AGRICULTURAL VARIATION AND SECOND-ORDER STATIONARY RANDOM MEASURES

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SUMMARY

Experimental measurements associated with n-dimensional regions or "plots" are regarded as observations on random variables indexed by the bounded Borel subsets of $\mathbb{R}^m$; these random variables having finite second moments and satisfying an obvious finite additivity property. Certain stationarity and continuity assumptions about the first two moments allow spectral representations to be derived which are analogous to those already familiar from the theory of generalized stationary random processes. The theory is used to justify four models of two-dimensional spatial variation which are fitted to some well-known uniformity trial data.

Some key words: Spectral measure; Transformably bounded measure; Isotropy.

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1. INTRODUCTION

The success of split-plot and incomplete block types of experimental design in increasing the precision of estimates of various "treatment" effects in agricultural experiments can frequently be explained by the observation that the yields of neighbouring plots are more highly correlated than those further apart. Fairly extensive data are available substantiating this observation, and also the accompanying observation that long narrow plots tend to be less variable than square plots of the same area (see Matérn (1960) § 6.12, and the references quoted therein).

The variation in the plot yields of an agricultural experiment is an example of two-dimensional spatial variation. The aim of this paper is to present some models of such variation, and to fit them to the uniformity trial data of H. Fairfield Smith (1938). To this end, we draw on some theory concerning second-order stationary random measures, the basic ideas for which are discussed by Daley (1971) in the point process context, and more generally by Vere-Jones (1974) and Yaglom (1958).

2. FORMULATION

Let N denote a reasonable class of subsets of $\mathbb{R}^n$, and for each $A \in N$, let $w(A)$ denote some real or complex measurement associated with A. In $\mathbb{R}^2$ for example, N might denote the class of all rectangles, while $w(A)$ could denote this season's wheat yield from the plot determined by A. With this example in mind the

following two assumptions about the function w seem plausible.

Assumption 2′. w is finitely additive, i.e., $w(A \cup B) = w(A) + w(B)$ for any two disjoint sets A and B.

Assumption 2″. $w(A)$ is small whenever A is small. Somewhat more precisely, if $A_1 \supset A_2 \supset \ldots$ is a non-increasing sequence of sets in N whose Lebesgue measures approach zero, then $w(A_j)$ approaches zero, in some sense, as $j \to \infty$.

The first of these assumptions is clearly easier to live with if we require N to be closed with respect to finite unions. Without further ado then, we introduce out principal assumptions.

Assumption 1. N consists of the bounded Borel subsets of $\mathbb{R}^n$.

Assumption 2. The collection $\{w(A), A \in N\}$ is a realization of a stochastic process $\{W(A), A \in N\}$ for which

(2a) $E(|W(A)|^2) < \infty$ for all $A \in N$ (second-order property);

(2b) if for any $y \in \mathbb{R}^n$, $A \in N$, we put $A+y = \{x+y, x \in A\}$, then we have $E(W(A+y)) = E(W(A))$ and $E(W(A+y)\overline{W(B+y)}) = E(W(A)\overline{W(B)})$ for all $y \in \mathbb{R}^n$, and $A, B \in N$ (stationarity property);

(2c′) $W(A \cup B) = W(A) + W(B)$ whenever A and B are disjoint (additivity property); and

(2c″) if $A_1 \supset A_2 \supset \ldots$ is a non-increasing sequence of sets in N whose Lebesgue measures approach zero, then $\lim_{j\to\infty} E(|W(A_j)|^2) = 0$ (continuity property).

Clearly, (2c′) and (2c″) are equivalent to the single condition, labelled (2c), that if $A_1, A_2, \ldots$ are disjoint members of N whose

union is also in N, then

$$W\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} W(A_j).$$

As usual, the equalities expressed in (2c') and (2c) are between equivalence classes of square integrable random variables.

2.1 Definition. A stochastic process $\{W(A), A \in N\}$ satisfying (2a), (2b), and (2c) will be referred to as a second-order stationary random measure.

It would be more in keeping with the terminology of Vere-Jones (1974) to call $\{W(A), A \in N\}$ a second-order stationary random complex measure, but we shall not do this. In fact, from now on, we shall simply use the term "random measure" to indicate a process of the above type. Similarly, the word "measure" will embrace only the complex Borel measures on $\mathbb{R}^n$, i.e., complex measures on the Borel subsets of $\mathbb{R}^n$ which take finite values on N.

