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EXPLICIT CHARACTERIZATION OF OPTIMAL STOPPING TIMES*

by

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ABSTRACT

A large class of continuous time optimal stopping problems is shown to have solutions explicitly determined by roots of equations $xH(x) = 1$ where H involves Laplace transforms. These results motivate the specification of discrete time optimal stopping problems whose solutions are approximated by solutions to corresponding continuous time problems, making rigorous a procedure sometimes employed in the literature. A fairly self-contained treatment of continuous time optimal stopping is also included, albeit for highly structured situations.

INTRODUCTION

(I) This paper is centered around Section (2) where we determine explicit solutions for the problem of optimally stopping $e^{-\lambda t}X_t$ where X is a Hunt process. More precisely, we determine F and T_∞ where

$$F(t,x) = \sup_T E\left\{e^{-\lambda T}X_T \mid X_t = x\right\}$$
$$= E\left\{e^{-\lambda T_\infty}X_{T_\infty} \mid X_t = x\right\}$$

for a representative class of Hunt processes. It is shown in the theorems of Section (2) that T_∞ satisfies

$$T_\infty = \inf t \quad \text{such that} \quad X_t \geq x_0$$

where x_0 is the largest x for which

$$x\,H(x) \leq 1$$

where, if T_x is the hitting time for x, then

$$H(x) = \lim_{y \to x} \frac{d}{dy} \int e^{-\lambda T_x} dPy \;, \quad y < x \;.$$

In particular, when X has independent homogeneous increments, continuous paths:

$$x_0 = \frac{1}{K(\lambda)} \quad \text{where}$$

$$\int e^{-\lambda T_x} dPy = e^{-K(\lambda)(x-y)} \;.$$

The technique determining these results can be partially adapted to more general situations, for instance, it is easily established that the optimal hitting time T_∞ for the return function $\frac{X_t}{a+t}$ where X is standard Brownian motion has form

$$T_\infty = \inf t \text{ such that } X_t \geq K\sqrt{a+t}$$

although the constant K must be determined by other methods.

(II) Section (1) represents an attempt to provide a simple treatment of continuous time optimal stopping by imposing strong conditions on the processes and payoffs under consideration. In many applications these restrictive conditions are shown to be more apparent than real — this is made precise in Remarks (1.8).

(III) Section (3) provides two representative situations in which the solution to a continuous time optimal stopping problem is a good approximation to the solution of a discrete time optimal stopping problem when the underlying processes are close in the sense of weak convergence.

The first two sections were motivated by a close reading of Taylor [13]. The last section developed from remarks made in Chernoff [3] and Shepp [12].

Section (1). Continuous Time Optimal Stopping

We restrict attention to a class of Markov processes $X = \{X_t, t \geq 0\}$ which are real valued and governed by an intial distribution u and a transition p in the sense that for all t, s, x, y:

$$P\{X_{t+s} \leq y | X_t = x\} = P_x\{X_s \leq y\} = \int_{\infty}^{y} p(s,x,dz)$$

and

$$P_u\{X_t \le y\} = \int_{-\infty}^{y} p(t,x,dz)u(dx) \; .$$

The subclass of such processess to be considered is a restricted class of *Hunt processes* which satisfies the following conditions .

(1.1) X is strong Markov and $p_x\{X_0 = x\} = 1$

(1.2) X has right continuous paths with left limits

(1.3) Feller property: For all bounded continuous f, $(P^t f)(x) = \int f(X_t)dP_x$ is bounded continuous.

(1.4) X is extended quasi-left continuous: If T_n is an increasing sequence of stopping times and if $T_n \nearrow T_\infty \le \infty$, then X_{T_∞} exists as a finite value P_x a.e. and $P_x\{X_{T_n} \to X_{T_\infty}\} = 1$.

Taylor [13] restricts $X_{T_n} \to X_{T_\infty}$ to the set where $T_\infty < \infty$. Our more restrictive condition, implying at least that X_∞ exist a.e. is convenient for the development of the theory of this section and presents no real obstacle for applications of the usual sort, as will soon become evident. We remark also that the conclusion $X_{T_n} \to X_{T_\infty}$ or $T_\infty < \infty$ obtains whenever one has the following (see Dynkin [7], Volume 1, p. 104):

$$\lim_{t \downarrow 0} \sup_{x \in K} p(t,x,[x-\varepsilon, x+\varepsilon]^c) = 0$$

all compact K.

In all that follows our state space S will be the time-space structure $[0,\infty] \times R$ with its product topology and Borel sets. The points in this space will usually be denoted $z = (t,x)$, and our process X will be thought of as a process $Z = \{(t,X_t), t \ge 0\}$ still governed by our transition p in the sence that $p(s,x,dy)$ becomes $p(s,(t,x),(t+s,dy))$. We now define our optimal stopping problem. The elements are:

(1.5) f, a non-negative and continuous function on S.

(1.6) For all $(t,x) \in S$, $f \in L_1(t,x)$ where this means

$$\int \sup_s f(t+s,\ X_{t+s})dP_{(t,x)} < \infty ,$$

Let's call f the *return function*. The optimal stopping problem is the determination of the *optimal payoff* F and the optimal stopping time T_∞ where

(1.7) $$F(t,x) = \sup_T \int f(T,\ X_T)dP_{(t,x)} = \int f(T_\infty,\ X_{T_\infty})dP_{(t,x)}$$

where T runs through all stopping times.

(1.8) <u>Remarks</u>

The determination of

$$\sup_{T \geq t} \int c(t+T)X^+{}_T dP_{(t,x)} ,\quad c(t) \geq 0$$

where is the exclusive concern of the applications in later sections is the problem determined by the process $X = \{X_t,\ t \geq 0\}$ and the return $f(t,x) = c(t)x^+$. It is seldom the case that X_∞ exists, i.e., that (1.4) holds. Our development of the theory of continuous time optimal stopping uses (1.4) essentially. This discrepancy between theory and application is only apparent, not substantial. The accomodation of such applications under our theory merely involves a redefinition of processes, that is, we define $Y = \{Y_t,\ t \geq 0\}$ where $Y_t = c(t)X_t$ and consider

$$\sup_{T\geq t} \int Y_T^+ \, dP_{(t,y)}$$

where $c(t)$ decreases rapidly enough so that $c(t)X_t \to 0$, i.e., so that $Y_\infty = 0$.

