ON COMPARING TWO GAMMAS FROM ONE SAMPLE

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Let there be given N observations on 4 ordered-categorical variables U_1, U_2, U_3, U_4, with M_1, M_2, M_3, M_4 categories respectively. It is desired to compare the value of Goodman and Kruskal's gamma G_{12} calculated from variables U_1 and U_2 with the gamma G_{34} calculated from variables U_3 and U_4. The 4-way contingency table of U_1, U_2, U_3, U_4, with $M_1 M_2 M_3 M_4$ cells, may be arranged as a 2-way table in which there are $M_1 M_2$ columns representing the X_1 and X_2 cross-classification and there are $M_3 M_4$ rows representing the X_3 and X_4 cross-classification. Let the row totals be $R_1,\ldots,R_{M_3M_4}$ and let the column totals be $C_1,\ldots,C_{M_1M_2}$. Then G_{12} and its standard error S_{12} can be calculated from the C's, and G_{34} and its standard error S_{34} from the R's, in accordance with well-known methods. To compare G_{12} and G_{34}, however, we need the standard error S of the difference $(G_{12}-G_{34})$. We have

$$S^2 = S_{12}^2 + S_{34}^2 - 2\mathrm{cov}(G_{12},G_{34}) \;;$$

but this involves the covariance of G_{12} and G_{34}, for which no formula has been published.

A comprehensive methodology which may be used for obtaining S was first presented by Grizzle, Starmer, and Koch (1969), and extended by

Forthofer and Koch (1973); it is known as "analysis of categorical data by weighted least squares", or more popularly "GSK analysis". The GSK approach begins by rewriting the two-way contingency table as a vector. Specifically, if the table has entries F_{ij} for $i=1,\ldots,r$ and $j=1,\ldots,c$ and $\Sigma\Sigma F_{ij} = N$, then the vector $\underline{G}$ has elements G_k for $k = 1,\ldots,rc$ where $G_1 = F_{11}$, $G_2 = F_{12},\ldots,G_c = F_{1c}$, $G_{c+1} = G_{21},\ldots,G_{rc} = F_{rc}$. Next GSK expresses the statistic to be studied by means of combinations of three operators applied to the vector $\underline{G}$:

(simple linear)

$$\underline{Y} = A\underline{G} \;, \quad \text{with} \quad V(\underline{Y}) = A[V(\underline{G})]A'$$

(logarithmic)

$$\underline{Y} = A\ln(\underline{G}) \;, \quad \text{with} \quad V(\underline{Y}) \doteq AD_G^{-1}[V(\underline{G})]D_G^{-1}A'$$

(exponential)

$$\underline{Y} = A\exp(\underline{G}) \;, \quad \text{with} \quad V(\underline{Y}) \doteq AD_Q[V(\underline{G})D_Q A'$$

In these formulas, the operator "ln" transforms each element G_k of the vector $\underline{G}$ into its natural logarithm, $\ln(G_k)$, and the operator "exp" converts each element G_k into its antilogarithm $\exp(G_k)$. A is any matrix having as many columns as $\underline{G}$ has elements, and A' is its transpose. D_G is the diagonal matrix whose diagonal elements are the elements of $\underline{G}$, and D_Q is the diagonal matrix whose diagonal elements are the antilogarithms of the elements of G. Finally, V(•) denotes the asymptotic variance matrix (also called variance-covariance matrix) of the random vector indicated within the parentheses. The three operators can be compounded as desired. For example, one may study

$$\underline{Y} = C\ \exp(B\ln(A\underline{G})) \;,$$

for which the variance matrix may be derived in stages as follows:

Let $\underline{Y}_1 = A\underline{G}$, then $V(\underline{Y}_1) = A[V(\underline{G})]A'$;

Let $\underline{Y}_2 = B\ln(\underline{Y}_1)$, then $V(\underline{Y}_2) \doteq BD_{Y_1}^{-1}[V(Y_1)]D_{Y_1}^{-1}B'$;