Regarding the first assumption, the two dimensional version of such an N would appear to contain all the subsets which could have any agricultural interest, eg., rectangles, circles, other convex sets and their unions. The second assumption indicates that our study of spatial variation will be a study of the first and second moments of random measures, with the implication that, for inference purposes, we would be making the traditional assumption of joint normality of the random variables $W(A)$, $A \in N$.

The relevance of random measures to agricultural experimentation may be seen as follows: Let $y_1,\ldots,y_k$ be yields from k (disjoint) experimental plots $A_1,\ldots,A_k$. Then $y_1,\ldots,y_k$ perhaps edited and

transformed, may be regarded as observations on jointly normally distributed random variables $Y_1,\ldots,Y_k$ of the form

$$Y_j = f_j(\beta_1,\ldots,\beta_p) + W(A_j) ,$$

$j=1,\ldots,k$, where $f_1,\ldots,f_k$ are known functions of unknown parameters $\beta_1,\ldots,\beta_p$ which are related to the various treatments applied to the plots, and $\{W(A),\ A \in N\}$ is a random measure. Without loss of generality, it may be supposed that $E(W(A)) = 0$ for all $A \in N$. The suggestion is that if such a model is confirmed to be appropriate for a series of experiments of the same type, then this would allow a better choice of design for the estimation of the main parameters of interest, $\beta_1,\ldots,\beta_p$, in the experiment of current interest.

3. THE FIRST TWO MOMENTS OF RANDOM MEASURES

Let $\{W(A),\ A \in N\}$ be a random measure, and for $A,\ B \in N$, put $\langle A\rangle = E(W(A))$ and $\langle A,B\rangle = E(W(A)\overline{W(B)})$.

3.1 Theorem There exists a complex constant c such that $\langle A\rangle = c\lambda_n(A)$ for all $A \in N$, where λ_n denotes n-dimensional Lebesgue measure.

The proof of this theorem is an obvious simplification of the proof of the following theorem. In fact, both results can be established by arguments similar to those used by Yaglom (1958) in the proof of the theorem on p. 91 for the case $n = 1$ (for both our and Yaglom's interpretation of n). Before stating this next theorem however, some further notation is required.

For any $A \in N$ let $[A]$ denote its indicator function, and let $[A]\hat{}$ denote the corresponding Fourier transform, i.e.,

$$[A](x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$[A]\hat{}\,(u) = \int_A e^{iu.x}\, dx$$

where, as usual, $u.x = \sum_{j=1}^{n} u_j x_j$ for $u = (u_1,\ldots,u_n)$, $x = (x_1,\ldots,x_n)$. Note that for any $y \in \mathbb{R}^n$.

$$[A+y](x) = [A](x-y)$$

for all $x \in \mathbb{R}^n$. Finally, let R denote the set of all non-negative measures ν such that

(i) $\int |[A]\hat{}\,|^2\, d\nu < \infty$ for all $A \in N$, and

(ii) if $A_1 \supset A_2 \supset \cdots$ is any non-increasing sequence of sets in N such that $\lim_{j\to\infty} \lambda_n(A_j) = 0$ then $\lim_{j\to\infty} \int |[A_j]\hat{}\,|^2 d\nu = 0$.

3.2 Theorem There exists $\nu \in R$ with $\nu(\{0\}) \geq |c|^2$ such that

$$\langle A,B\rangle = \int [A]\hat{}\,(u)\,[B]\hat{}\,(-u)\, d\nu(u) \qquad (3.1)$$

for all $A,B \in N$.

Proof. The main step is establishing the identity

$$\iint [A](x)\,[A](y)\, \langle B+x+t,\, B+y\rangle\, dx dy = \iint [B](x)\,[B](y)\, \langle A+x+t, A+y\rangle\, dx\, dy \qquad (3.2)$$

for all $A, B \in N$ and $t \in \mathbb{R}^n$. To that end, for each integer $m \geq 1$, let $C_{mj}, j = 0,1,2,\ldots,$ denote a listing of the mutually exclusive and exhaustive subsets of $\mathbb{R}^n$ of the form

$$C_{mj} = \{(x_1,\ldots,x_n),\ 3^{-m}(p_{mjk} - \tfrac{1}{2}) < x_k \le 3^{-m}(p_{mjk} + \tfrac{1}{2}),\ k = 1,\ldots,n\}$$