We now consider classes of processes X and well-behaved non-negative decreasing $c(t)$ for which (1.1) through (1.3) hold and for which (1.4) and (1.6) hold in the form

(1.9)
$$\begin{cases} P_{(0,0)}\left\{\lim\limits_{t\uparrow\infty} c(t)X_t = 0\right\} = 1 \\ \int \sup\limits_t c(t)|X_t| dP_{(0,0)} < \infty . \end{cases}$$

These are the versions of (1.4) and (1.6) proper to $f(t,y) = y^+$ where $Y_t = c(t)X_t$. We consider only the starting point $(t,x) = (0,0)$ for computational convenience.

Class (1)

(1.10)
$$X_t = X_0 + \int_0^t \sigma(X_s)dW_s + \int_0^t m(X_s)ds$$

where

σ and m are bounded, X_0 is

square integrable, W is standard Brownian motion and X_0 in independent of W.

Class (2)

X_t is non-explosive pure jump on the integers with bounded intensities, i.e., there exists $\{\lambda_x$, x an integer$\}$ and $\{Q_x$, x an integer$\}$ where if T_x is the jump time from x, then

$$
(1,11) \quad \begin{cases} P_x\{T_x > t\} = e^{-\lambda_x t} \\ P_x\{X_{T_x} = x+y\} = Q_x(y) \\ \sup_x \lambda_x \le \lambda_\infty < \infty \\ \sup_x \int y^2 Q_x(dy) < \infty \,. \end{cases}
$$

Both classes of processes are easily seen to satisfy the decomposition

$$
(1.12) \begin{cases} X_t = X_0 + Y_t + Z_t \quad \text{where} \\ X_0 \quad \text{is square integrable} \\ Y_t \quad \text{is a martingale and} \quad \int (Y_{t+s} - Y_t)^2 \le Ks \,, \quad \text{small} \quad s \\ |Z_t| \le Mt \end{cases}
$$

Clearly, for decreasing $c(t)$, $\int \sup_t c(t)|X_t| dP_{(0,0)} < \infty$ provided

$$
(1.13) \qquad \begin{cases} \qquad \int \sup_t c(t)|Y_t| dP_{(0,0)} < \infty \\ \text{and} \\ \qquad \sup_t c(t)t < \infty \,. \end{cases}
$$

Now, by right continuity and with obvious notation

$$
\begin{aligned}
\int \sup_t c(t)|Y_t| dP_{(0,0)} &= \lim_{\Delta t \downarrow 0} \int \sup_k c(k\Delta t)|Y_{k\Delta t}| dP_{(0,0)} \\
&= \lim_{\Delta t \downarrow 0} \int_0^\infty P_{(0,0)}\left\{\sup_k c(k\Delta t)|Y_{k\Delta t}| > a\right\} da \\
&\le \lim_{\Delta t \downarrow 0} \int_0^\infty \left\{\frac{1}{a^2} \sum_k c^2(k\Delta t) \int (Y_{(k+1)\Delta t} - Y_{k\Delta t})^2 dP_{(0,0)}\right\} da
\end{aligned}
$$

by the Hajek-Renyi-Chow inequality, (see Chow, Robins, Siegmund [5]).

Using (1.12):

$$\int \sup_t c(t)|Y_t|dP_{(0,0)} \le 1 + \int_1^\infty \left\{K\int_0^\infty c^2(t)dt\right\} \frac{da}{a^2}$$

from which

$$\int_0^\infty c^2(t)dt < \infty \Rightarrow \int \sup_t c(t)|Y_t|dP_{(0,0)} < \infty , \tag{1.14}$$

It is easily seen that $\int_0^\infty c^2(t)dt < \infty \Rightarrow \sup_{t\ge 0} c(t)t < \infty$,

Further, it is shown in Chow [4] that the hypothesis in (1.14) also implies $c(t)Y_t \to 0$. Thus, in order that (1.9) hold, we need $c(t)Z_t \to 0$, which certainly holds if $tc(t) \to 0$. We collect these criteria for ready reference.

<u>Proposition (1.1)</u>

Let $X_t = X_0 + Y_t + Z_t$ satisfy (1.12). Then

(A) If $Z_t \equiv 0$, and $\int_0^\infty c^2(t)dt < \infty$, then

$$\begin{cases} \int \sup_s c(t+s)|X_{t+s}|dP_{(t,x)} < \infty , \quad \text{all} \quad (t,x) \\ \text{and} \\ P_{(t,x)}\left\{\lim_{s\to\infty} c(t+s)X_{t+s} = 0\right\} = 1 , \quad \text{all} \quad (t,x) \end{cases} \tag{1.15}$$

(B) The conclusions (1.15) obtain if the hypotheses in (A) are replaced by

$$tc(t) \to 0 . \tag{1.16}$$

Remarks

The stationary Ornstein-Uhlenback process has representation

$$X_t = e^{-at}\left(X_0 + \sqrt{2ab}\int_0^t e^{as}\, dW_s\right)$$

and appears in the applications. One can model the arguments above, via the Hajek-Renyi-Chow inequality to determine that

$$\int \sup_s e^{-\lambda(t+s)} |X_{t+s}| dP_{(t,x)} < \infty$$

and

$$P_{(t,x)} \left\{\lim_{s\to\infty} e^{-\lambda(t+s)} X_{t+s} = 0\right\} = 1 .$$

All these results overlap those of Walker [14].

We now turn to the determination of F and T_∞ from (1.7). Our development is a modification and generalization of that found in Taylor [13] and Dynkin [8]. We always assume (1.1) through (1.6).