Let $\underline{Y}_3 = C\exp(\underline{Y}_2)$, then $V(\underline{Y}_3) \doteq CD_{Y_2}[V(Y_2)]D_{Y_2}C'$;

then, substituting back, we find

$$\underline{Y}_3 = C\exp(B\ln(\underline{Y}_1)) = C\exp(B\ln(AG)) \text{ , which is } \underline{Y} \text{ ,}$$

and

$$V(\underline{Y}_3) = V(\underline{Y}) \doteq CD_{Y_2}BD_{Y_1}^{-1}A[V(G)]A'D_{Y_1}^{-1}B'D_{Y_2}C' \text{ .}$$

These formulas are more fully explained, and illustrated, in the papers cited above.

In applying this method to the problem at hand we shall first assume, for convenience of exposition, that the 4 original variables X_1, X_2, X_3, X_4 all have 3 categories. Then the two-way contingency table has 9 rows and 9 columns; examples are given as Table 1, containing the tooth data from Davis and Quade (1968), and Table 2, containing data for a cross-lagged panel analysis from Brenner (1976). The vector $\underline{G}$ derived from such a table has 81 elements; for the tooth data

$$\underline{G} = (24, 0, 0, 0, 0, 0, 0, 0, 0, 0, 4, 0, 5, \ldots, 0, 0, 69)'$$

and for the Brenner data

$$\underline{G} = (55, 27, 2, 6, 3, 0, 0, 0, 0, 5, \ldots, 0, 1, 4)' \text{ .}$$

TABLE 1

Numbers of Carious Second Molars in Different Parts of the Mouth derived from Davis and Quade (1968), Table 1

U_1 = Left Side		0			1			2			
U_2 = Right Side		0	1	2	0	1	2	0	1	2	Total
U_3 Upper Jaw	U_4 Lower Jaw										
0	0	24	-	-	-	-	-	-	-	-	24
	1	-	4	-	5	-	-	-	-	-	9
	2	-	-	-	-	26	-	-	-	-	26
1	0	-	3	-	6	-	-	-	-	-	9
	1	-	-	4	-	10*	-	0	-	-	14
	2	-	-	-	-	-	13	-	7	-	20
2	0	-	-	-	-	5	-	-	-	-	5
	1	-	-	-	-	-	9	-	7	-	16
	2	-	-	-	-	-	-	-	-	69	69
Total		24	7	4	11	41	22	0	14	69	192

Notes:

- = impossible event with tooth data

0 = possible event which did not happen to occur

* = the sum of 5 occurrences each of Outcomes #7 and #10 in Table 1 of Davis and Quade

TABLE 2

Data for Cross-lagged Panel Analysis

	U_1	R			O			M			
	U_2	R	O	F	R	O	F	R	O	F	Total
U_3	U_4										
	R	55	27	2	6	3	0	0	0	0	93
R	O	5	6	3	1	6	2	0	0	0	23
	M	0	0	0	0	0	0	0	0	0	0
	R	33	40	5	7	15	4	0	1	1	106
O	O	4	16	1	2	25	3	0	1	0	52
	M	0	0	0	0	2	0	0	0	0	2
	R	19	25	9	2	5	4	1	1	1	67
F	O	1	9	4	0	11	3	0	1	3	32
	M	0	0	0	0	0	1	0	1	4	6
Total		117	123	24	18	67	17	1	5	9	381

Notes:

U_1 = How often felt depressed during week before first interview

U_2 = How often had psychophysiologic problems during week before second interview

U_3 = How often had psychophysiologic problems during week before first interview

