/2} |y+x|^{-1/2} dy] \\ &\leq \text{const.} \times [1 + \sum_{i,j=1}^N |\log|x-(A_i-A_j)||] \end{aligned}$$

since $\int_U |y+\epsilon|^{-1/2} |y|^{-1/2} dy = O(\log \frac{1}{|\epsilon|})$ as $\epsilon \rightarrow 0$.

As for (3b), let $H(x) = 1 + \sum_{i=1}^L |\log|x-B_i||$. Then $I(\psi;F) < \infty \quad \forall \psi \in L^1(\text{Hdm})$ if and only if

$$\int_{-\infty}^{\infty} \frac{\psi(x)}{H(x)} \Lambda(x) dx < \infty \quad \forall \psi \in L^1(dx),$$

if and only if $\text{ess sup}_x \frac{\Lambda(x)}{H(x)} < \infty$. Since Λ, H are continuous from \mathbb{R} to $\mathbb{R} \cup \{\infty\}$, this is the same as $\sup_x \frac{\Lambda(x)}{H(x)} < \infty$. In other words, the "lim" part of (3a) is equivalent to " $I(\psi;F) < \infty \quad \forall \psi \in L^1(\text{Hdm})$." Now if $I(\psi;F) < \infty$ and $\psi \in L^1(dx)$, then it is easy to see, using the "lim" part of (3a) that ψH is integrable. The last statement of the theorem follows from (3b) and the aforementioned fact that $I(\psi;F) < \infty$ and $\psi \in L^1$ imply $\psi \in L^1[0,1]$.

§4. Let $F(t) = t^2$. Then $D_0 = \{B_i\} = \{0\}$ and

$$\Lambda(x) = \frac{1}{2} \log \left\{ \frac{1 + \sqrt{1-|x|}}{\sqrt{|x|}} \right\}, \quad |x| \leq 1.$$

For $F(t) = \sin 2\pi t$, $\Lambda(x)$ is an elliptic integral (of the first kind). I would give more examples, especially in "closed form" and with $L > 1$, if I could; the computations (even for F a third degree polynomial) are formidable.

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