RANK ANALYSIS OF COVARIANCE UNDER PROGRESSIVE CENSORING

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Institute of Statistics Mimeo Series No. 1118

May 1977

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ABSTRACT

In the context of survival analysis under a progressive censoring scheme, a class of analysis of covariance tests based on suitable linear rank statistics is proposed and studied. Some invariance principles for certain (multivariate) progressively censored rank order processes are established and incorporated in the study of the asymptotic properties of the proposed tests.

AMS 1970 Classification Nos: 60F05, 62G10, 62L99.

Key Words & Phrases: Analysis of covariance, asymptotic power; asymptotic relative efficiency; Brownian motions, censoring, invariance principles; linear rank statistics; permutational invariance; progressive censoring.

*Work supported partially by the Air Force Office of Scientific Research, U.S.A.F., A.F.S.C., Grant No. AFOSR-74-2736 and partially by the (U.S.) National Heart, Lung, and Blood Institute, Contract NIH-NHLI-71-2243 from the National Institutes of Health.

1. INTRODUCTION

In clinical trials and life testing problems, due to practical limitations, often, statistical tests are based on censored or truncated data. In this context, a progressive censoring scheme (PCS) incorporates a continuous monitoring of experimentation from the beginning with the objective of an early termination depending on the accumulated statistical evidence. In this sense, a PCS involves a *time-sequential* test. For censored or truncated tests based on ranks, we may refer to Breslow (1970) and Halperin and Ware (1974) where other references are cited. For nonparametric testing under PCS, Chatterjee and Sen (1973) have formulated a general class of tests based on linear rank statistics, censored at successive failures. Further works in this direction are due to Sen (1976a,b), Majumdar and Sen (1977) and Davis (1977), among others. In all these studies, PCS tests have been developed only for the simple analysis of variance problem. In application to clinical trials, besides the primary variate [viz., failure time due to a heart-attack], there may be other concomitant variates [viz., initial blood pressure, cholesterol level etc.] which possibly accounts for some assignable variations in the observed responses. Thus, in this setup, an analysis of covariance (ANOCA) in the context of survival analysis is deemed appropriate. In the present paper, we desire to develop such rank based ANOCA tests under PCS.

A variety of rank ANOCA tests, considered by Quade (1967), Puri and Sen (1969a) and Sen and Puri (1970), being based on the complete sample observations, is not directly usable in a PCS. Basically, the repeated significance testing involved in a PCS (relating to an increasing dimension of dependent data) introduces extra complications in a valid statistical analysis and calls for more refinements. The theory developed by Chatterjee and Sen (1973) is extended here to the general ANOCA model. The proposed tests are based on progressively censored linear rank statistics in a multivariate setup and rest on a permutational-invariance principle developed earlier by Chatterjee and Sen (1964) [and incorporated in the ANOCA problem by Puri and Sen (1969a)].

Along with the preliminary notions, the basic problem is formulated in Section 2. The proposed PCS tests are developed in Section 3. Asymptotic distribution theory of the test statistics, under the null and local alternative hypotheses, are developed in Sections 4 and 5. The concluding section deals with some general remarks.

2. PRELIMINARY NOTIONS

Let $\{\underset{\sim}{X}_i^* = (X_{0i}, \underset{\sim}{X}_i')'; \underset{\sim}{X}_i' = (X_{1i}, \ldots, X_{pi}); i \geq 1\}$ be a sequence of independent random vectors (rv), where p is a positive integer, the X_{0i} are the *primary variables* and the $\underset{\sim}{X}_i$ are the *concomitant variates*. It is assumed that $\underset{\sim}{X}_i^*$ has a continuous $(p+1)$-variate distribution function (df) $F_i^*(\underset{\sim}{x})$, and we denote by

$$\underset{\sim}{F}_i(\underset{\sim}{x}) = F_i^*(\infty, \underset{\sim}{x}) = P\{\underset{\sim}{X}_i \leq \underset{\sim}{x}\}, \quad \underset{\sim}{x} \in E^p; \tag{2.1}$$

$$F_i^0(x_0|\underset{\sim}{x}) = P\{X_{0i} \leq x_0 | \underset{\sim}{X}_i = \underset{\sim}{x}\}, \quad x_0 \in E, \quad \underset{\sim}{x} \in E^p; \tag{2.2}$$

$$F_{i0}(x) = P\{X_{0i} \le x\} = F_i(x, \underset{\sim}{\infty}) , \quad x \in E , \quad \forall\ i \ge 1 . \tag{2.3}$$

As is usually the case with ANOCA models [with stochastic covariates; cf. Scheffé (1959, Ch. 6)], we assume that the df F_i in (2.1) does not depend on $i(\ge 1)$, so that

$$F_i(\underset{\sim}{x}) = F(\underset{\sim}{x}) , \quad \forall\ i \ge 1 , \quad \underset{\sim}{x} \in E^p . \tag{2.4}$$

Our basic problem is to test the null hypothesis

$$H_0 : F_i^0(x_0|\underset{\sim}{x}) = F^0(x_0|\underset{\sim}{x}) , \quad \forall\ i \ge 1 , \quad (x_0, \underset{\sim}{x}')' \in E^{p+1} . \tag{2.5}$$

Note that in view of (2.4), H_0 implies that

$$F_i^*(\underset{\sim}{x}) = F^*(\underset{\sim}{x}) , \quad \forall\ i \ge 1 , \quad \underset{\sim}{x} \in E^{p+1} . \tag{2.6}$$

Keeping in mind a simple (ANOCA) linear model, we set

$$F_i^0(x_0|\underset{\sim}{x}) = F^0(x_0 - \beta c_i|\underset{\sim}{x}) , \quad i \ge 1 , \quad (x_0, \underset{\sim}{x}')' \in E^{p+1} , \tag{2.7}$$

where the c_i are known constants (not all equal) and β is an unknown parameter. For example, for the so called two-sample problem, we may set $n = n_1 + n_2$, $n_1 \ge 1$, $n_2 \ge 1$, $c_1 = \ldots = c_{n_1} = 0$ and $c_{n_1+1} = \ldots = c_n = 1$, so that β stands for the difference of locations of the two conditional df's F_1^0 and F_n^0. In this setup, (2.5) reduces to $H_0^0 : \beta = 0$.

In the context of a life testing problem, we conceive of the set $\{\underset{\sim}{X}_i^*;\ i = 1, \ldots, n\}$ and assume that the covariates $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n$ are all observable at the start of the experimentation, but the primary variates $X_{01}, \ldots, X_{0n}$ are not so. Let $Z_{n,1}^0 \le \ldots \le Z_{n,n}^0$ be the ordered rv's corresponding to $X_{01}, \ldots, X_{0n}$; by virtue of the assumed continuity of the df's, ties among the X_{0i} (and hence, the $Z^0_{n,i}$) are neglected, in probability. Let then

$$Z^0_{n,i} = X_{0S_i} \ , \quad \text{for} \quad i = 1,\dots,n \ , \tag{2.8}$$

so that $\underset{\sim}{S}_n = (S_1,\dots,S_n)'$ represents the vector of *anti-ranks* of the X_{0i} and is a (random) permutation of $(1,\dots,n)$. In a life testing situation, at the k-th failure $Z^0_{n,k}$, the observable rv's are

$$\underset{\sim}{Q}_i = (S_i, Z^0_{n,i}, \underset{\sim}{X}'_{S_i}) \ , \quad \text{for} \quad i = 1,\dots,k \ ; \ \ k \le n \ . \tag{2.9}$$

In a fixed-point censoring scheme, for some pre-fixed $r(\le n)$, experimentation is curtailed when $Z^0_{n,r}$ has been observed and a test for H_0 in (2.5) is then based on $Q_1,\dots,Q_r$. In contrast, in a PCS, one proceeds to construct a time-sequential test based on the progressively available (partial) sequence $\underset{\sim}{Q}_1,\dots,\underset{\sim}{Q}_r$, so that the option of stopping experimentation prior to observing $Z^0_{n,r}$ is left open. In this setup, depending on the accumulated statistical evidence at the various failures, one may stop when, for some $k \le r$, $Z^0_{n,k}$ is observed.