U_4 = How often felt depressed during week before second interview

R = Rarely

O = Occasionally

F = Frequently

M = Mostly

The first stage of the GSK analysis is to apply a linear operator which generates the row and column totals of the two-way contingency table. Such an operator is determined by the matrix A, of 18 rows and 81 columns, as follows:

$$A = \left(\begin{array}{c:c:c:c} 1\;1\;1\;1\;1\;1\;1\;1\;1 & 0\;0\;0\;0\;0\;0\;0\;0\;0 & & \\ & 1\;1\;1\;1\;1\;1\;1\;1\;1 & & \\ {}_{8\times 9}0 & {}_{7\times 9}0 & \cdots & {}_{8\times 9}0 \\ & & & 1\;1\;1\;1\;1\;1\;1\;1\;1 \\ \hdashline I_9 & I_9 & \cdots & I_9 \end{array}\right),$$

where I_9 indicates the 9×9 identity matrix, and ${}_k0_9$ indicates a null matrix of k rows and 9 columns. Then we obtain the vector

$$\underset{18\times 1}{\underline{Z}} = \underset{18\times 81}{A}\;\underset{81\times 1}{\underline{G}} = \begin{pmatrix} R_1 \\ R_2 \\ \cdot \\ \cdot \\ \cdot \\ R_9 \\ C_1 \\ C_2 \\ \cdot \\ \cdot \\ \cdot \\ C_9 \end{pmatrix}.$$

The variance matrix of $\underline{Z}$ has the form

$$V(Z) = \left(\begin{array}{c:c} D_R & F \\ \hdashline F' & D_C \end{array}\right) - \frac{1}{N}\,\underline{Z}\,\underline{Z}' ,$$

where the matrix F is the original two-way contingency table (and F' is its transpose), and D_R and D_C are diagonal matrices containing as their diagonal elements the row totals and column totals, respectively, of F.

The next stage of the GSK analysis is to apply a logarithmic operator which generates the individual products which are summands of the numbers of concordant and discordant pairs. Specifically, we take

$$\underset{36\times 1}{Y} = \underset{36\times 18}{B} \; \underset{18\times 1}{\ln(Z)} \qquad \text{where} \qquad B = \left[\begin{array}{c|c} Q & 0 \\ \hline 0 & Q \end{array}\right] .$$

Here 0 is a null matrix of 18 rows and 9 columns, and Q is the matrix of 18 rows and 9 columns given as Table 3. Then Y is a vector of 36 elements as follows:

$Y_1 = \ln(R_1R_5)$	$Y_{10} = \ln(R_3R_4)$	$Y_{19} = \ln(C_1C_5)$	$Y_{28} = \ln(C_3C_4)$
$Y_2 = \ln(R_1R_6)$	$Y_{11} = \ln(R_3R_5)$	$Y_{20} = \ln(C_1C_6)$	$Y_{29} = \ln(C_3C_5)$
$Y_3 = \ln(R_1R_8)$	$Y_{12} = \ln(R_3R_7)$	$Y_{21} = \ln(C_1C_8)$	$Y_{30} = \ln(C_3C_7)$
$Y_4 = \ln(R_1R_9)$	$Y_{13} = \ln(R_3R_8)$	$Y_{22} = \ln(C_1C_9)$	$Y_{31} = \ln(C_3C_8)$
$Y_5 = \ln(R_2R_6)$	$Y_{14} = \ln(R_2R_4)$	$Y_{23} = \ln(C_2C_6)$	$Y_{32} = \ln(C_2C_4)$
$Y_6 = \ln(R_2R_9)$	$Y_{15} = \ln(R_2R_7)$	$Y_{24} = \ln(C_2C_9)$	$Y_{33} = \ln(C_2C_7)$
$Y_7 = \ln(R_3R_8)$	$Y_{16} = \ln(R_6R_7)$	$Y_{25} = \ln(C_3C_8)$	$Y_{34} = \ln(C_6C_7)$
$Y_8 = \ln(R_3R_9)$	$Y_{17} = \ln(R_6R_8)$	$Y_{26} = \ln(C_3C_9)$	$Y_{35} = \ln(C_6C_8)$
$Y_9 = \ln(R_5R_9)$	$Y_{18} = \ln(R_5R_7)$	$Y_{27} = \ln(C_5C_9)$	$Y_{36} = \ln(C_5C_7)$