(1.17) Definition

(A) $G\colon\ S \to R_+$ is called *excessive* if G is universally measurable and if

$$G \geq P^t G \quad \text{all} \quad t \geq 0$$

and

$$\lim_{t\downarrow 0} P^t G = G ,$$

(B) An excessive G is called an *excessive majorant* for f if $G \geq f$,

(C) An excessive majorant G for f is called the *least excessive majorant* if $H \geq G$ for all excessive majorants H of f .

Grigelionis-Shiryaev [9] show that f has a least excessive majorant, f_∞, determined by the recursion

$$f_0 = f\ ,\quad f_n = \sup_t P^t f_{n-1}\ ,\quad f_\infty = \uparrow f_n\ .$$

Further, the continuity of f and the Feller property make f_∞ lower semi-continuous — see Taylor [13]. We define (allowing $t = \infty$ as a value):

(1.18) $$\Gamma_\infty = \{(t,x);\ f(t,x) = f_\infty(t,x)\}\ .$$

Again, the lower semi-continuity of f_∞ makes Γ_∞ a closed set. The hitting time

(1.19) $$T_\infty = \inf_t\ (t,X_t) \in \Gamma_\infty$$

will be achieved with probability one from any starting point (t,x) if the following holds

(1.20) $$P_{(t,x)}\{f(\infty\ ,X_\infty) = f_\infty(\infty\ ,X_\infty)\} = 1$$

We prove this by adapting Neveu [11] to our context:

For fixed $t < t_0 < r$:

$$\begin{aligned} f_\infty(r,X_r) &\le E_{(t,x)}\left\{\sup_{s\ge r} f(s,X_s)\,|\,B_r\right\} \\ &\le E_{(t,x)}\left\{\sup_{s\ge t_0} f(s,X_s)\,|\,B_r\right\}\ , \end{aligned}$$

Let $r \to \infty$, we have

$$E_{(t,x)}\left\{\sup_{s\ge t_0} f(s,X_s)\,|\,Br\right\} \to \sup_{s\ge t_0} f(s,Xs)\ P_{(t,x)}\quad \text{a.e.}$$

so that, by lower-continuity of f_∞:

$$f_\infty(\infty, X_\infty) \le \lim_{r\to\infty} f_\infty(r,X_r) \le \sup_{s\ge t_0} f(s,X_s) \quad P_{(t,x)} \text{ a.e.}$$

Letting $t_0 \to \infty$ and using the continuity of f we have

$$f_\infty(\infty, X_\infty) \le f(\infty, X_\infty)$$

and the other direction is obvious. We turn now to our principal result.

Theorem (1.1)

$$f_\infty(t,x) = \int f_\infty(T_\infty, X_{T_\infty}) dP_{(t,x)} , \quad \text{all} \quad (t,x) \in S .$$

Proof:

Dynkin [8] shows the following. If $\varepsilon > 0$ and $\Gamma_\varepsilon = \{(t,x);\ f_\infty(t,x) \le f(t,x) + \varepsilon\}$, then with T_ε the hitting time for Γ_ε and with f bounded:

$$f_\infty(t,x) = \int f_\infty(T_\varepsilon, X_{T_\varepsilon}) dP_{(t,x)} .$$

It is clear that Γ_ε is closed, that $\Gamma_\varepsilon \downarrow \Gamma_\infty$ and that $T_\varepsilon \uparrow T_\infty$ so that by our quasi-left continuity assumptions $X_{T_\varepsilon} \to X_{T_\infty}$. Further,

$$f_\infty(t,x) \le \int f(T_\varepsilon, X_{T_\varepsilon}) dP_{(t,x)} + \varepsilon .$$

Letting $\varepsilon \to 0$ and using the quasi-left continuity of X, and continuity and boundedness of f, we have

$$f_\infty(t,x) \le \int f(T_\infty, X_{T_\infty}) dP_{(t,x)} .$$

Since $f \le f_\infty$ and since f_∞ is excessive and bounded, we've established our result in the bounded case. We now consider the unbounded case, subject as usual to (1.6). Set, for each $a > 0$;

$$f_a = \min(f,a)$$

$$\tilde{f}_a = \text{least excessive majorant of } f_a$$

$$\Gamma_a = \{f_a = \tilde{f}_a\}$$

$$\tilde{\Gamma}_a = \{f \geq \tilde{f}_a\}$$

$$T_a = \text{hitting time for } \Gamma_a$$

$$N_a = \text{hitting time for } \tilde{\Gamma}_a .$$

Note that $\Gamma_a \subset \tilde{\Gamma}_a$, that $N_a \leq T_a$ and that $b \geq a$ implies $\tilde{f}_b \geq \tilde{f}_a$ by properties of least excessive majorants. Let $\tilde{f}_\infty = \uparrow \tilde{f}_a$ as $a \to \infty$. Clearly $\tilde{f}_\infty$ is excessive and $\tilde{f}_\infty \geq f$, therefore $\tilde{f}_\infty \geq f_\infty$. On the other hand, $f_\infty \geq f_a$, hence $f_\infty \geq \tilde{f}_a$, from which $f_\infty = \tilde{f}_\infty$, so that $N_a \nearrow T_\infty$, for if we set $N_\infty = \uparrow N_a$, then $N_\infty \leq T_\infty$ while

$$f(N_\infty, X_{N_\infty}) = \lim_{a\uparrow\infty} f(N_a, X_{N_a}) \geq \underline{\lim}_{a\uparrow\infty} \tilde{f}_a(N_a, X_{N_a})$$

$$\geq \underline{\lim}_{a\uparrow\infty} \tilde{f}_b(N_a, X_{N_a}) \geq \tilde{f}_b(N_\infty, X_{N_\infty})$$

by lower semi-continuity of $\tilde{f}_b$. But then

$$f(N_\infty, X_{N_\infty}) \geq \lim_{b\uparrow\infty} \tilde{f}_b(N_\infty, X_{N_\infty}) = f_\infty(N_\infty, X_{N_\infty})$$

which implies $(N_\infty, X_{N_\infty}) \in \Gamma_\infty$, thus $N_\infty \geq T_\infty$.