To formulate suitable rank-based PCS tests, let us denote by $c(u) = 1$ or 0 according as u is $\ge$ or < 0 and let

$$R_{ji} = \sum_{\alpha=1}^{n} c(X_{ji} - X_{j\alpha}) = \text{Rank of } X_{ji} \text{ among } X_{j0},\dots,X_{jn} \ , \tag{2.10}$$

for $i = 1,\dots,n$ and $j = 0,1,\dots,p$. Thus, for the complete sampling scheme, we obtain the *rank-collection matrix* (of order $(p+1) \times n$):

$$\underset{\sim}{R}_n = ((R_{ji}))_{j=0,\dots,p;\, i=1,\dots,n} \ . \tag{2.11}$$

Consider now a permutation of the columns of $\underset{\sim}{R}_n$ so that the top row is in the natural order [viz. $1,\ldots,n$], and denote the resulting matrix [termed the *reduced rank-collection matrix*] by $\underset{\sim}{R}_n^*$, so that

$$\underset{\sim}{R}_n \to \underset{\sim}{R}_n^* = \begin{pmatrix} 1 & 2 & \ldots & n \\ R_{11}^*, & R_{12}^*, & \ldots, & R_{1n}^* \\ \ldots & \ldots & \ldots & \ldots \\ R_{p1}^*, & R_{p2}^*, & \ldots, & R_{pn}^* \end{pmatrix} . \tag{2.12}$$

Note that by (2.8), (2.11) and (2.12),

$$R_{0S_i} = i \quad \text{and} \quad R_{ji}^* = R_{jS_i} \quad \text{for} \quad j = 0,1,\ldots,p \quad \text{and} \quad i = 1,\ldots,n . \tag{2.13}$$

Thus, corresponding to the partial collection $(\underset{\sim}{Q}_1,\ldots,\underset{\sim}{Q}_k)$, we have the *partial reduced rank-collection matrix*

$$\underset{\sim}{R}_{n,k}^* = ((R_{ji}^*))_{j=0,1,\ldots,p;i=1,\ldots,k} , \quad \text{for} \quad k = 1,\ldots,n . \tag{2.14}$$

We also denote by

$$\underset{\sim}{S}_{n,k} = (S_1,\ldots,S_k)' , \quad 1 \le k \le n , \quad \text{so that} \quad \underset{\sim}{S}_n = \underset{\sim}{S}_{n,n} . \tag{2.15}$$

Since we are interested in developing a rank test under PCS, we confront the problem of constructing a (sequential) test based on the partial sequence $\{\underset{\sim}{R}_{n,k}^*;\ k \le r\}$ (where, we may even let $r = n$). [Note that as the covariates are all observable at the beginning of the experiment, we have the knowledge of $\underset{\sim}{S}_{n,k}$ and $\underset{\sim}{R}_{n,k}^*$ at the k-th failure $\underset{\sim}{Z}_{n,k}$, $k \le n$.] As the S_i or the corresponding $(R_{jS_i},\ 0 \le j \le p)$ are not independent for different i [and the distribution of $\underset{\sim}{R}_n$ (or any $\underset{\sim}{R}_{n,k}^*$, $k \le n$) depends on the underlying F^* (even under H_0), unless the coordinates of $\underset{\sim}{X}_i^*$

are mutually independent], in a PCS, the repeated significance tests involve an increasing sequence of dependent data, and thereby, poses additional complications.

Our proposed tests are based on linear rank statistics. In view of (2.7), we consider the statistics

$$\underset{\sim}{T}_n = (T_n^{(0)},\ldots,T_n^{(p)})' = \sum_{i=1}^n (c_i - \bar{c}_n)(a_{n,0}(R_{0i}),\ldots,a_{n,p}(R_{pi}))' , \tag{2.16}$$

where the c_i are known (regression) constants, not all equal, $\bar{c}_n = n^{-1}\sum_{i=1}^n c_i$, the R_{ij} are defined by (2.10), and for each $j(=0,\ldots,p)$, $a_{n,j}(1),\ldots,a_{n,j}(n)$ are (real valued) *scores* (not all equal) which we shall define more formally later on. When all the $\underset{\sim}{X}_i^*$ are observable, the usual rank-ANOCA tests are based on $\underset{\sim}{T}_n$ [viz., Puri and Sen (1969a) for motivations]. In the current setup, at the k-th failure $Z_{n,k}^0$, we are provided only with $\underset{\sim}{R}_{n,k}^*$ and $\underset{\sim}{S}_{n,k}$ $(k \le n)$, and hence, we need some modifications. We rewrite $\underset{\sim}{T}_n$ as

$$\underset{\sim}{T}_n = \sum_{i=1}^n (c_{S_i} - \bar{c}_n)(a_{n,0}(R_{0i}^*),\ldots,a_{n,p}(R_{pi}^*))' . \tag{2.17}$$

Note that under H_0 in (2.5), $\underset{\sim}{X}_1^*,\ldots,\underset{\sim}{X}_n^*$ are independent and identically distributed (i.i.d.) rv, and hence, their joint distribution remains invariant under any permutation of the n vectors among themselves. Let $\mathcal{R}_n^*$ be the set of $n!$ matrices obtained from $\underset{\sim}{R}_n^*$ by all possible permutations of the columns of the latter. Then, as in Chatterjee and Sen (1964) [or Puri and Sen (1969b)], under (2.4)-(2.5), the conditional distribution of $\underset{\sim}{R}_n$ over $\mathcal{R}_n^*$ is uniform [with each element of $\mathcal{R}_n$ having the common conditional probability $(n!)^{-1}$]. Let use denote this conditional probability measure by $\mathcal{P}_n$. In passing, we may remark that the unconditional df of

$\underset{\sim}{T}_n$ generally depends on the unknown F^*, and hence, $\underset{\sim}{T}_n$ is not genuinely distribution-free under H_0. Nevertheless, it is conditionally distribution-free under H_0. Then, motivated by the line of attack of Chatterjee and Sen (1973), we define

$$\begin{aligned}\underset{\sim}{T}_{n,k} &= (T_{n,k}^{(0)},\ldots,T_{n,k}^{(p)})' = E_{P_n}(\underset{\sim}{T}_n|\underset{\sim}{S}_{n,k}) \\ &= \sum_{i=1}^{k}(c_{S_i}-\bar{c}_n)(a_{n,0}(i),a_{n,1}(R_{1S_i}),\ldots,a_{n,p}(R_{pS_i}))' \\ &\qquad + \left\{\sum_{i=k+1}^{n}(c_{S_i}-\bar{c}_n)\right\}(a_{n,0}^*(k),\ldots,a_{n,p}^*(k))' \qquad (2.18) \\ &= \sum_{i=1}^{k}(c_{S_i}-\bar{c}_n)(a_{n,0}(i)-a_{n,0}^*(k),a_{n,1}(R_{1S_i}) \\ &\qquad -a_{n,1}^*(k),\ldots,a_{n,p}(R_{pS_i})-a_{n,p}^*(k))'\end{aligned}$$

where for $1 \le k \le n-1$,

$$\begin{aligned}a_{n,j}^*(k) &= (n-k)^{-1}\sum_{i=k+1}^{n}a_{n,j}(R_{jS_i}) \\ &= (n-k)^{-1}\left\{\sum_{i=1}^{n}a_{n,j}(R_{ji}^*) - \sum_{i=1}^{k}a_{n,j}(R_{ji}^*)\right\} \qquad (2.19) \\ &= (n-k)^{-1}\left\{n\bar{a}_{n,j} - \sum_{i=1}^{k}a_{n,j}(R_{ji}^*)\right\}, \quad j = 0,1,\ldots,p\ .\end{aligned}$$

and

$$\bar{a}_{n,j} = n^{-1}\sum_{i=1}^{n}a_{n,j}(i)\ , \quad j = 0,1,\ldots,p\ ; \quad \underset{\sim}{\bar{a}}_n = (\bar{a}_{n,0},\ldots,\bar{a}_{n,p})'\ . \qquad (2.20)$$

Conventionally, we let $a_{n,j}^*(n) = 0$, $0 \le j \le p$, so that $\underset{\sim}{T}_{n,n} = \underset{\sim}{T}_n$, and we let $\underset{\sim}{T}_{n,0} = 0$. Thus, $\underset{\sim}{T}_{n,k}$ is defined for every k: $0 \le k \le n$.

Our task is to consider a suitable sequence of covariate-adjusted statistics, viz.,

$$L_{n,k} = L_n(\underset{\sim}{T}_{n,k})\ , \quad k = 0,1,\ldots,r(\le n)\ , \qquad (2.21)$$

and continue experimentation as long as $L_{n,k}$ lies below a critical value $\ell_{\alpha}^{(n)}$ (where α: $0 < \alpha < 1$ is the desired level of significance of the PCS test); if, for the first time, for some $k = N(\le r)$, $L_{n,N} \ge \ell_{\alpha}^{(n)}$, the experimentation is curtailed at the N-th failure $Z_{n,N}^{0}$ along with the rejection of H_0, and if no such $k(\le r)$ exists, experimentation stops at the preplanned r-th failure $Z_{n,r}^{0}$ along with the acceptance of H_0. Thus, $N(\le r)$ is the *stopping number* and $Z_{n,N}^{0}$ is the *stopping time* of the time-sequential procedure in the prescribed PCS. Finally,

$$P\{L_{n,k} \ge \ell^{(n)} \text{ for some } k \le r | H_0\} \le \alpha \ . \tag{2.22}$$

Our basic problem is to construct $\{L_{n,k};\ k \le r(\le n)\}$ and to choose $\ell_{\alpha}^{(n)}$, such that (2.22) holds.