TABLE 3

The Matrix Q

Rows	1	2	3	4	5	6	7	8	9
1	1	0	0	0	1	0	0	0	0
2	1	0	0	0	0	1	0	0	0
3	1	0	0	0	0	0	0	1	0
4	1	0	0	0	0	0	0	0	1
5	0	1	0	0	0	1	0	0	0
6	0	1	0	0	0	0	0	0	1
7	0	0	0	1	0	0	0	1	0
8	0	0	0	1	0	0	0	0	1
9	0	0	0	0	1	0	0	0	1
10	0	0	1	1	0	0	0	0	0
11	0	0	1	0	1	0	0	0	0
12	0	0	1	0	0	0	1	0	0
13	0	0	1	0	0	0	0	1	0
14	0	1	0	1	0	0	0	0	0
15	0	1	0	0	0	0	1	0	0
16	0	0	0	0	0	1	1	0	0
17	0	0	0	0	0	1	0	1	0
18	0	0	0	0	1	0	1	0	0

Notes:

1. Each row contains 2 1's and 7 0's, and each column contains 4 1's and 15 0's
2. The first 9 rows of the matrix produce the numbers of concordant pairs; their internal ordering is entirely arbitrary. Similarly, the last 9 rows produce the numbers of discordant pairs, and their internal ordering is also arbitrary.

Note that the elements in the first column above are the logarithms of the products required for the concordant pairs in G_{34}, the second column for the discordant pairs in G_{34}, the third column for the concordant pairs in G_{12}, and the fourth column for the discordant pairs in G_{12}. The variance matrix of Y, using the general formula, is

$$V(\underline{Y}) = B\, D_Z^{-1}\, [V(\underline{Z})] D_Z^{-1}\, B' \quad ,$$

where D_Z is the diagonal matrix whose diagonal elements are the elements of $\underline{Z}$. A computational complication arises, by the way, if any element of the vector $\underline{Z}$ is zero (which happens for both our examples), since then D_Z^{-1} and $\ln(\underline{Z})$ do not exist. However, the same result is obtained if in writing down D_Z the zero elements are replaced by arbitrary non-zero numbers, since these numbers will all be multiplied by zero elements from $V(\underline{Z})$ and thus have no effect on $V(\underline{Y})$.

The GSK analysis continues with application of an exponential operator which generates the numerator and denominator of each of G_{12} and G_{34}. Specifically, we take

$$\underset{4\times 1}{\underline{X}} = \underset{4\times 36}{C} \exp \underset{36\times 1}{(\underline{Y})} \quad \text{where} \quad C = \left[\begin{array}{c|c} H & 0 \\ \hline 0 & H \end{array}\right]$$

and H is the following matrix of 2 rows and 18 columns:

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Then the elements of $\underline{X}$ are

X_1 = (number of pairs of observations which are concordant with respect to U_3 and U_4) — (number of pairs of observations which are discordant with respect to U_1 and U_2)

= numerator of G_{34} ,

and similarly

X_2 = denominator of G_{34},

X_3 = numerator of G_{12} ,

X_4 = denominator of G_{12} .

The variance matrix of $\underline{X}$, from the general formula, is

$$V(\underline{X}) = CD_Y[V(\underline{Y})]D_YC'$$
$$= CD_YBD_Z^{-1}[V(Z)]D_Z^{-1}B'D_YC' \quad ,$$

where D_Y is the diagonal matrix whose diagonal elements are the anti-logarithms of the elements of $\underline{Y}$.