Next,

$$\int f(N_a, X_{N_a})\, dP_{(t,x)} \geq \int \tilde{f}_a(N_a, X_{N_a})\, dP_{(t,x)}$$

$$\geq \int \tilde{f}_a(T_a, X_{T_a})\, dP_{(t,x)}$$

since $\tilde{f}_a$ is excessive and $T_a \geq N_a$.

Since $\tilde{f}_a$ is bounded, we have by Dynkin's result,

$$\int \tilde{f}_a(T_a, X_{T_a})dP_{(t,x)} = \tilde{f}_a(t,x) ,$$

so that

$$\int f(N_a, X_{N_a})dP_{(t,x)} \geq \tilde{f}_a(t,x) .$$

Now, (1.6) allows us to use Lebesgue dominated convergence;

$$\int f(T_\infty, X_{T_\infty})dP_{(t,x)} = \int \lim_{a\uparrow\infty} f(N_a, X_{N_a})dP_{(t,x)}$$

$$= \lim_{a\uparrow\infty} \int f(N_a, X_{N_a})dP_{(t,x)} \geq \lim_{a\uparrow\infty} \tilde{f}_a(t,x)$$

$$= f_\infty(t,x) = \tilde{f}_\infty(t,x) ,$$

Since $f_\infty \geq f$ and f_∞ is excessive, the result follows. Q.E.D.

<u>Theorem (1.2)</u>

$$F(t,x) = \int f(T_\infty, X_{T_\infty})dP_{(t,x)} , \text{ all } (t,x) \in S .$$

<u>Proof</u>:

$$F(t,x) = \sup_T \int f(T, X_T)dP_{(t,x)} \geq \int f(T_\infty, X_{T_\infty})dP_{(t,x)}$$

$$= \int f_\infty(T_\infty, X_{T_\infty})dP_{(t,x)} = f_\infty(t,x)$$

$$\geq \sup_T \int f_\infty(T, X_T)dP_{(t,x)}$$

$$\geq \sup_T \int f(T, X_T)dP_{(t,x)}$$

$$= F(t,x) ,$$

Q.E.D.

(1.21) Remarks on Discrete Time Optimal Stopping

Let $X = \{X_{kr}, k = 0,1,2,\ldots\}$ where $r > 0$ be a Markov process which moves on the discrete time lattice $\{kr\}$ and which is governed by the transition

$$P(kr, x, dy) = P_x\{X_{kr} \in dy\} .$$

In the case $r = 1$, we write

$$p(r,x,dy) = p(x,dy) .$$

Suppose

(A) If f is bounded and continuous, then

$$(Pf)(x) = \int f(t+r,y)p(x,dy)$$

is bounded and continuous.

(B) $P_{(t,x)}\left\{\lim_{k\to\infty} X_{kr} = X_\infty \text{ exists and is finite}\right\} = 1$, all (t,x).

(C) f is positive continuous and $\int \sup_k f(t+kr, X_{t+kr})dP_{(t,x)} < \infty$.

Under (A), (B), and (C) the previous development for continuous time will hold for discrete time — it will in fact be simpler. Restricting stopping times, T, to live on the lattice $\{kr\}$, we have

$$F(t,x) = \sup_T \int f(T,X_T)dP_{(t,x)} = \int f(T_\infty,X_{T_\infty})dP_{(t,x)}$$

where T_∞ is the hitting time for $\Gamma_\infty = \{f = f_\infty\}$ where

$$f_0 = f , \quad f_n = \max(f,Pf_{n-1}) , \quad f_\infty = \uparrow f_n ,$$

A self-contained and much more general development of the discrete-time theory appears in Neveu [11] and in Chow, Robbins, Siegmund [5].

Section (2). Determination of Optimal Hitting Sets and Payoffs

Let $X = \{X_t, t \geq 0\}$ be a Hunt process such that $P_{(t,x)} = P_x$ and

$$(2.1)\qquad \begin{cases} \text{For all } \lambda > 0, \text{ and all } (t,x) \in S \\ \text{(a)} \int \sup_s e^{-\lambda(t+s)} |X_{t+s}| dP_{(t,x)} < \infty \\ \text{(b)} P_{(t,x)}\left\{\lim_{s\uparrow\infty} e^{-\lambda(t+s)} X_{t+s} = 0\right\} = 1 \end{cases}$$

We want to determine F, Γ_∞ and T_∞ for $f(t,x) = e^{-\lambda t}x^+$. Our theory applies since (1.1) through (1.6) obtain. It is easily seen that we can just as well consider $f(t,x) = e^{-\lambda t}x$ since (2.1b) implies that it is unreasonable to examine stopping times T where X_T is negative, given that we can do better by continuing on to time infinity. Thus, we'll investigate the structure of F, Γ_∞, T_∞ where

$$F(t,x) = \sup_{T\geq t} \int e^{-\lambda T} X_T dP_{(t,x)} = \int e^{-\lambda T_\infty} X_{T_\infty} dP_{(t,x)} .$$

We assume throughout that $P_{(t,x)} = P_x$.

Theorem (2.1)

(A) Let $X = \{X_t, t \geq 0\}$ be a Hunt process satisfying (2.1). Suppose that for all $0 \leq a < b < \infty$, there exists $x \in (a,b)$ such that if $T_{a,b}$ is the exist time from $[a,b]$, then

$$(2.2)\qquad \begin{cases} P_x\{T_{a,b} < \infty\} = 1 \\ x \geq \int e^{-\lambda T_{a,b}} X_{T_{a,b}} dP_x . \end{cases}$$

Then there exists $x_0 \geq 0$ such that $\Gamma_\infty = \{(t,x) ; x \geq x_0\}$.