3. THE PROPOSED PCS TESTS

Let us define for every k: $1 \le k \le n$,

$$v_{n,j\ell}^{(k)} = \frac{1}{n}\left\{\sum_{i=1}^{k} a_{n,j}(R_{ji}^{*})a_{n,\ell}(R_{\ell i}^{*}) + (n-k)a_{n,j}^{*}(k)a_{n,\ell}^{*}(k)\right\} - \bar{a}_{n,j}a_{n,\ell} , \tag{3.1}$$

for $j, \ell = 0,1,\dots,p$, and set

$$\underset{\sim}{V}_{n,k} = ((v_{n,j\ell}^{(k)}))_{j,\ell=0,\dots,p} , \quad 1 \le k \le n ; \quad \underset{\sim}{V}_{n,0} = \underset{\sim}{0} . \tag{3.2}$$

Also, let

$$C_n^2 = \sum_{i=1}^{n}(c_i - \bar{c}_n)^2 . \tag{3.3}$$

Then, following the line of attack of Chatterjee and Sen (1965) and Puri and Sen (1969b), we have by (2.18) and (3.1),

$$E_{P_n}\{\underset{\sim}{T}_{n,k}\} = E\{E_{P_n}[\underset{\sim}{T}_n | \underset{\sim}{S}_{n,k}]\}$$

$$= E_{P_n}(\underset{\sim}{T}_n) = 0 \ , \quad \forall\ k \le n \ ; \tag{3.4}$$

$$V_{P_n}\{\underset{\sim}{T}_{n,k}\} = C_n^2 \cdot \underset{\sim}{V}_{n,k} \quad \text{for} \quad k = 0,.,,\ldots,n \ . \tag{3.5}$$

To utilize the information contained in the concomitant variates, we fit a linear regression of $T_{n,k}^{(0)}$ on $(T_{n,k}^{(1)},\ldots,T_{n,k}^{(p)})$ and work with the residuals. [In view of the asymptotic multinormality of $C_n^{-1}\underset{\sim}{T}_{n,k}$ (to be proved in Section 4), the fitting of linear regression seems justifiable.] Let then

$$T_{n,k}^{*} = T_{n,k}^{(0)} - (\text{fitted value of } T_{n,k}^{(0)} \text{ on } (T_{n,k}^{(1)},\ldots,T_{n,k}^{(p)}))$$

$$= \sum_{j=0}^{p} (v_{n,k}^{pj}/v_{n,k}^{00})T_{n,k}^{(j)} \ , \tag{3.6}$$

where

$$\underset{\sim}{V}_{n,k}^{-1} = ((v_{n,k}^{j\ell}))_{j,\ell=0,\ldots,p} \ ; \quad k \ge 1 \ , \tag{3.7}$$

and, for the time-being, we assume that $\underset{\sim}{V}_{n,k}$ is positive-definite (p.d.). Also, note that

$$V_{P_n}(T_{n,k}^{*}) = C_n^2/v_{n,k}^{00} \ ; \quad k \ge 1 \ . \tag{3.8}$$

Let us introduce the standardize variates

$$\xi_{n,k} = \begin{cases} C_n^{-1}(v_{n,k}^{00})^{\frac{1}{2}}T_{n,k}^{*} \ , & \text{if } \underset{\sim}{V}_{n,k} \text{ is p.d.} \ , \\ 0 \ , & \text{otherwise, for } 0 \le k \le n \ . \end{cases} \tag{3.9}$$

Then, depending on the one or two-sided alternatives, we may have in mind [viz., (2.7): $\beta > 0$ or $\beta \ne 0$], we set

$$L_{n,k} = \xi_{n,k} \quad \text{or} \quad |\xi_{n,k}| \ , \quad k \ge 0 \ . \tag{3.10}$$

Thus, our proposed PCS test is based on the standardized and covariate-adjusted linear rank statistics at the successive failures for the primary variates. Though, in the above formulation, we have made use of P_n, in order to enumerate the permutational distribution of $\max_{k\le r} L_{n,k}$, [required for the determination of $\ell_\alpha^{(n)}$ in (2.2)], we need the knowledge of $\underset{\sim}{R}^*_{n,r}$, so that we have to wait till the r-th failure $Z^0_{n,r}$. This is in contrary to our aim of progressively censoring from the very beginning. Also, $v^{00}_{n,r}$ is not known until the r-th failure has occurred, and hence, unlike in Chatterjee and Sen (1973), in (3.9), at the kth stage, we take $v^{00}_{n,k}$ but not $v^{00}_{n,r}$ (for $k \leq r$). For these reasons, we do not use the permutation distribution [as in Puri and Sen (1969a) dealing with the complete sample case]. Rather, we proceed to develop certain invariance principles for the partial sequences $\{T_{n,k};\ k \le n\}$ and $\{\xi_{n,k};\ k \le n\}$ and incorporate them in the study of the asumptotic form of $\ell_\alpha^{(n)}$.

4. ASYMPTOTIC DISTRIBUTIONS UNDER H_0

For the study of some weak convergence results for the partial sequences $\{\underset{\sim}{T}_{n,k};\ k \le n\}$, $\{T^*_{n,k};\ k \le n\}$ and $\{\xi_{n,k};\ k \le n\}$, we make the following assumptions:

(I) For the maximum censoring point $Z^0_{n,r}$, we assume that as $n \to \infty$, $r = r(n) \to \infty$ with

$$\lim_{n\to\infty} r(n)/n = \tau;\ 0 < \tau \le 1 . \tag{4.1}$$

(II) The scores $a_{n,j}(i)$ are generated by a *score function* ϕ_j in the following way

$$a_{n,j}(i) = \phi_j\left(\frac{i}{n+1}\right) \text{ or } E\phi_j(U_{ni}) \ , \quad 1 \le i \le n \ ; \quad j = 0,1,\dots,p \ , \tag{4.2}$$

where $U_{n1} < \dots < U_{nn}$ are the ordered rv's of a sample of size n from the rectangular $(0,1)$ df, and for each j $(0 \le j \le p)$,

$$\phi_j(u) = \phi_{j,1}(u) - \phi_{j,2}(u) \ , \quad 0 < u < 1 \ , \tag{4.3}$$

where $\phi_{j,k}(u)$ is non-decreasing, absolutely continuous and square integrable inside $(0,1)$.

(III) Concerning the c_i, we let

$$c^*_{ni} = (c_i - \bar{c}_n)/C_n \ , \quad 1 \le i \le n \ \left(\Longrightarrow \sum_{i=1}^n c^*_{ni} = 0 \ , \ \sum_{i=1}^n (c^*_{ni})^2 = 1\right) \ , \tag{4.4}$$

and assume that

$$\lim_{n\to\infty} \left\{ \max_{1\le i\le n} |c^*_{ni}| \right\} = 0 \ . \tag{4.5}$$

(IV) Let $F_{[j]}(x)$, $F_{[j,\ell]}(x,y)$ and $F_{[0,j,\ell]}(x,y,z)$ be respectively the marginal df of X_{ji}, joint df of $(X_{ji}, X_{\ell i})$ $(0 \le j \ne \ell \le p)$ and the trivariate (if $p \ge 2$) df of $(X_{0i}, X_{ji}, X_{\ell i})$, $1 \le j \ne \ell \le p$, all under H_0. Also, we assume that for every $0 < t < 1$, $F_{[0]}(\zeta_t^0) = t$ has a unique solution (ζ_t^0), and we let $\zeta_0^0 = -\infty$, $\zeta_1^0 = +\infty$. Let then

$$\bar{\phi}_{0t} = (1-t)^{-1}\int_t^1 \phi_0(u)du \ , \quad 0 \le t < 1 \ , \quad \bar{\phi}_{00} = \bar{\phi}_0 \ , \quad \bar{\phi}_{01} = 0 \ ; \tag{4.6}$$

$$\phi_{jt} = (1-t)^{-1}\int_{\zeta_t^0}^{\infty}\int_{-\infty}^{\infty} \phi_j(F_{[j]}(y))dF_{[0,j]}(x,y) \ , \quad 0 \le t < 1 \ , \quad j = 1,\dots,p \ , \tag{4.7}$$

so that $\bar{\phi}_j = \int_0^1 \phi_j(u)du = \bar{\phi}_{j0}$ and let $\bar{\phi}_{j1} = 0$, $\forall\ 1 \le j \le p$. Also, let

$$\nu_{00}(t) = \int_0^t \phi_0^2(u)du + (1-t)\bar{\phi}_{0t}^2 - \bar{\phi}_0^2 \ , \quad 0 \le t \le 1 \ , \tag{4.8}$$

$$\nu_{0j}(t) = \nu_{j0}(t) = \int_{-\infty}^{\zeta_t^0}\int_{-\infty}^{\infty}\phi_0(F_{[0]}(x))\phi_j(F_{[j]}(y))dF_{[0,j]}(x,y)$$

$$+ (1-t)\bar{\phi}_{0t}\bar{\phi}_{jt} - \bar{\phi}_0\bar{\phi}_j\ ,\quad j = 1,\dots,p\ ,\quad 0 \le t \le 1\ ; \tag{4.9}$$

$$\nu_{j\ell}(t) = \int_{-\infty}^{\zeta_t^0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi_j(F_{[j]}(y))\phi_\ell(F_{[\ell]}(z))dF_{[0,j,\ell]}(x,y,z)$$

$$+ (1-t)\bar{\phi}_{jt}\bar{\phi}_{\ell t} - \bar{\phi}_j\bar{\phi}_\ell\ ,\quad j,\ell = 1,\dots,p\ ,\quad 0 \le t \le 1\ ; \tag{4.10}$$

$$\underset{\sim}{\nu}(t) = ((\nu_{j\ell}(t)))_{j,\ell=0,\dots,p} \quad \text{for} \quad 0 \le t \le 1\ . \tag{4.11}$$