The final stages of the GSK analysis involve one further application each of the logarithmic and exponential operators. We take

$$\underset{2\times 1}{\underline{W}} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \qquad \ln(\underline{X}) = \begin{pmatrix} \ln(G_{34}) \\ \ln(G_{12}) \end{pmatrix}$$

and

$$\exp(W) = \begin{pmatrix} G_{34} \\ G_{12} \end{pmatrix} .$$

The asymptotic variance matrix of G_{12} and G_{34} is then

$$D_Q \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} D_X^{-1} \; [V(X)]D_X^{-1} \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} D_Q$$

where D_X is the diagonal matrix with diagonal elements (X_1, X_2, X_3, X_4) and D_Q the diagonal matrix with diagonal elements (G_{34}, G_{12}). Upon performing some of the indicated multiplications, we find

$$G_{34} = X_1/X_2 \ , \quad G_{12} = X_3/X_4 \ ,$$

and the variance matrix

$$\begin{pmatrix} \frac{1}{X_2} & -\frac{X_1}{X_2^2} & 0 & 0 \\ 0 & 0 & \frac{1}{X_4} & -\frac{X_1}{X_4^2} \end{pmatrix} [V(X)] \begin{pmatrix} 1/X_2 & 0 \\ -X_1/X_2^2 & 0 \\ 0 & 1/X_4 \\ 0 & -X_3/X_4^2 \end{pmatrix} = \begin{pmatrix} S_{34}^2 & cov \\ cov & S_{12}^2 \end{pmatrix}$$

where "cov" is the covariance of G_{12} and G_{34}. Thus the required S^2 can be calculated. The correlation between G_{12} and G_{34} is of course $R = cov/(S_{12}S_{34})$.

The listing of a Fortran IV program written to accomplish the above analysis, and its output when given as data the two examples presented above, form an Appendix to this paper. The results were:

(<u>Example 1</u>: tooth data)

$$G_{34} = .5076 \pm .0838 \ ,$$

$$G_{12} = .8479 \pm .0407 \ ,$$

$$G_{12} - G_{34} = .3403 \pm .0790 \ ;$$

(Example 2; Brenner data)

$$G_{34} = .2613 \pm .0838 ,$$

$$G_{12} = .5359 \pm .0725 ,$$

$$G_{12} - G_{34} = .2746 \pm .1030 .$$

A test of the hypothesis that G_{12} and G_{34} estimate identical population values can be obtained by taking the ratio of their difference to its standard error as a normal deviate; thus:

(Example 1): $Z_1 = .3403/.0790 = 4.31$,

(Example 2): $Z_1 = .2746/.1030 = 2.67$.

In both examples the difference is significant.

An alternative approach would be to use the general computer program GENCAT of Landis et al (1976) which was especially developed to perform GSK analyses. Unfortunately, the present problem, involving a contingency table of 81 cells, is already too large for a straightforward application of GENCAT, although it can be managed by using the "raw data option".

Let us consider briefly the modification which would be necessary should the numbers of categories of X_1, X_2, X_3 and X_4, namely M_1, M_2, M_3 and M_4, not all be equal to 3. In this general case the matrix F has $M_1 M_2$ columns and $M_3 M_4$ rows, and the vector G derived from it has $M_1 M_2 M_3 M_4$ elements. The matrix A has the form

$$A = \left(\begin{array}{c|c|c} H_1 & \cdots & H_{M_3 M_4} \\ \hline I & \cdots & I \end{array} \right)$$

where each I is the identity matrix of $M_1 M_2$ rows and columns, and H_k is a matrix of $M_1 M_2$ columns and $M_3 M_4$ rows such that its k-th row contains only 1's, and the remaining rows contain only 0's. Then the vector Z will have $(M_1M_2 + M_3M_4)$ elements, namely the row totals of F followed by the column totals of F, and its variance matrix has the same formula as given earlier. The matrix B will have the form

$$B = \left(\begin{array}{c|c} Q_1 & 0 \\ \hline 0 & Q_2 \end{array}\right) .$$

Here Q_2 has $M_1(M_1 - 1)M_2(M_2 - 1)/2$ rows and $M_1 M_2$ columns, where every row contains two 1's and the remainder 0's, and every column contains $(M_1 - 1)(M_2 - 1)$ 1's and the remainder 0's; similarly, Q_1 has $M_3(M_3 - 1)M_4(M_4 - 1)/2$ rows and $M_3 M_4$ columns, where every row contains two 1's and the remainder 0's, and every column contains $(M_3 - 1) \times (M_4 - 1)$ 1's and the remainder 0's. The exact forms of Q_1 and Q_2 may be obtained by analogy with the 3×3 case treated above.