(B) Let $X = \{X_t, t \geq 0\}$ be a Hunt process satisfying (2.1). Then the optimal hitting set for $f(t,x) = e^{-\lambda t}x$ again has form $\Gamma_\infty = \{(t,x): x \geq x_0\}$ if X has homogenous independent increments, i.e., if

(2.3) $$P_x\{X_t \leq a\} = P_0\{X_t \leq a-x\} = F_t(a-x)$$

where F_t is a distribution for each $t \geq 0$.

Proof:

We write $P_{(t,x)} = P_x$, $\tau = T-t$, $x_\tau = X_T-x$. Note then that if $(t,x) \notin \Gamma_\infty$, then there exists τ with

$$e^{-\lambda t}x < \int e^{-\lambda(t+\tau)}(x+X_\tau)dP_x .$$

If $(t,x) \in \Gamma_\infty$, then for all θ

$$e^{-\lambda t}x \geq \int e^{-\lambda(t+\tau)}(x+X_\tau)dP_x .$$

Consequently, by straightforward algebra

(2.4) $$(t,x) \in \Gamma_\infty \iff x \geq \sup_{\tau>0} \frac{\int e^{-\lambda\tau}x_\tau dP_x}{1-\int e^{-\lambda\tau}dP_x} \equiv H(x)$$

where we can restrict τ to run through hitting times of closed sets in S. If (2.3) holds, $H(x)$ is a constant and we're finished. So let's assume (2.2) holds. Now

$$\Gamma_\infty = \{(t,x);\ t \geq H(x)\}$$

and since Γ_∞ is closed and contains no (t,x) with x negative, if $\Gamma_\infty \neq \{(t,x): x \geq x_0\}$, then there must exist $0 \leq a < b < \infty$ with $x < H(x)$, $x \in (a,b)$ while $a \geq H(a)$, $b \geq H(b)$. For x as determined by (2.2), we'd have simultaneously

$$x \geq \int e^{-\lambda T_{a,b}} X_{T_{a,b}} dP_x$$

and

$$e^{-\lambda t} x < \int e^{-\lambda(t+T_{a,b})} X_{T_{a,b}} dP_x ,$$

a contradiction. Q.E.D.

An Explicit Characterization of x_0 for Diffusions

We restict attention to $X = \{X_t, t \geq 0\}$ satisfying the hypotheses of our theorem and having continuous paths. We impose a further reqirement

(2.5) Let τ_{x_0} be the time to reach x_0 and let $x < x_0$. Then for all $\lambda > 0$, the derivative $\frac{d}{dx} \int e^{-\lambda \tau_{x_0}} dP_x$ exists as a continuous function with limit

$$H_\lambda(x_0) = \lim_{x \uparrow x_0} \frac{d}{dx} \int e^{-\lambda \tau_{x_0}} dP_x .$$

Theorem (2.2)

Under the hypotheses in theorem (2.1) and the condition (2.5) we can characterize x_0 by

(2.6) $$x_0 = \sup_x \text{ such that } xH_\lambda(x) \leq 1 .$$

Proof:

If $x < x_0$, then

$$e^{-\lambda t} x < x_0 \int e^{-\lambda(t+\tau_{x_0})} dP_x$$

which we re-write as

$$x < (x_0-x)\,\frac{\int e^{-\lambda\tau_{x_0}}dP_x}{1-\int e^{-\lambda\tau_{x_0}}dP_x}\ .$$

By the Mean Value Theorem of Calculus

$$1-\int e^{-\lambda\tau_{x_0}}dP_x = (x_0-x)\,\frac{d}{dx}\int e^{-\lambda\tau_{x_0}}dP_y$$

where $x<y<x_0$. Letting $x \quad x_0$ we have our conclusion that

$$x_0 \le \frac{1}{H_\lambda(x_0)}\ .$$

On the other hand, if $x_0 \le x < x_1$, then

$$e^{-\lambda t}x \ge x_1 \int e^{-\lambda(t+\tau_{x_1})}\,dP_x$$

and repeating our reasoning we have

$$x_1 \ge \frac{1}{H_\lambda(x_1)}\ , \qquad \text{Q.E.D.}$$

Applications

(I) If $X = \{X_t,\ t\ge 0\}$ has homogeneous independent increments, continuous paths, then there exists non-negative $G(\lambda)$ with

$$\int e^{-\lambda\tau_{x_0}}dP_x = e^{-(x_0-x)G(\lambda)}\ .$$

Consequently

$$H_\lambda(x_0) = G(\lambda)$$

so that

$$x_0 = \frac{1}{G(\lambda)}\ ,$$

Thus, if X is Brownian Motion with variance σ^2 and drift m, then

$$G(\lambda) = \frac{\sqrt{m^2+2\lambda\sigma^2} - m}{\sigma^2}$$

from which

$$x_0 = \frac{\sigma^2}{\sqrt{m^2+2\lambda\sigma^2} - m}$$

and

$$F(t,x) = \begin{cases} e^{-\lambda t}x & \text{if } x \geq x_0 \\ x_0 e^{-\lambda t} e^{-(x_0-x)G(\lambda)} & \text{if } x < x_0 . \end{cases}$$

This agrees with Taylor [13].

(II) The stationary Ornstein-Uhlenbeck process has representation

$$X_t = e^{-\alpha t}\left(X_0 + \sqrt{2\alpha\beta}\int_0^t e^{\alpha s} dW_s\right)$$

where W is standard Brownian motion, X_0 is $n(0,\beta)$, and X_0 and W are independent. Conditions (2.1) and (2.2) are easily seen to hold for this process; for (2.2) one uses

$$P_x\{X_{T_{a,b}} = a\} = \frac{\int_x^b e^{t^2/2\beta}\, dt}{\int_a^b e^{t^2/2\beta}\, dt} .$$

We consider the optimization

$$\sup_{T\geq t} \int e^{-T} X_T \, dP_{(t,x)}$$

solved by Taylor [13]. The transform $\int e^{-\lambda\tau_{x_0}}\, dP_x \equiv \phi(x,x_0)$ satisfies

$$ab\ \phi'' - ax\ \phi' = \lambda\phi\ , \quad \phi(x_0) = 1$$

which is solved in Darling-Siegert [6] and which, for the case $\lambda = 1$, takes the relatively simple form

$$(x,x_0) = \frac{\int_0^\infty e^{xt - \frac{t^2}{2}}\, dt}{\int_0^\infty e^{x_0 t - \frac{t^2}{2}}\, dt} \ .$$