Note that $\underset{\sim}{\nu}(0) = \underset{\sim}{0}$. We assume that

$$\underset{\sim}{\nu}(t) \text{ is p.d. for every } 0 < t \le 1\ , \tag{4.12}$$

and denote the reciprocal matrix by

$$\underset{\sim}{\nu}^{-1}(t) = ((\nu^{j\ell}(t)))_{j,\ell=0,1,\dots,p} \quad \text{for} \quad 0 < t \le 1\ . \tag{4.13}$$

Theorem 4.1. *Under* (2.6), (4.2), (4.3) *and for continuous* F^*,

$$\max_{k\le n}\left\{\max_{0\le j,\ell\le p}\left|v_{n,j\ell}^{(k)} - \nu_{j\ell}\left(\frac{k}{n}\right)\right|\right\} \xrightarrow{p} 0\ ,\quad \textit{as} \quad n \to \infty\ . \tag{4.14}$$

Outline of the proof. It sufficies to show that for every (j,ℓ): $0 \le j$, $\ell \le p$, $\max_{k\le n}|v_{n,j\ell}^{(k)} - \nu_{j\ell}\left(\frac{k}{n}\right)| \xrightarrow{p} 0$ as $n \to \infty$, and we prove this for some $1 \le j$, $\ell \le p$; the case of either of j or ℓ being 0 follows by comparatively simpler manipulations. Let us define

$$F_{n[j]}(x) = n^{-1}\sum_{i=1}^{n}c(x-X_{ji})\ ,\quad F_{n[j,\ell]}(x,y) = n^{-1}\sum_{i=1}^{n}c(x-X_{ji})c(y-X_{\ell i})\ , \tag{4.15}$$

for $j \neq \ell = 0,1,\ldots,p$, and also for $j \neq \ell = 1,\ldots,p$, we let

$$F_{n[0,j,\ell]}(x,y,z) = n^{-1}\sum_{i=1}^{n} c(x-X_{0i})c(Y-X_{ji})c(z-X_{\ell i}) \ . \tag{4.16}$$

Further, we let

$$\phi_j^{(n)}(u) = a_{n,j}(i) \quad \text{for} \quad \frac{i-1}{n} < u \leq \frac{i}{n} \ , \quad 1 \leq i \leq n \ , \quad j = 0,1,\ldots,p \ . \tag{4.17}$$

Then, by (3.1), (4.15), (4.16) and (4.17), we have for $k < n$,

$$v_{n,j\ell}^{(k)} = \int_{-\infty}^{Z_{n,k}^0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi_j^{(n)}(F_{n[j]}(y))\phi_\ell^{(n)}(F_{n[\ell]}(z))dF_{n[0,j,\ell]}(x,y,z) \tag{4.18}$$

$$+ \frac{n}{n-k}\left[\int_{Z_{n,k}^0}^{\infty}\int_{-\infty}^{\infty}\phi_j^{(n)}(F_{n[j]}(y))dF_{n[0,j]}(x,y)\right]\left[\int_{Z_{n,k}^0}^{\infty}\int_{-\infty}^{\infty}\phi_\ell^{(n)}(F_{n[\ell]}(z)dF_{n[0,\ell]}(x,z)\right]$$

$$- \left[\int_{-\infty}^{\infty}\phi_j^{(n)}(F_{n[j]}(x))dF_{n[j]}(x)\right]\left[\int_{-\infty}^{\infty}\phi_\ell^{(n)}(F_{n[\ell]}(y))dF_{n[\ell]}(y)\right] ,$$

and for $k = n$, the second term on the right hand side of (4.18) droppsout.

Now, under (4.2) and (4.3), we may use the Hájek (1968) polynomial approximation for the ϕ_j (and hence, the $a_{n,j}(i)$), and proceeding as in the first part of the proof of Theorem 3.1 of Puri and Sen (1969b), for every $\eta > 0$, we have a decomposition of ϕ_j into a polynomial part (say, ϕ_j^η) and a residual part (say, $\hat{\phi}_j$), such that, definining the $v_{n,j\ell}^{\eta(k)}$ as in (4.18) with the ϕ_j being replaced by ϕ_j^η, $0 \leq j \leq p$,

$$|v_{n,j\ell}^{\eta(k)} - v_{n,j\ell}^{(k)}| < \eta \ , \quad \forall\ k \leq n \ , \quad j,\ell = 0,1,\ldots,p \ . \tag{4.19}$$

Similarly, if in (4.8)-(4.10), we replace the ϕ_j by ϕ_j and denote the resulting quantities by $\nu_{j\ell}^{\eta}(t)$, $0 \leq j,\ \ell \leq p$, $0 \leq t \leq 1$, then we have

$$|\nu_{j\ell}(t) - \nu^{\eta}_{j\ell}(t)| < \eta\ , \quad \forall\ 0 \le t \le 1 \quad \text{and} \quad 0 \le j\ ,\ \ell \le p\ . \tag{4.20}$$

Hence, it suffices to show that (4.14) holds with $v^{(k)}_{n,j\ell}$ and $\nu_{j\ell}\left(\frac{k}{n}\right)$ being replaced by $v^{\eta(k)}_{n,j\ell}$ and $\nu^{\eta}_{j\ell}\left(\frac{k}{n}\right)$, respectively. Note that the ϕ^{η}_{j} are all polynomial (and hence, are bounded and continuously differentiable) and the Glivenko-Cantelli Lemma insures the almost sure (a.s.) convergnece of $\sup_x |F_{n[j]}(x) - F_{[j]}(x)|$ (to 0), $\forall\ 0 \le j \le p$; similar a.s. convergence results also hold for the bivariate and trivariate df's in (4.15)-(4.16). Finally, $\max_{k\le n} |F_{n[0]}(Z^0_{n,k}) - F_{[0]}(Z^0_{n,k})| = \max_{k\le n}|F_{[0]}(Z^0_{n,k}) - k/n| \to 0$ a.s., as $n \to \infty$. Hence, by some standard steps, it can be shown that as $n \to \infty$,

$$\max_{k\le n}\left| v^{\eta(k)}_{n,j\ell} - \nu^{\eta}_{j\ell}\left(\frac{k}{n}\right)\right| \to 0 \quad \text{a.s, for} \quad \forall\ 0 \le j\ ,\ \ell \le p\ , \tag{4.21}$$

and the desired result follows from (4.19)-(4.21). Q.E.D.

Note that by the continuity of F^* and the score functions, $\underset{\sim}{\nu}(t)$ is a continuous function of $t \in [0,1]$ and $\underset{\sim}{\nu}(1) = \underset{\sim}{\nu} = ((\nu_{j\ell}))$, where

$$\nu_{j\ell} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_j(F_{[j]}(x))\phi_\ell(F_{[\ell]}(y))dF_{[j,\ell]}(x,y) - \bar{\phi}_j\bar{\phi}_\ell\ , \quad 0 \le j\ ,\ \ell \le p\ . \tag{4.22}$$

It follows therefore that

$$\max_{0\le j,\ell\le p}\left\{\text{Sup}[|\nu_{j\ell}(t) - \nu_{j\ell}(s)| : 0 \le s < t \le s + \delta \le 1]\right\} \to 0 \quad \text{as} \quad \delta \downarrow 0\ . \tag{4.23}$$

Further, note that by virtue of (4.12) and Theorem 4.1, for every ε: $0 < \varepsilon < 1$, the matrices in the partial sequence $\{\underset{\sim}{V}_{n,k};\ n\varepsilon \le k \le n\}$ are all positive definite, in probability, as $n \to \infty$.

Let now $\mathcal{B}_{n,k}$ be the sigma-field generated by $\underset{\sim}{S}_{n,k}$ (under P_n), for $k = 1,\dots,n$ and let $\mathcal{B}_{n,0}$ be the trivial sigma-field. Then, for every $n(\ge 1)$, $\mathcal{B}_{n,k}$ is non-decreasing in $k(\le n)$. Note that by (2.18) and (3.4),

$$E_{P_n}(\underset{\sim}{T}_n | \mathcal{B}_{n,k}) = \underset{\sim}{T}_{n,k} \;, \quad \forall\ 0 \le k \le n \;, \tag{4.24}$$

and hence, for every $n(\ge 1)$, under P_n, $\{\underset{\sim}{T}_{n,k}, \mathcal{B}_{n,k};\ 0 \le k \le n\}$ is a martingale, closed to the right by $\underset{\sim}{T}_n$.

Lemma 4.2. *For every* (k,n): $0 \le k \le n-1$, $\underset{\sim}{V}_{n,k+1} - \underset{\sim}{V}_{n,k}$ *is positive semi-definite (p.s.d.).*

Proof. By (3.5) and (4.24), for every $0 \le k \le n-1$,

$$\begin{aligned}\underset{\sim}{V}_{n,k+1} - \underset{\sim}{V}_{n,k} &= C_n^{-2}[\underset{\sim}{V}_{P_n}(\underset{\sim}{T}_{n,k+1}) - \underset{\sim}{V}_{P_n}(\underset{\sim}{T}_{n,k})] \\ &= C_n^{-2} \cdot \underset{\sim}{V}_{P_n}(\underset{\sim}{T}_{n,k+1} - \underset{\sim}{T}_{n,k}) \\ &= C_n^{-2} E_{P_n}\{(\underset{\sim}{T}_{n,k+1} - \underset{\sim}{T}_{n,k})(\underset{\sim}{T}_{n,k+1} - \underset{\sim}{T}_{n,k})'\}\end{aligned} \tag{4.25}$$

and hence, is p.s.d. Q.E.D.