Then

$$C = \left(\begin{array}{c|c} H_1 & 0 \\ \hline 0 & H_2 \end{array}\right)$$

where H_1 and H_2 each have two rows, but H_2 has $M_1(M_1 - 1)M_2(M_2 - 1)/2$ columns and H_1 has $M_3(M_3 - 1)M_4(M_4 - 1)/2$ columns; in each of H_1 and H_2 the second half of the first row consists of -1's and the rest of the matrix consists of 1's. The remainder of the analysis is unchanged.

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```
          C SE_GAM1
0001            REAL*8 N,Z(18),VZ(18,18),F(9,9)
0002            DIMENSION Q(18,9)
0003            REAL*8 H1(2,9), H2(2,9), VX(4,4), X(4), Y(4),EXPY(36)
0004            REAL*8 R,C,RI,RJ,CI,CJ,W
0005            DATA Q/
               -1.,1.,1.,1.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,
               -1.,1.,0.,0.,0.,0.,0.,0.,0.,1.,1.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,
               -0.,1.,1.,1.,1.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,1.,1.,0.,1.,0.,0.,
               -0.,1.,0.,0.,0.,0.,1.,0.,0.,0.,0.,0.,0.,0.,1.,0.,1.,0.,0.,0.,0.,0.,
               -0.,1.,0.,1.,0.,0.,1.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,1.,1.,0.,0.,0.,
               -0.,0.,0.,0.,0.,0.,0.,0.,0.,1.,0.,0.,1.,1.,0.,1.,0.,0.,1.,0.,0.,0.,
               -1.,0.,0.,0.,0.,0.,1.,0.,0.,0.,1.,0.,0.,0.,0.,1.,0.,1.,0.,1.,1.,0.,
               -0.,0.,0.,0.,0.,0.,0.,0./
0006        1   NX=NX + 1
0007            DIMENSION A(15)
0008            READ(5,111,END=999) A
0009      111   FORMAT(20X,15A4)
0010            DO 66 I=1,9
0011            READ(5,112,END=999) (F(I,J),J=1,9)
0012     112    FORMAT(16F5.0)
0013      66    CONTINUE
0014            WRITE(6,166) A
0015      166   FORMAT('1',15A4/)
0016            N=0
0017            WRITE(6,201)
0018      201   FORMAT(' ORIGINAL DATA MATRIX:'/)
0019            DO 202 I=1,9
0020      202   WRITE(6,203) (F(I,J),J=1,9)
0021      203   FORMAT(9F8.0)
          C CALCULATE VECTOR Z OF ROW + COLUMN TOTALS OF F
0022            DO 3 I =1,9
0023            R=0
0024            C=0
0025            DO 2 J=1,9
0026            R=R+F(I,J)
0027            C=C+F(J,I)
0028        2   CONTINUE
0029            Z(I)=R
0030            Z(I+9)=C
0031        3   N=N+R
          C CALCULATE VARIANCE MATRIX VZ OF VECTOR Z
0032            DO 4 I=1,9
0033            RI=Z(I)
0034            CI=Z(I+9)
0035            DO 4 J=1,9
0036            RJ=Z(J)
0037            CJ=Z(J+9)
0038            VZ(I,J)=-RI*RJ/N
0039            VZ(I+9,J+9) = -CI*CJ/N
0040            VZ(J+9,I) = F(I,J)-RI*CJ/N
0041        4   VZ(I,J+9) = VZ(J+9,I)
0042            DO 99 I=1,18
0043            VZ(I,I) = VZ(I,I) + Z(I)
```