where the p_{mjk} are suitably determined integers, and let x_{mj} be the mid-point of C_{mj}. Using the fact that

$$[C_{mj}](x) = [C_{mr}](x - x_{mj} + x_{mr})$$

for all $x \in \mathbb{R}^n$ it may be shown that (3.2) holds when both A and B are finite unions of the sets C_{mj}, i.e.,

$$[A](x) = \sum_j a_{mj}[C_{mj}](x) \quad \text{and} \quad [B](x) = \sum_k b_{mk}[C_{mk}](x)$$

where the a_{mj}'s, b_{mk}'s only take the values 0 or 1, and only a finite number of them are non-zero. To extend the identity to all $A,B \in N$ we use (2c'') in conjunction with the fact that for each $A \in N$ there exists an m such that

$$[A](x) = \sum_j a_{mj}[C_{mj}](x) + [A_m^*](x)$$

where $\lambda_n(A_m^*)$ is arbitrarily small.

Next it may be shown that for each A, $\langle A+y, A\rangle$ is a continuous positive-definite function of y, and so, by Bochner's theorem, there exists a finite non-negative measure ν_A such that

$$\langle A+y, A\rangle = \int e^{iu.y}\, d\nu_A(u).$$

This together with (2b) and Fubini's theorem allows (3.2) to be re-written as

$$\int e^{iu.t}|[A]\hat{}(u)|^2 d\nu_B(u) = \int e^{iu.t}|[B]\hat{}(u)|^2 d\nu_A(u).$$

The measure ν we seek is then uniquely determined by the relationship

$$\nu(D) = \int_D |[A]\hat{}(u)|^{-2} d\nu_A(u)$$

for all $D \in N$, where A may be chosen so that $[A]^{\wedge}(u) \neq 0$ for $u \in D$. If we now suppose B to be a "square" centered at the origin, then putting $t = 0$ in the right-hand side of (3.2) we get the identity

$$\beta^{-2}\iint [B](x)[B](y) < A+x, A+y > dx\, dy = \beta^{-2}\int |[A]^{\wedge}|^2 \, |[B]^{\wedge}|^2 \, d\nu$$

where $\beta = \lambda_n(B)$. Letting $\beta \to 0$ yields

$$<A,A> = \int |[A]^{\wedge}|^2 \, d\nu$$

which, by the usual inner-product or variance-covariance manipulations, implies

$$<A,B> = \int [A]^{\wedge}(u)\,[B]^{\wedge}(-u)\, d\nu(u)$$

for all $A, B \in N$. That $\nu \in R$ follows directly from (2a) and (2c"), and that $\nu(\{0\}) \geq |c|^2$ is a consequence of letting $\beta \to \infty$ in the inequality

$$\beta^{-2}<B,B> \geq \beta^{-2}|<B>|^2 \quad ,$$

B and β being as introduced above.

3.3 Definition The measure $\nu \in R$ constructed above will be called the spectral measure of the random measure $\{W(A), A \in N\}$.

4. CHARACTERIZING SPECTRAL MEASURES

The results of the previous section are of course completely analogous to the spectral representations in the Itô-Gel'fand theory of generalized stationary random processes (see Itô (1953), Gel'fand & Vilenkin (1964), and Yaglom (1957)). In this theory the class of spectral measures consists of the so-called tempered measures, i.e.,

those measures ν satisfying

$$\int (1+|u|^2)^{-k}\, d\nu(u) \quad < \quad \infty$$

for some $k \geq 0$, where $|u|^2 = u \cdot u$. The author is not aware of any correspondingly simple necessary and sufficient characterization of the members of R. However, some sufficient characterizations can be formulated from the following considerations. Let N denote the space of complex-valued functions of the form $f(x) = \sum_{j=1}^{\infty} a_j [A_j](x)$

where $A_1, A_2, \ldots$, are pairwise disjoint members of N, and $a_1, a_2, \ldots$ are complex numbers, only a finite number of which are non-zero. The Fourier transform of $f \in N$ will be denoted by $\hat{f}$, and for $f,g \in N$, $f*g^*$ will stand for the function

$$f*g^*(x) = \int f(y)\overline{g(y-x)}\, dy.$$

4.1 Definition (i) A measure μ is said to be positive-definite if

$$\int f*f^*\, d\mu \geq 0$$

for all $f \in N$.