Theorem 4.3. *Under Assumptions* (II), (III) *and* (IV), *for every (fixed)* $m(\ge 1)$, $(0 \le) t_1 < \dots < t_m (\le 1)$ *and every* $\{k_1, \dots, k_m\}$ *satisfying* $\lim_{n\to\infty} n^{-1} k_r = t_r$, $r = 1,\dots,m$, *when* H_0 *holds*,

$$\mathcal{L}(C_n^{-1}(\underset{\sim}{T}_{n,k_1}, \dots, \underset{\sim}{T}_{n,k_m})) \to \mathcal{N}(0, \underset{\sim}{\Gamma}_m) \;. \tag{4.26}$$

where

$$\underset{\sim}{\Gamma}_m = ((\gamma_{jj',rr'} = \nu_{jj'}(t_r \wedge t_{r'})))_{j,j'=0,\dots,p;\, r,r'=1,\dots,m} \;. \tag{4.27}$$

Outline of proof. For simplicity, we take $m = 1$; a similar but somewhat lengthy proof holds for any $m(\ge 1)$. Note that by (2.18) and (4.4) for any $\underset{\sim}{\lambda} = (\lambda_0, \dots, \lambda_p)'$ $(\ne 0)$ and $k(1 \le k \le n)$,

$$Y_{n,k} = C_n^{-1}\underset{\sim}{\lambda}'(\underset{\sim}{T}_{n,k} - \underset{\sim}{T}_{n,k-1})$$

$$= \left[c_{nS_k}^* - \frac{1}{n-k+1}\sum_{j=k}^{n} c_{nS_j}^*\right]\left[\sum_{j=0}^{p}\lambda_j\{a_{n,j}(R_{jk}^*) - a_{n,j}^*(k)\}\right] \tag{4.28}$$

and $Y_{n,0} = 0$. Then, by (4.24) and (4.25),

$$E_{P_n}(Y_{n,k}|\mathcal{B}_{n,k-1}) = 0 \ , \quad \forall\ 1 \le k \le n \ ; \tag{4.29}$$

$$E_{P_n}(\ Y_{n,k}^2\ ) \ = \ \underset{\sim}{\lambda}'(\underset{\sim}{V}_{n,k+1} - \underset{\sim}{V}_{n,k})\underset{\sim}{\lambda}(\ge 0) \ , \ \forall\ 1 \le k \le n \ . \tag{4.30}$$

Therefore, by Theorem 4.1 and (4.30),

$$\lim_{n\to\infty} k/n = t \Rightarrow V_{P_n}\left(\sum_{i=1}^{k} Y_{n,i}\right) \to \underset{\sim}{\lambda}'\underset{\sim}{\nu}(t)\underset{\sim}{\lambda} \ (> 0, \ \forall\ 0 < t \le 1) \ . \tag{4.31}$$

First, we establish (4.26) under P_n. By virtue of (4.29), (4.31) and the main theorem of Dvoretzky (1972), it suffices to show that for $k/n \to t \in (0,1]$,

$$(\underset{\sim}{\lambda}'\underset{\sim}{\nu}(t)\underset{\sim}{\lambda})^{-1}\left\{\sum_{i=1}^{k} E_{P_n}[Y_{n,i}^2|\mathcal{B}_{n,i-1}]\right\} \overset{P}{\to} 1 \quad \text{as} \quad n \to \infty \tag{4.32}$$

and, for every $\varepsilon > 0$,

$$\sum_{i=1}^{k} E_{P_n}\{Y_{n,i}^2 I(Y_{n,i}^2 > \varepsilon)\} \overset{P}{\to} 0 \ , \quad \text{as} \quad n \to \infty \ , \tag{4.33}$$

where $I(A)$ stands for the indicator function of a set A.

Now, under assumption (II), denoting $T_{n,k}^{\eta}$, the vector of rank statistics when the scores are generated by the polynomial score functions ϕ_j^{η}, $0 \le j \le p$, we have

$$C_n^{-2}E_{P_n}\{\underset{\sim}{\lambda}'(\underset{\sim}{T}_{n,k} - \underset{\sim}{T}_{n,k}^{\eta})\}^2 = \underset{\sim}{\lambda}'[\underset{\sim}{V}_{n,k} - \underset{\sim}{V}_{n,k}^{\eta}]\underset{\sim}{\lambda} \ , \tag{4.34}$$

and hence, by (4.19), (4.34) can be made arbitrarily small, by choosing

$\eta(>0)$ arbitrarily small. Hence, $C_n^{-1}\underset{\sim}{\lambda}'(\underset{\sim}{T}_{n,k} - \underset{\sim}{T}^{\eta}_{n,k}) \overset{p}{\to} 0$, under P_n, and to prove (4.32)-(4.33), it suffices to take the score functions ϕ_m, $0 \le m \le p$ all polynomials inside (0,1) [which are therefore boundedly continuous inside (0,1)]. As such, for polynomial score functions, it follows by some standard setps that with probability 1,

$$\max_{0\le j\le p}\left\{\max_{1\le k\le n}|a_{n,j}(R^*_{jk}) - a^*_{n,j}(k)|\right\} = 0(1) , \tag{4.35}$$

so that by (4.5), (4.28) and (4.35), for every $\varepsilon > 0$, there exists a positive integer n_ε, such that for $n \ge n_\varepsilon$,

$$\max_{1\le i\le n}\{Y^2_{n,i}\} < \varepsilon , \text{ with probability } 1 , \tag{4.36}$$

and (4.36) insures (4.33). Also, by (4.28), we have

$$E_{P_n}(Y^2_{n,i}|\mathcal{B}_{n,i-1}) = \left[\sum_{j=0}^{p}\lambda_j\{a_{n,j}(R^*_{ji}) - a^*_{n,j}(i)\}\right]^2 \tag{4.37}$$

$$(n-i+1)^{-1}\left[\sum_{s=i}^{n}\{c^*_{nS_s} - \frac{1}{n-i+1}\sum_{\alpha=i}^{n}c^*_{nS_\alpha}\}^2\right] , \; i \le i \le n .$$

Thus, for every k: $1 \le k \le n$,

$$\sum_{i=1}^{k}E_{P_n}(Y^2_{n,i}|\mathcal{B}_{n,i-1}) = g_{n,k}(c^*_{nS_1},\ldots,c^*_{nS_n}) , \tag{4.38}$$

where $g_{n,k}(\cdot)$ is a quadratic form in its n arguments. Note that

$$E_{P_n}g_{n,k}(c^*_{nS_1},\ldots,c^*_{nS_n}) = E_{P_n}(\underset{\sim}{\lambda}'\underset{\sim}{T}_{n,k})^2 = \underset{\sim}{\lambda}'\underset{\sim}{V}_{n,k}\underset{\sim}{\lambda} , \tag{4.39}$$

while by using (4.5), (4.14), (4.37) and following some standard steps, it follows that (for polynomial score functions)

$$E_{P_n}g^2_{n,k}(c^*_{nS_1},\ldots,c^*_{nS_n}) = (\underset{\sim}{\lambda}'\underset{\sim}{V}_{n,k}\underset{\sim}{\lambda})^2 + o(1) , \text{ as } n \to \infty . \tag{4.40}$$

Hence, (4.32) follows from (4.38), (4.39) and (4.40). Thus,

$$\left[\frac{k}{n} \to t\right] \Rightarrow L_{P_n}(C_n^{-1}\underset{\sim}{T}_{n,k}) \to N(\underset{\sim}{0}, \underset{\sim}{V}_{n,k}) \ . \tag{4.41}$$

Since, $P\{C_n^{-1}\underset{\sim}{T}_{n,k} \le \underset{\sim}{x} | H_0\} = E[P\{C_n^{-1}\underset{\sim}{T}_{n,k} \le \underset{\sim}{x},$ under $P_n\}]$, $\forall\ \underset{\sim}{x}$, the desired result follows from (4.41) and Theorem 4.1. Q.E.D.