```
   99 CONTINUE
C CALCULATE VECTOR EXP(Y)
      DO 6 I=1,18
      R=1
      C=1
      DO 5 J=1,9
      IF(Q(I,J).LE.0) GO TO 5
      R=R*Z(J)
      C=C*Z(J+9)
    5 CONTINUE
      EXPY(I)=R
    6 EXPY(I+18)=C
C PRE- AND POST-MULTIPLY MATRIX VZ BY INV(DZ)
      DO 8 I=1,18
      DO 7 J=1,18
      W=Z(I)*Z(J)
      IF(W.LE.0) GO TO 7
      W=VZ(I,J)/W
    7 VZ(I,J)=W
    8 CONTINUE
C CALCULATE VECTOR X
      DO 11 I=1,2
      DO 11 J=1,9
      R=0
      C=0
      DO 10 K=1,18
      W=H(I,K)*Q(K,J)
      R=R+EXPY(K)*W
      C=C+EXPY(K+18)*W
   10 CONTINUE
      H1(I,J)=R
      H2(I,J)=C
   11 CONTINUE
      DO 13 I=1,2
      R=0
      C=0
      DO 12 J=1,18
      R=R+H(I,J)*EXPY(J)
   12 C=C+H(I,J)*EXPY(J+18)
      X(I)=R
   13 X(I+2)=C
C CALCULATE VARIANCE MATRIX VX OF VECTOR X
      DO 15 I=1,2
      DO 15 J=1,2
      R=0
      C=0
      W=0
      DO 14 K=1,9
      DO 14 L=1,9
      R=R + H1(I,K)*VZ(K,L)*H1(J,L)
      C=C + H2(I,K)*VZ(K+9,L+9)*H2(J,L)
   14 W=W + H1(I,K)*VZ(K,L+9)*H2(J,L)
      VX(I,J)=R
      VX(I+2,J+2)=C
```

```
      REAL FUNCTION H*4(I,J)
      H=1.
      IF(1-I)9,10,10
   10 IF(J-10)9,11,11
   11 H=-1.
    9 RETURN
      END
```

OUTPUT FOR TOOTH DATA

ORIGINAL DATA MATRIX:

24.	0.	0.	0.	0.	0.	0.	0.	0.
0.	4.	0.	5.	0.	0.	0.	0.	0.
0.	0.	0.	0.	26.	0.	0.	0.	0.
0.	3.	0.	6.	0.	0.	0.	0.	0.
0.	0.	4.	0.	10.	0.	0.	0.	0.
0.	0.	0.	0.	0.	13.	0.	7.	0.
0.	0.	0.	0.	5.	0.	0.	0.	0.
0.	0.	0.	0.	0.	9.	0.	7.	0.
0.	0.	0.	0.	0.	0.	0.	0.	69.

	GAMMA	SE GAMMA	GAMMA/SE GAMMA
GAMMA 1,2	0.84787	0.04069	20.83501
GAMMA 3,4	0.50755	0.08382	6.05526
DIFF(G12-G34)	0.34031	0.07897	4.30939
COR(G12,G34)	0.35849		

OUTPUT FOR BRENNER DATA

ORIGINAL DATA MATRIX:

55.	27.	2.	6.	3.	0.	0.	0.	0.
5.	6.	3.	1.	6.	2.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.
33.	40.	5.	7.	15.	4.	0.	1.	1.
4.	16.	1.	2.	25.	3.	0.	1.	0.
0.	0.	0.	0.	2.	0.	0.	0.	0.
19.	25.	9.	2.	5.	4.	1.	1.	1.
1.	9.	4.	0.	11.	3.	0.	1.	3.
0.	0.	0.	0.	0.	1.	0.	1.	4.

	GAMMA	SE GAMMA	GAMMA/SE GAMMA
GAMMA 1,2	0.53585	0.07251	7.38954
GAMMA 3,4	0.26127	0.08381	3.11737
DIFF(G12-G34)	0.27458	0.10296	2.66681
COR(G12,G34)	0.1[illegible]834		