Straightforward calculations lead to

$$H(1,x_0) = \lim_{x \uparrow x_0} \frac{d}{dx}\ (x,x_0) = x_0 + \frac{e^{-\frac{1}{2}x_0^2}}{\int_{-x_0}^{\infty} e^{-\frac{1}{2}t^2}\, dt} \ .$$

Thus, from (2.6), we need

$$x_0^2 + \frac{x_0 e^{-\frac{1}{2}x_0^2}}{\int_{-x_0}^{\infty} e^{-\frac{1}{2}t^2}\,dt} = 1$$

which has solution $x_0 \sim .839$, agreeing with Taylor [ ]. The payoff F has form

$$F(t,x) = \begin{cases} e^{-\lambda t}x & \text{if } x \geq x_0 \\ x_0\phi(x,x_0) & \text{if } x < x_0 \end{cases} .$$

An Explicit Characterization of x_0 for Compound Poisson Processes

Let $X = \{X_t,\ t \geq 0\}$ live on the integers as a pure jump process which has independent homogeneous increments. Thus, if T_x is the jump time from x to some other state, then

$$P_x\{T_x > t\} = e^{-at}\ ,\quad \text{some finite positive } a$$

$$P_x\{X_{T_x} = x + Z\} = \theta(Z)\ ,\quad \theta \text{ a fixed probability},$$

Let's assume that $\theta(Z) = 0$ if $Z \leq 0$ so that the process is increasing. We assume further that $\sum Z^2\theta(Z) < \infty$ so that, using Section (1), Remarks (1.8), it is clear that X qualifies for the hypotheses of Theorem (2.1), in fact, the existence of second moments is much more that is needed. Interpreting (2.4) in the present context, it is seen that

$$x_0-1 < \frac{\int e^{-\lambda T_{x_0-1}} X_{T_{x_0-1}} \, dP_{x_0-1}}{1-\int e^{-\lambda T_{x_0-1}} \, dP_{x_0-1}}$$

$$x_0 \geq \frac{\int e^{-\lambda T_{x_0}} X_{T_{x_0}} \, dP_{x_0}}{1-\int e^{-\lambda T_{x_0}} \, dPx_0}$$

where X_{T_x} is the size of the jump taken at jump time T_x. Now the right side in the inequalities above is constant since the process has homogeneous independent increments. This constant is

$$\frac{\int e^{-\lambda T_0} X_{T_0} \, dP_0}{1-\int e^{-\lambda T_0} \, dP_0} = \frac{a}{\lambda} \sum Z\theta(Z) \ .$$

Thus x_0 is the smallest integer for which

$$x_0 \geq \frac{a}{\lambda} \sum Z\theta(Z) \ ,$$

The optimal payoff for the case $\theta(1) = 1$ is

$$F(t,x) = \begin{cases} e^{-\lambda t} x & \text{if } x \geq x_0 \\ x_0 \left(\frac{a}{a+\lambda}\right)^{x_0-x} e^{-\lambda t} & \text{if } x < x_0 \ . \end{cases}$$

These results agree with Taylor [13].

Remarks

Let $c(t)$ be smooth and decreasing to zero as $t \to \infty$, let X be a Hunt process and consider

$$F(t,x) = \sup_{T} \int C(T)X_T \, dP_{(t,x)} .$$

Arguing as before, $(t,x) \in \Gamma_\infty$ iff

$$x \geq \sup_{\tau>0} \frac{\int \frac{c(t)}{c(t+\tau)} X_\tau \, dP_{(t,x)}}{1-\int \frac{c(t)}{c(t+\tau)} \, dP_{(t,x)}} .$$

If the process X has homogeneous independent increments, then $P_{(t,x)} = P_0$ and the right side above is a function of t alone. The case most thoroughly treated in the literature has $c(t) = \frac{1}{1+t}$ and X standard Brownian motion.

A generalization treated by Walker [15], slightly modified here, has X standard Brownian motion and

$$c(t) = \frac{1}{(A+Bt)^r}, \quad r > \frac{1}{2} .$$

The characterization of Γ_∞ for this case is $(t,x) \in \Gamma_\infty$ iff

$$x \geq \sup_{\tau} \frac{(A+Bt)^r \int \frac{X_\tau}{(A+B(t+\tau))^r}}{1-(A+Bt)^r \int \frac{1}{(A+B(t+\tau))^r}} .$$

Using the fact that $X_t \overset{\mathcal{D}}{=} a X_{\frac{t}{a^2}}$ for X standard Brownian motion, and making this substitution above with $a = \sqrt{\frac{A+Bt}{B}}$, we calculate that

$(t,x) \in \Gamma_\infty$ iff

$$x \geq K_r \cdot \sqrt{\frac{A+Bt}{B}}$$

where

$$K_r = \sup_\tau \frac{\int \frac{x_\tau}{(1+\tau)^r}}{1-\int \frac{1}{(1+\tau)^r}},$$

Thus, T_∞ is the hitting time for the square root boundary $K_r\sqrt{\frac{A+Bt}{B}}$, Kr some appropriate constant. The determination of Kr is achieved by Walker [15] and in special cases by Shepp [13], Taylor [12], and others; we won't pursue this problem here. We remark only that on the basis of this example and those considered previously, it is not too much to expect that the time T_∞ is the hitting time for some smooth curve $G(t)$. This assumption is exploited in the next section.