Now, by virtue of Lemma 4.2, whenever properly defined

$$v_{n,k}^{00} \text{ is } \searrow \text{ in } k \text{ for } 1 \le k \le n \ , \tag{4.42}$$

and we take $v_{n,k}^{00}$ to be equal to $+\infty$ whenever $\underset{\sim}{V}_{n,k}$ is not p.d. Then, whenever $\underset{\sim}{V}_{n,r}$ is p.d., we consider a stochastic process ${}_{\varepsilon}W_{n,r}^{*} = \{W_{n,r}^{*}(t);\ \varepsilon \le t \le 1\}$ (where $0 < \varepsilon < 1$), by letting

$$\begin{aligned} W_{n,r}^{*}(t) &= C_n^{-1}(v_{n,k_{n,r}(t)}^{00})^{\frac{1}{2}} T_{n,k_{n,r}(t)}^{*} \\ &= \xi_{n,k_{n,r}(t)} \ , \quad \varepsilon \le t \le 1 \end{aligned} \tag{4.43}$$

where

$$k_{n,r}(t) = \max\{k\colon v_{n,r}^{00} \le t v_{n,k}^{00}\}, \quad \varepsilon \le t \le 1 \ . \tag{4.44}$$

Then, whenever, $v_{n,r}^{00}$ is > 0 and finite, for every $0 < \varepsilon < 1$, the process ${}_{\varepsilon}W_{n,r}^{*}$ belongs to the space $D[\varepsilon,1]$, endowed with the Skorohod J_1-topology. Let $W = \{W(t),\ 0 \le t \le 1\}$ be a standard Wiener process on $[0,1]$, and for every $0 < \varepsilon < 1$, we define

$${}_{\varepsilon}W^{*} = \{W^{*}(t) = t^{-\frac{1}{2}}W(t);\ \varepsilon \le t \le 1\} \ . \tag{4.45}$$

Our basic contention is to show that ${}_{\varepsilon}W_{n,r}^{*}$ weakly converge to ${}_{\varepsilon}W^{*}$.

By virtue of Theorems 4.1 and 4.3 and the definitions in (4.43) and (4.44), we arrive at the following theorem by some routine steps.

<u>Theorem 4.4</u>. *Under* H_0 *and the Assumptions* (I), (II), (III) *and* (IV), *for every (fixed)* $m(\geq 1)$, $\varepsilon(0<\varepsilon<1)$ and $(\varepsilon \leq)\ t_1 < \dots < t_m (\leq 1)$,

$$\{W^*_{n,r}(t_1),\dots,W^*_{n,r}(t_m)\} \overset{\mathcal{D}}{\to} \{W^*(t_1),\dots,W^*(t_m)\} \ . \tag{4.46}$$

Let us now consider $(p+1)$ stochastic processes $W^{(j)}_{n,r} = \{W^{(j)}_{n,r}(t);\ 0 \leq t \leq 1\}$, $j=0,1,\dots,p$, by letting

$$W^{(j)}_{n,r}(t) = C_n^{-1}[v^{(r)}_{n,jj}]^{-\frac{1}{2}} T^{(j)}_{n,k^{(j)}_{n,r}(t)} \ ; \tag{4.47}$$

$$k^{(j)}_{n,r}(t) = \max\{k:\ v^{(k)}_{n,jj} \leq t v^{(r)}_{n,jj}\} \ , \quad 0 \leq t \leq 1 \ , \quad j=0,1,\dots,p \ . \tag{4.48}$$

Then, each of these processes belongs to the $D[0,1]$ space. By virtue of (4.23), the martingale property (4.24) and Theorems 4.1 and 4.3, we may proceed as in the proof of Theorem 4.2 of Chatterjee and Sen (1973) and show that on defining (for $0<\delta<1$)

$$\omega_\delta(x) = \sup\{|x(t) - x(s)|;\ 0 \leq s < t \leq s+\delta \leq 1\} \ , \tag{4.49}$$

that for every $\varepsilon > 0$ and $\eta > 0$, there exist a δ: $0<\delta<1$ and a positive number n_0, such that for $n \geq n_0$, under H_0,

$$P\{\omega_\delta(W^{(j)}_{n,r}) > \varepsilon\} < (p+1)^{-1}\eta \ , \quad \text{for} \quad j=0,1,\dots,p \ . \tag{4.50}$$

The convergence of finite-dimensional distributions (f.d.d.) of $W^{(j)}_{n,r}$ to those of W, the standard Wiener process, follows readily from Theorem 4.3, and hence, by (4.50), we claim that under the hypothesis of Theorem 4.4,

$$W_{n,r}^{(j)} \overset{\mathcal{D}}{\to} W \ , \quad \text{in the } J_1\text{-topology on } D[0,1] \ , \quad \forall \ 0 \le j \le p \ , \tag{4.51}$$

and note that (4.51) insures that

$$\max_{0\le j\le p} \left\{ \sup_{0\le t\le 1} |W_{n,r}^{(j)}(t)| \right\} = O_p(1) \ . \tag{4.52}$$

Now, by virtue of (4.12) and (4.23), we claim that for every ε; $0<\varepsilon<1$,

$$\max_{0\le j\le p} \left\{ \text{Sup}[|\nu^{0j}(t) - \nu^{0j}(s)| ; \ \varepsilon \le s < t \le s+\delta \le 1] \right\} \to 0 \quad \text{as} \quad \delta \downarrow 0 \ , \tag{4.53}$$

and by (4.12) and Theorem 4.1,

$$\max_{0\le j\le p} \left\{ \max_{n\varepsilon\le k\le n} \left| v_{n,k}^{0j} - \nu^{0j}\left(\frac{k}{n}\right) \right| \right\} \overset{p}{\to} 0 \ , \quad \text{as} \quad n\to\infty \ . \tag{4.54}$$

As such, for $k \le q \le r$ and $v_{n,k}^{00} < \infty$ $(\Rightarrow v_{n,q}^{00} < \infty)$,

$$\begin{aligned} |\xi_{n,q} - \xi_{n,k}| &= C_n^{-1} |\textstyle\sum_{j=0}^{k} \{ (v_{n,q}^{00})^{-\frac{1}{2}} v_{n,q}^{0j} T_{n,q}^{(j)} - (v_{n,k}^{00})^{-\frac{1}{2}} v_{n,k}^{0j} T_{n,k}^{(j)} \}| \\ &\le \textstyle\sum_{j=0}^{p} |(v_{n,q}^{00})^{-\frac{1}{2}} v_{n,q}^{0j} - (v_{n,k}^{00})^{-\frac{1}{2}} v_{n,k}^{0j}| C_n^{-1} |T_{n,q}^{(j)}| \\ &\quad + \textstyle\sum_{j=0}^{p} (v_{n,k}^{00})^{-\frac{1}{2}} |v_{n,k}^{0j}| C_n^{-1} |T_{n,q}^{(j)} - T_{n,k}^{(j)}| \ . \end{aligned} \tag{4.55}$$

Hence, by using the definitions in (4.43), (4.44), (4.47), (4.48) and (4.49) along with (4.50), (4.52) and (4.53)-(4.54), it follows from (4.55) that

$$\text{Sup}\{|W_{n,r}^{*}(t) - W_{n,r}^{*}(s)| : \ \varepsilon \le s \le t \le s+\delta \le 1\} \overset{p}{\to} 0 \quad \text{as } \delta\downarrow 0 \ , \quad \forall \ 0<\varepsilon<1 \ . \tag{4.56}$$

From Theorem 4.4 and (4.56), we obtain the following:

Theorem 4.5. *Under the hypothesis of Theorem 4.4, for every* ε: $0<\varepsilon<1$,

$$_{\varepsilon}W_{n,r}^{*} \overset{\mathcal{D}}{\to} {}_{\varepsilon}W^{*} \ , \quad \textit{in the } J_1\textit{-topoloty on } D[\varepsilon,1] \ . \tag{4.57}$$

For our proposed PCS tests, (4.57) provides the key result. For the procedure sketched in (2.21)-(2.22) with the $L_{n,k}$ defined by (3.9)-(3.10), we conceive of an initial failure number $k_0(\geq 2)$, such that the PCS is implemented only when the failure Z^0_{n,k_0} has occurred; the basic idea is not to reject H_0 until at least a few observations are at hand. In order to choose k_0 properly, we note that though $v^{(r)}_{n,00}$ is known in advance, $v^{00}_{n,r}$ is not known until the r-th failure has taken place. But, we have the inequality that $v^{(r)}_{n,00} \geq 1/v^{00}_{n,r}$, so that for every k; $n\varepsilon \leq k \leq r$,

$$v^{00}_{n,r}/v^{00}_{n,k} \geq 1/(v^{(r)}_{n,00}v^{00}_{n,k}) \ , \tag{4.58}$$

where the right hand side is observable at the k-th failure $Z^0_{n,k}$. Hence, if we define for any given ε: $0 < \varepsilon < 1$,

$$k_0 = \min\{k;\ (v^{(r)}_{n,00}v^{00}_{n,k}) \leq \varepsilon^{-1}\} \ , \tag{4.59}$$

then we have

$$\max_{k_0 \leq k \leq r} \xi_{n,k} \leq \sup_{\varepsilon \leq t \leq 1} W^*_{n,r}(t) \ , \tag{4.60}$$

and a similar inequality holds for the two-sided case. Thus, under the hypothesis of Theorem 4.5, for every real λ,

$$\lim_{n\to\infty} P\left\{\max_{k_0 \leq k \leq r} \xi_{n,k} > \lambda \,\middle|\, H_0\right\} \leq P\left\{\sup_{\varepsilon \leq t \leq 1} W^*(t) > \lambda\right\} \ , \tag{4.61}$$

$$\lim_{n\to\infty} P\left\{\max_{k_0 \leq k \leq r} |\xi_{n,k}| > \lambda \,\middle|\, H_0\right\} \leq P\left\{\sup_{\varepsilon \leq t \leq 1} |W^*(t)| > \lambda\right\} \ . \tag{4.62}$$

Thus, if $\lambda^+_\alpha(\varepsilon)$ and $\lambda_\alpha(\varepsilon)$ be the values of λ for which the right hand sides of (4.61) and (4.62) are both equal to α, for $0 < \alpha < 1$, $0 < \varepsilon < 1$, then in the asymptotic case, in (2.22), we may take $\ell^{(n)}_\alpha = \lambda^+_\alpha(\varepsilon)$ or $\lambda_\alpha(\varepsilon)$.