(ii) A non-negative measure ν is said to be transformable if there exists a measure μ such that

$$\int |[A]^\wedge|^2\, d\nu = \int [A]*[A]^*\, d\mu \qquad (4.1)$$

for all $A \in N$.

(iii) A non-negative measure ν is said to be transformably bounded if there exists a transformable measure ν_1 such that

$$\int |[A]^\wedge|^2\, d\nu \leq \int |[A]^\wedge|^2\, d\nu_1$$

for all $A \in N$

We use R_t to denote the set of non-negative measures which are

transformable, while R_{tb} will denote those that are transformably bounded.

It is easy to see that if $\nu \in R_t$ then the measure μ in (4.1) is positive-definite. Conversely, if μ is positive-definite then Theorem 3.2 can be invoked to show that there exists $\nu \in R_t$ such that (4.1) is true. For present purposes however, the most important observation is that

$$R_t \subset R_{tb} \subset R,$$

i.e., if a non-negative measure ν is transformable, or failing that, is transformably bounded, then ν can be the spectral measure for some random measure. The first of these inclusions is strict. To see this, consider the measure ν given by

$$\nu(D) = \int_D h(u)\, du \quad ,$$

$D \in N$, where h is a bounded continuous non-negative function which is not the Fourier transform of any finite measure. Then $\nu \notin R_t$ but is transformably bounded by

$$\nu_1 = \sup_u h(u)\, \lambda_n.$$

It is an open question whether or not R_{tb} is a proper subset of R_t.

The following theorem characterizes R_t:

4.2 Theorem A necessary and sufficient condition for the spectral measure of a random measure $\{W(A), A \in N\}$ to be transformable is that for every $A \in N$ there exists a positive constant M_A such that for any selection $A_1, A_2, \ldots$ of pairwise disjoint members of N whose union is A,

$$\sum_j \sum_k |\langle A_j, A_k \rangle| \leq M_A \quad .$$

Proof. Necessity follows by observing that we can take

$$M_A = \int [A]^* [A]^* \, d|\mu| < \infty \quad .$$

For sufficiency, let C_{mj} and x_{mj} be as in the proof of Theorem 3.2, and for $A \in N$ let

$$a_{mj} = \begin{cases} 1 & \text{if } C_{mj} \subset A, \\ 0 & \text{otherwise} \end{cases} .$$

Consider the function $h_{A,m}$ defined by

$$h_{A,m}(t,u) = \sum_j a_{mj} e^{-it.x_{mj}} [C_{mj}]\hat{}(u).$$

Then (i) $\lim_{m\to\infty} h_{A,m}(t,u) = [A]\hat{}(u-t)$;

(ii) $\int |h_{A,m}(t,u)|^2 \, d\nu(u) \leq M_A$; and

(iii) for any function f integrable with respect to λ_n we have

$$\left|\int f(t)\left\{\int |h_{A,m}(t,u)|^2 d\nu(u)\right\}dt\right| \leq M_A \sup_u |\hat{f}(u)| .$$

It follows then that

$$h_A(t) = \int |[A]\hat{}(u-t)|^2 \, d\nu(u)$$

is a bounded continuous non-negative function satisfying

$$\left|\int f(t) h_A(t) dt\right| \leq M_A \sup_u |\hat{f}(u)|$$

for any f integrable with respect to λ_n. Thus (see Eberlein (1955), Theorem 2) there exists a bounded measure μ_A such that

$$h_A(t) = \int e^{-ix.t} \, d\mu_A(x) \quad .$$

The identity

$$\int |[B]\hat{}(u+t)|^2 h_A(t)dt = \int |[A]\hat{}(u+t)|^2 h_B(t)\, dt$$

may then be employed in the same manner as (3.2) to show that the measure μ constructed according to

$$\mu(D) = (2\pi)^n \int_D \{[A]*[A]^*(x)\}^{-1} d\mu_A(x)$$

for all $D \in N$, satisfies (4.1) ..