Section (3). Weak Convergence in Optimal Stopping

We consider a class of discrete time optimal stopping problems whose solutions are approximated by solutions of related continuous time problems. To begin with, suppose $\{X_n\}$ to be a sequence of processes in discrete time where $X_n(t)$ has t restricted to the lattice $\{m\phi(n), m=0,1,2,..\}$ where $\phi(n) \to 0$. Let f be our return function having its usual domain $S = [0,\infty] \times R$ and define

$$(3.1) \qquad F_n(t,x) = \sup_T \int f(T,X_n(T))dP_{(t,x)}$$

where T runs through stopping times on the lattice $\{m\phi(n)\}$. Suppose we interpolate X_n so that it is a process in $C[0,\infty)$, i.e., we set

$$X_n(t) = \frac{((m+1)\phi(n)-t)}{\phi(n)} X_n(m\phi(n)) + \frac{(t-m\phi(n))}{\phi(n)} X_n((m+1)\phi(n)) \tag{3.2}$$

for

$$t \in [m\phi(n), (m+1)\phi(n)] .$$

Each discrete time process X_n is Markovian and satisfies, for all t,s in the lattice $\{m\phi(n)\}$

$$P\{X_n(t+s) \in dy | X_n(t) = x\} = p_n(s,x,dy) \tag{3.3}$$

for appropriate fixed transition probabilities p_n. For each fixed (t,x), the continuous time process X_n conditioned on $X_n(t) = x$ will be called X_n under initial (t,x) and the resulting probability on $C[0,\infty)$ will be denoted either $P_{(t,x)}$ or P_x, the latter denotation being adequate in view of (3.3). We assume in what follows that there exists a Markov process X_∞ on $C[0,\infty)$ governed by a transition probability $p_\infty(t,x,dy)$ such that

$$X_n \xrightarrow{\mathcal{D}} X_\infty$$

under all initial (t,x) where the indicated convergence is weak convergence — see Billingsley [2] and Whitt [16].

Suppose there exists a closed set Γ_∞ in S whose hitting time T_∞ satisfies

$$F_\infty(t,x) = \sup_T \int f(T,X_\infty(T))dP_{(t,x)} = \int f(T_\infty,X_\infty(T_\infty))dP_{(t,x)} ,$$

We define T_n as the approximate hitting time of Γ_∞ for the discrete time process X_n;

(3.4) T_n = minimal $m\phi(n)$ such that the continuous time process X_n hits Γ_∞ in $((m-1)\phi(n), m\phi(n)]$.

It is reasonable to expect that T_n is approximately optimal for discrete time X_n, i.e.,

$$F_n(t,x) \sim \int F(T_n, X_n(T_n))dP_{(t,x)} .$$

We now proceed to lay down assumptions which given this approximation a precise meaning. It is not assumed in this development that $\lim_{t\to\infty} X_n(t)$ exists.

<u>Assumptions</u>

(3.5) For all $n \leq \infty$ and under initial (t,x):

$$P_{(t,x)}\left\{\lim_{s\to\infty} f(s,X_n(s)) = 0\right\} = 1 .$$

Thus, we interpret $f(T,X_n(T)) = 0$ on $\{T = \infty\}$.

(3.6) $f(t,x)$ is increasing in x, decreasing in t, uniformly continuous in t.

(3.7) There exists $G \in C[0,\infty)$, non-decreasing in t with

$$\Gamma_\infty = \{(t,x);\ x \geq G(t)\} ,$$

We interpret $T_\infty = \infty$ on paths where $X_\infty(t) < G(t)$, all t.

(3.8) For all $(t,x) \in S$

$$P_{(t,x)}\left\{\bigcap_{\varepsilon>0}\ \bigcup_{0<s<\varepsilon}\left\{X_\infty(T_\infty+s) \quad G(T+s)\right\}\right\} = 1$$

conditional on $\{T_\infty < \infty\}$.

(3.9) For all large n, F_∞ is excessive for discrete time X_n, i.e.,

$$F_\infty(m\phi(n),x) \geq \int F_\infty((m+1)\phi(n),y)p_n(\phi(n),x,dy) \ .$$

Theorem (3.1)

Let X_n be the discrete time Markov process on the time lattice $\{m\phi(n)\}$ whose continuous time interpolation converges weakly to X_∞ under all initial (t,x). Under assumptions (3.5) through (3.8) we have it that for any $\varepsilon > 0$, there exists $N(\varepsilon)$ such that $n \geq N(\varepsilon)$ implies

$$\begin{aligned} 0 &\leq F_n(t,x) - \int f(T_n,X_n(T_n))dP_{(t,x)} \\ &\leq F_\infty(t,x) - \int f(T_n,X_n(T_n))dP_{(t,x)} \\ &\leq \varepsilon \end{aligned}$$

Proof:

By Skorokhod's theorem — see Billingsley [2] — there exists a probability triple (Ω,B,P) supporting random elements $Y_n: \Omega \to C[0,\infty)$, $n \leq \infty$ such that

(3.10)
$$Y_n \overset{\mathcal{D}}{=} X_n \ , \quad n \leq \infty$$

$$Y_n \to Y_\infty \ , \quad \text{all } \omega \in \Omega \ .$$

Since the metric on $C[0,\infty)$ is uniform convergence on all closed intervals

$[0,t_0]$, it is easily seen from assumptions (3.7) and (3.8) that if V_n is the hitting time of $G(t)$ for the continuous time process X_n and if S_n is the corresponding hitting time for Y_n, then

$$(3.11) \qquad V_n \overset{\mathcal{D}}{=} S_n \ , \quad \text{all} \quad n \le \infty$$

and

$$(3.12) \qquad S_n \to S_\infty \quad \text{on} \quad \{S_\infty < \infty\} \ .$$

Clearly (3.8) implies (3.12) by making hitting times $C[0,\infty)$ continuous where they occur. But then, by continuity of f and Y_n, we have

$$(3.13) \qquad f(S_n, Y_n(S_n)) \to f(S_\infty, Y_\infty(S_\infty)) \ , \quad \text{all} \quad \omega \in \Omega \ .$$