Analytical solutions for $\lambda_\alpha^+(\varepsilon)$ and $\lambda_\alpha(\varepsilon)$ are difficult to workout. Majumdar and Sen (1977) have made some simulation studies and we report some of their values here.

TABLE 1

Simulated values of $\lambda_\alpha^+(\varepsilon)$ and $\lambda_\alpha(\varepsilon)$ for some typical (ε,α)

	$\lambda_\alpha^+(\varepsilon)$			$\lambda_\alpha(\varepsilon)$		
ε	$\alpha = .01$	$\alpha = .05$	$\alpha = .10$	$\alpha = .01$	$\alpha = .05$	$\alpha = .10$
0.005	3.32	2.76	2.46	3.48	3.04	2.79
0.01	3.28	2.72	2.42	3.46	3.01	2.76
0.05	3.19	2.60	2.29	3.39	2.89	2.62
0.10	3.05	2.50	2.19	3.26	2.79	2.52

For larger values of ε, $\lambda_\alpha^+(\varepsilon)$ and $\lambda_\alpha(\varepsilon)$ are not very sensitive to ε.

5. ASYMPTOTIC DISTRIBUTION THEORY UNDER LOCAL ALTERNATIVES

We shall now consider the non-null case where (2.5) does not hold, but (2.7) holds for some $\beta > 0$ (or $\neq 0$). For any fixed non-null β, the consistency of the proposed test can be proved along the same line as in Chatterjee and Sen (1973). Hence, for the study of the asymptotic power of the proposed test, we confine ourselves to some local alternatives where we allow β to be close to 0 (or the c_i to be so), Specifically, we frame a sequence $\{K_n\}$ of alternative hypotheses, where

$$K_n: F_i^0(x^0|\underset{\sim}{x}) = F_{i,n}^0(x^0|\underset{\sim}{x})$$
$$= F^0(x_0 - \theta c_{ni}^*|\underset{\sim}{x}) \ , \quad 1 \le i \le n \ , \tag{5.1}$$

with a real θ (fixed) and the c_{ni}^* given by (4.4).

Under (5.1) [and (2.4)], for $\theta \neq 0$, the basic permutational invariance structure of Section 2 does not hold, and, as a result, neither (4.24) nor the martingale-proof of Theorem 4.3 holds. A general proof of the asymptotic normality of $C_n^{-1}(\underset{\sim}{T}_{n,k} - E\underset{\sim}{T}_{n,k})$ may be worked out by using the basic projection technique of Hájek (1968) [as extended to the multivariate case by Puri and Sen (1969b)]. However, in the absence of the martingale property (4.24), this approach does not yield the *tightness* property of ${}_{\varepsilon}W_{n,r}^*$ [in (4.56)] when H_0 may not hold. For this reason, we employ here the notion of *contiguity* [viz., Chapter VI of Hájek and Šidák (1967)] which provides (in a reasonably simple way) a natural extension of Theorem 4.5 under $\{K_n\}$.

We assume the the df F_i^* possesses an absolutely continuous probability density function (pdf) $f_i^*(\underset{\sim}{x})$, $\underset{\sim}{x} \in E^{p+1}$, $\forall\ i \ge 1$ and let $f^*(\underset{\sim}{x})$ be the pdf when $\theta = 0$ [in (5.1)]. Also, let $f_{[0,j]}(x_1,x_2)$ be the bivariate pdf for the $(0,j)$th variates, for $1 \le j \le p$, and we assume that $\dot{f}_{[0,j]}^0(x_1,x_2) = -(\partial/\partial x_1)\log f_{[0,j]}(x_1,x_2)$ exists (a.e.) and

$$J_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\dot{f}_{[0,j]}^0(x_1,x_2)]^2 dF_{[0,j]}(x_1,x_2) < \infty \ , \quad \forall\ j = 1,\ldots,p \ . \tag{5.2}$$

Let then

$$\mu_j(t) = -\int_{-\infty}^{\zeta_t^0}\int_{-\infty}^{\infty}\phi_j(F_{[j]}(y))\dot{f}^0_{[0,j]}(x,y)dF_{[0,j]}(x,y) + f_{[0]}(\zeta_t^0)\bar{\phi}_{jt} , \quad (5.3)$$

for $0 \le t \le 1$ and $j = 1,\ldots,p$;

$$\mu_0(t) = \int_0^t \phi_0(u)\psi_0(u)du + (1-t)\bar{\phi}_{0t}\bar{\psi}_{0t} , \quad 0 \le t \le 1 , \quad (5.4)$$

where

$$\psi_0(u) = -f'_{[0]}(F^{-1}_{[0]}(u))/f_{[0]}(F^{-1}_{[0]}(u)) , \quad 0 < u < 1 , \quad (5.5)$$

and $\bar{\psi}_{0t}$ is defined by (4.6) for ϕ_0 being replaced by ψ_0. Further, note that $\nu^{00}(t)$ is a continuous and non-increasing function of $t \in (0,1]$. Defining τ as in (4.1), we let

$$\pi_\tau(t) = \inf\{s: \nu^{00}(\tau)/\nu^{00}(s) \le t\} , \quad 0 < t \le 1 , \quad (5.6)$$

Finally, for every ε: $0 < \varepsilon < 1$, we define ${}_\varepsilon\mu^*_\tau = \{\mu^*_\tau(t);\ \varepsilon \le t \le 1\}$ by letting

$$\mu^*_\tau(t) = \theta\{\nu^{00}(\pi_\tau(t))\}^{-\frac{1}{2}}\sum_{j=0}^{p}\nu^{0j}(\pi_\tau(t))\mu_j(\pi_\tau(t)) , \quad \varepsilon \le t \le 1 . \quad (5.7)$$

Theorem 5.1. *Under Assumptions* (I), (II), (III), (IV) *of Section* 4, $\{K_n\}$ *in* (5.1) *and* (5.2) *for every* (fixed) $m(\ge 1)$ *and* $(0 <)t_1 < \ldots < t_m(\le 1)$,

$$\{W^*_{n,r}(t_1),\ldots,W^*_{n,r}(t_m)\} \xrightarrow{\mathcal{D}} \{W^*(t_1) + \mu^*_\tau(t_1),\ldots,W^*(t_m) + \mu^*_\tau(t_m)\} , \quad (5.8)$$

where $W^*_{n,r}$ *and* W^* *are defined by* (4.43)-(4.45).

Proof. Here also, we prove (5.8) for $m = 1$ only; a similar proof holds for any $m \ge 1$. Recall that for every j: $0 \le j \le p$ and $k(\le n)$,

$$C_n^{-1}T_{n,k}^{(j)} = \sum_{i=1}^n c_{ni}^*\{\phi_j^{(n)}(F_{n[j]}(X_{ji}))c(Z_{n,k}^0-X_{0i}) + a_{n,j}^*(k)c(X_{0i}-Z_{n,k}^0-)\} . \quad (5.9)$$

Define then for every $0 < t \le 1$ and $0 \le j \le p$,

$$h_{n,j}^*(t) = \sum_{i=1}^n c_{ni}^*\{\phi_j(F_{[j]}(X_{ji}))c(\zeta_t^0-X_{0i}) + \bar{\phi}_{jt}c(X_{0i}-\zeta_t^0-)\} . \quad (5.10)$$

As in Theorems 4.1 and 4.3, under Assumption II, we use the polynomial decomposition of ϕ_m, $0 \le m \le p$, and then it can be shown by some routine steps that whenever $k/n \to t \in [0,1]$, as $n \to \infty$,

$$|C_n^{-1}T_{n,k}^{(j)} - h_{n,j}^*(t)| \overset{p}{\to} 0 , \quad 0 \le j \le p , \quad \text{under } H_0 . \quad (5.11)$$

We denote the joint distribution of $\{\underset{\sim}{X}_1^*,\ldots,\underset{\sim}{X}_n^*\}$ under K_n (and H_0) by $P_{n,n}$ (and $P_{n,0}$, respectively). Then, from the results of Chapter IV of Hájek and Šidák (1967), it follows that under the hypothesis of Theorem 5.1, $\{P_{n,n}\}$ is contigious to $\{P_{n,0}\}$, and hence, by (5.11),

$$|C_n^{-1}T_{n,k}^{(j)} - h_{n,j}^*(t)| \overset{p}{\to} 0 , \quad 0 \le j \le p , \quad \text{under } \{K_n\} \text{ as well} . \quad (5.12)$$