Here we use (3.5) on $\{S_\infty = \infty\}$. Now (3.11) demands that

$$(3.14) \qquad \int f(V_n, X_n(V_n))dP_{(t,x)} = \int f(S_n, Y_n(S_n))dP \ , \quad \text{all} \quad n \le \infty$$

and then Fatou's lemma applied to the right and interpreted for the left yields (note $V_\infty = T_\infty$):

$$(3.15) \qquad \int f(T_\infty, X_\infty(T_\infty))dP_{(t,x)} \le \underline{\lim}_{n} \int f(V_n, X_n(V_n))dP_{(t,x)} \ ,$$

Now we want to replace V_n by T_n so that we can relate behaviour of discrete time X_n to continuous time X_∞. Since $|T_n - V_n| \le \phi(n)$, and $\phi(n) \to 0$, the uniform continuity of f in t demands that for large n

$$(3.16) \qquad f(V_n, X_n(V_n)) \le f(T_n, X_n(V_n)) + \varepsilon \ .$$

Further, since G is non-decreasing and V_n is the hitting time for G, and f is increasing in x:

$$(3.17) \qquad f(T_n, X_n(V_n)) \le f(T_n, X_n(T_n)) \ ,$$

Consequently:

$$\int f(T_\infty, X_\infty(T_\infty))dP_{(t,x)} \le \underline{\lim_n} \int f(T_n, X_n(T_n))dP_{(t,x)} .$$

Finally, assumption (3.9) and the properties of least excessive majorants gives $F_\infty \ge F_n$ so that, using (3.18)

$$(3.19) \quad F_\infty(t,x) = \int f(T_\infty, X_\infty(T_\infty))dP_{(t,x)} \le \underline{\lim_n} \int f(T_n, X_n(T_n))dP_{(t,x)}$$

$$\le \underline{\lim_n} F_n(t,x) \le F_\infty(t,x) \quad .$$

Q.E.D.

A Related Result for $D[0,\infty)$

Let $\{X_n\}$ be a discrete time sequence of integer-valued processes where X_n has time lattice $\{m\phi(n)\}$, and where X_n has transitions governed by p_n where, for all integer x,y:

$$P\{X_n((m+1)\phi(n)) = y | X_n(m\phi(n)) = x\} = p_n(\phi(n),x,y) .$$

Let the continuous time version of X_n be defined by

$$X_n(t) = X_n(m\phi(n)) \quad \text{if} \quad t \in [m\phi(n),(m+1)\phi(n)) .$$

Each X_n is a pure-jump process with jumps occurring only at time values $m\phi(n)$. Let X_∞ be a pure-jump continuous time process which lives on the integers, hence is a random element in $D[0,\infty)$, and assume that X_∞ is non-explosive, i.e., X_∞ has at most finitely many jumps in finite time. That is,

$$(3.20) \qquad X_\infty(t) = \sum_m X_\infty(s_m) I_{[s_m, s_{m+1})}(t)$$

where $\{s_m\}$ is the sequence of jump times for X_∞ and this sequence satisfies

(3.21) $$P_x\left\{\lim_{n\to\infty} s_n = \infty\right\} = 1 \text{ , all integer } x \text{ ,}$$

Now suppose $X_n \overset{\mathcal{D}}{\to} X_\infty$ on $D[0,\infty)$; this type of weak convergence is treated in Whitt [17] and Lindvall [10]. We want then to relate F_n to F_∞ where, as usual

$$F_n(t,x) = \sup_{T\geq t} \int f(T,X_n(T))dP_{(t,x)} \text{ , } n \leq \infty \text{ .}$$

Assumptions

(3.22) $f(t,x)$ is increasing in x, decreasing in t, continuous in t.

(3.23) $P_{(t,x)}\left\{\lim_{s\to\infty} f(s,x_n(s)) = 0\right\} = 1$, all initial (t,x) .

(3.24) There exists non-decreasing $G \in C[0,\infty)$, with

$$\Gamma_\infty = \{(t,x): x \geq G(t)\}$$

and we again interpret $T_\infty = \infty$ on paths where $X_\infty(t) < G(t)$, all t .

(3.25) For all large n, F_∞ is excessive for discrete time X_n.

Theorem (3.2)

If X_n, X_∞ above satisfy $X_n \overset{\mathcal{D}}{\to} X_\infty$ on $D[0,\infty)$, and if (3.22) through (3.25) hold, then if T_n = minimal $m\phi(n)$ where $X_n(m\phi(n)) \geq G(t)$, we have:

$$F_\infty(t,x) = \lim_{n\to\infty} F_n(t,x) = \lim_{n\to\infty} \int f(T_n,X_n(T_n))dP_{(t,x)} \text{ .}$$

Proof;

Just as before, we replace X_n be Y_n, $n \le \infty$ where $X_n \overset{\mathcal{D}}{=} Y_n$ and $Y_n \to Y_\infty$ in the $D[0,\infty)$ metric for each path. Since all paths are integer valued, and since Y_∞ executes at most finitely many jumps in finite time, we see that for a particular path ω, $Y_n(\omega) \to Y_\infty(\omega)$ if and only if Y_n eventually makes exactly the same jumps as Y_∞, i.e., $Y_n(s_m^n) = Y_\infty(s_m^\infty)$ where s_m^n is the time of the m-th jump and further $s_m^n \to s_m^\infty$. Since G is continuous and since Y_n looks like Y_∞ except for a slight distortion of the time axis, we have $S_n \to S_\infty$ on $\{S_\infty < \infty\}$ where S_n, S_∞ are defined as in the last theorem. Also, since $Y_n(S_n) = Y_\infty(S_\infty)$ for large n on $\{S_\infty < \infty\}$, and since f is continuous in t, we have, given (3.23), that $f(S_n, Y_n(S_n)) \to f(S_\infty, Y_\infty(S_\infty))$. Continuing the logic and notation of the previous theorem, noting that $T_n = V_n$, we see that

$$\int f(T_\infty, X_\infty(T_\infty))\,dP_{(t,x)} \le \underline{\lim_n} \int f(T_n, X_n(T_n))\,dP_{(t,x)}$$

and that

$F_\infty \ge F_n$, so that all conclusions follow . Q.E.D.

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