On the other hand, for any $\underset{\sim}{\lambda} \ne \underset{\sim}{0}$, $\sum_{j=0}^p \lambda_j h_{n,j}^* = \sum_{i=1}^n c_{ni}^*\{c(\zeta_t^0-X_{0i})\sum_{j=0}^p \lambda_j\phi_j(F_{[j]}(X_{ji}) + c(X_{0i}-\zeta_t^0-)\sum_{j=0}^p\lambda_j\bar{\phi}_{jt}\}$ involves a linear combination of independent rv's, and hence, by the central limit theorem, under $\{K_n\}$, $\sum_{j=0}^p \lambda_j h_{n,j}^*$ is asymptotically normal with mean $\theta\sum_{j=0}^p\lambda_j\mu_j(t)$ and variance $\underset{\sim}{\lambda}'\underset{\sim}{\nu}(t)\underset{\sim}{\lambda}$. Further, by (4.14) and the contiguity of $\{P_{n,n}\}$ to $\{P_{n,0}\}$, we claim that (4.14) also holds under $\{K_n\}$. As such, by (4.12), for every $0 < t \le 1$ and $k/n \to t$,

$$v_{n,k}^{0j}/(v_{n,k}^{00})^{\frac{1}{2}} \overset{p}{\to} \{\nu^{00}(t)\}^{-\frac{1}{2}} \cdot \nu^{0j}(t) , \quad 0 \le j \le p , \quad \text{under } \{K_n\} . \quad (5.13)$$

Hence, by the Slutzky theorem, under $\{K_n\}$, $k/n \to t \Rightarrow$

$$C_n^{-1}(v_{n,k}^{00})^{-\frac{1}{2}}\sum_{j=0}^{p} v_{n,k}^{0j} T_{n,k}^{(j)} \xrightarrow{\mathcal{D}} \{\nu^{00}(t)\}^{-\frac{1}{2}}\sum_{j=0}^{p}\nu^{0j}(t)h_{n,j}^{*}(t) \ , \tag{5.14}$$

where the right hand side of (5.14) is asymptotically normal (under $\{K_n\}$) with mean $\theta\{\nu^{00}(t)\}^{-\frac{1}{2}}\sum_{j=0}^{p}\nu^{0j}(t)\mu_j(t)$ and variance 1. The desired result then follows by using the definitions in (4.43)-(4.44) and (5.6)-(5.7). Q.E.D.

Note that (4.56) insures the tightness of ${}_{\varepsilon}W_{n,r}^{*}$ under H_0, and hence, proceeding as in the proof of Theorem 2 of Sen (1976a), we conclude that by the contiguity of $\{P_{n,n}\}$ to $\{P_{n,0}\}$, (4.56) and (5.8), ${}_{\varepsilon}W_{n,r}^{*}$ remains tight under $\{K_n\}$ as well. Hence, we have the following.

Theorem 5.2. *Under the hypothesis of Theorem* 5.1, *for every* $0 < \varepsilon < 1$,

$${}_{\varepsilon}W_{n,r}^{*} \xrightarrow{\mathcal{D}} {}_{\varepsilon}W^{*} + {}_{\varepsilon}\mu_{\tau}^{*} \ , \quad \textit{in the } J_1\textit{-topology on } D[\varepsilon,1] \ . \tag{5.15}$$

Defining τ by (4.1), for every $0 < \varepsilon < 1$, we let

$$t_0 = \inf\{t: \nu_{00}(\tau)\nu^{00}(t_0) \le \varepsilon^{-1}\} \ , \quad t^0 = \nu^{00}(\tau)/\nu^{00}(t_0) \ . \tag{5.16}$$

Then, $t^0 = \nu^{00}(\tau)/\nu^{00}(t_0) \ge 1/[\nu_{00}(\tau)\nu^{00}(t_0)] \ge \varepsilon$. As such, by (4.58), (4.59) and (5.16), we claim that under $\{K_n\}$,

$$n^{-1}k_0 \xrightarrow{p} t_0 \quad \text{and} \quad v_{n,r}^{00}/v_{n,k_0}^{00} \xrightarrow{p} t^0 \ , \quad \text{as} \quad n \to \infty \ , \tag{5.17}$$

and hence, by Theorem 5.2 and (4.58)-(4.55), we conclude that

$$\begin{aligned} &\lim_{n\to\infty} P\left\{\max_{k_0\le k\le r} \xi_{n,k} > \lambda_{\alpha}^{+}(\varepsilon)\,|\,K_n\right\} \\ &= P\{W^{*}(t) > \lambda_{\alpha}^{+}(\varepsilon) - \mu_{\tau}^{*}(t) \ , \quad \text{for some} \quad t^0 \le t \le 1\} \qquad (5.18) \\ &= P\{W(t) > t^{\frac{1}{2}}[\lambda_{\alpha}^{+}(\varepsilon) - \mu_{\tau}^{*}(t)] \ , \quad \text{for some} \quad t^0 \le t \le 1\} \ ; \end{aligned}$$

$$\lim_{n\to\infty} P\left\{\max_{k_0 \le k \le r} |\xi_{n,k}| > \lambda_\alpha \; \varepsilon | K_n\right\}$$

$$= P\{W(t) \notin t^{\frac{1}{2}}[\pm \lambda_\alpha(\varepsilon) - \mu_\tau^*(t)] \;, \quad \text{for some} \quad t;\ t^0 \le t \le 1\} \;, \tag{5.19}$$

where $W = \{W(t),\ t \ge 0\}$ is a standard Wiener-process. Thus, in either case, the asymptotic power function is expressible in terms of the boundary crossing probabilities of a standard Wiener process (on $[t^0,1]$), where the boundaries are generally non-linear.

6. SOME CONCLUDING REMARKS

Note that for $\tau = t = 1$, $\mu_1^*(1) = \theta[\nu^{00}(1)]^{\frac{1}{2}}\mu_0(1)$, as all the $\mu_j(1)$, $1 \le j \le p$ are equal to 0. Similarly, for the ANOVA test (based on $T_n^{(0)}$ alone), we have the asymptotic mean of $C_n^{-1}[v_{n,00}^{(n)}]^{-\frac{1}{2}}T_n^{(0)}$ (under $\{K_n\}$) equal to $\theta[\nu_{00}(1)]^{-\frac{1}{2}}\mu_0(1)$. Hence, for the complete sample (non-sequential) case, the asymptotic relative efficiency (A.R.E.) of the ANOCA test with respect to the ANOVA test (based on the same score function) is given by

$$e_1 = \nu^{00}(1)\nu_{00}(1) \quad (\ge 1) \;, \tag{6.1}$$

where the quality sign holds only when $\nu_{0j}(1) = 0$, $\forall\ 1 \le j \le p$. In the context of PCS, we note that for $t < 1$ (even when $\tau = 1$), $\mu_\tau^*(t)$ is rather a complicated function (of τ, t, $\phi_0,\ldots,\phi_p$ and $f_{[0,j]}$, $1 \le j \le p$). If, we do not have any concomitant variate, we may develop a parallel PCS ANOVA test, where in (3.9), we shall take

$$\xi_{n,k} = \begin{cases} (v_{n,00}^{(k)})^{-\frac{1}{2}} C_n^{-1} T_{n,k}^{(0)} \;, & k \ge k_0 \;, \\ 0 \;, & \text{otherwise} \end{cases} \tag{6.2}$$

where $v_{n,00}^{(k)} > 0$, $\forall\ k_0 \le k \le n$. In that event, we need not use the inequality (4.58), so that in (4.61)-(4.62), the $\le$ sign may be replaced by an $=$ sign. Also, in (5.18)-(5.19), we will have then $t^0 = \varepsilon$ while for $\mu_\tau^*(t)$, given by

$$\mu_\tau^0(t) = \theta\{\nu_{00}(\tau)\}^{-\frac{1}{2}}\mu_0(\pi_\tau^0(t)) \ , \quad \varepsilon \le t \le 1 \ ; \tag{6.3}$$

$$\pi_\tau^0(t) = \inf\{s: \nu_{00}(s)/\nu_{00}(\tau) \ge t\} \ , \quad 0 \le t \le 1 \ , \tag{6.4}$$

In general, neither $\pi_\theta(t)$ and $\pi_\tau^0(t)$ are equal (for all $t \le 1$), nor $\mu_\tau^*(t)$ and $\mu_\tau^0(t)$ are either dominated by the other. Hence, unlike (6.1), in general, it is difficult to conclude whether the ANOCA always performs (asymptotically) better than the ANOVA rank test.

Chatterjee and Sen (1973) have studied the Bahadur-ARE of the PCS ANOVA rank test with respect to the single-point censoring test and have shown that under very general conditions, the PCS test performs better. In the current setup, such a study can be made provided we are able to show that for (4.61) or (4.62), as $\lambda \to \infty$, $-\log P\{\ldots\} = c\lambda^2 + o(\lambda^2)$, where $c > 0$; this remains an open problem.

Finally, batch-arrival models or staggering entry plans are often more flexible to suit practical problems. In such a case, as in Sen (1976a), one needs a two-dimensional time-parameter stochastic process to formulate the ANOCA tests. This is possible — though the formulation needs much more complications, and will not be done here. Some simulation studies for the asumptotic power of the proposed tests are intended to be made and published in a separate communication.

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