In many agricultural situations it would be reasonable to suppose that $\langle A,B\rangle \geq 0$ for all $A, B \in N$. Under this assumption the condition of the previous theorem is met with

$$M_A = \langle A,A\rangle \quad ,$$

so that $\nu \in R_t$. In addition, the positive-definite measure μ corresponding to ν must itself be non-negative. The role of such measures - called p.p.d. measures - in the theory of second-order stationary point processes has been pointed out by Vere-Jones (1974). Note also that if ν is bounded, then it is transformable. It is clear then that although R_t does not exhaust R, its two-dimensional version is a source of many useful models of agricultural variation.

The arguments used in proving Theorems 3.2 and 4.2 are tailored versions of arguments well-known in abstract harmonic analysis, e.g., see the second chapter of Argabright and de Lamadrid (1974).

5. ISOTROPIC RANDOM MEASURES

Let G denote the group of all orthogonal transformations

(rotations and reflections) in $\mathbb{R}^n$, and for $g \in G$, $A \in \mathcal{N}$, let $gA = \{gx, x \in A\}$.

5.1 Definition A random measure $\{W(A), A \in \mathcal{N}\}$ is said to be isotropic if $\langle gA \rangle = \langle A \rangle$ and $\langle gA, gB \rangle = \langle A, B \rangle$ for all $A, B \in \mathcal{N}$, $g \in G$.

Isotropy is an appealing assumption in modelling instances of natural variation, though perhaps not so appealing in the case of crops planted according to a rectangular grid pattern. Even here it offers a simplification which should be considered before looking at models which are not orientation-free.

Since the relationship $\langle gA \rangle = \langle A \rangle$ is always true for any random measure, the condition of isotropy relates only to the spectral measure of such a process. The following characterization of isotropic spectral measures, as well as its justification, is identical with the corresponding result for generalized stationary random processes (See Yaglom (1957), Theorem 11, or Gel'fand & Vilenkin (1964), p. 294). To that end let J_r denote the Bessel function of the first kind of order r, Γ denote the gamma function, and define the function $\Omega_n: \mathbb{R} \to \mathbb{R}$ by

$$\Omega_n(t) = (2/t)^{(n-2)/2} \Gamma(n/2) J_{(n-2)/2}(t) \quad .$$

5.2 Theorem (Yaglom) A necessary and sufficient condition for a spectral measure ν to be isotropic is that $\nu(D) = \nu(gD)$ for all $g \in G$, $D \in \mathcal{N}$.

In this case we have

$$\langle A, B \rangle = \int_0^\infty \{\iint [A](x)[B](y)\, \Omega_n(s|x-y|)dx\, dy\}\, d\eta(s) \qquad (5.1)$$

for all $A, B \in \mathcal{N}$ where $d\eta(s)$ denotes the ν-measure of the spherical

shell $s \le |u| < s+ds$ in $\mathbb{R}^n$.

6. SOME TWO-DIMENSIONAL MODELS

In the remaining sections our attention is confined solely to $\mathbb{R}^2$. To avoid subscripts we shall denote typical points in $\mathbb{R}^2$ by $x = (y,z)$ and $u = (v,w)$ where y, z, v, w, as well as $r = (y^2+z^2)^{1/2}$ and $s = (v^2 + w^2)^{1/2}$ are all real. Since only variances and covariances are of interest, the quantity c of Theorem 3.1 will be taken to be zero. The versions of $\nu, \eta,$ and μ in the models considered - - each ν is transformable - are all absolutely continuous with respect to Lebesgue measure of the appropriate dimension, so to save notation, the same symbols will be used to represent the respective derivatives with respect to Lebesgue measure. A positive parameter σ^2, representing a scaling factor or variance component, appears in all models.

The model $M\alpha$ $\eta(s) = \sigma^2(2\pi)^{-1} s e^{-\alpha s}$, $\alpha \ge 0$,

with

$$\mu(y,z) = \sigma^2(2\pi)^{-1}\alpha(\alpha^2 + r^2)^{-3/2} .$$

The model $M\beta$ $\eta(s) = \sigma^2(2\pi)^{-1} s\beta \int_0^\infty e^{-\beta t} t^3 (t^2+s^2)^{-3/2} dt$, $\beta \ge 0$,

with

$$\mu(y,z) = \sigma^2\pi^{-1}\beta(\beta + r)^{-3} .$$

The model $M\gamma$ $\nu(v,w) = \{\sigma \sin(\gamma\pi/2)\ \Gamma(\gamma+1)/(2\pi)\}^2 |vw|^{1-\gamma}$ $1 \le \gamma < 2$,

with

$$\mu(y,z) = \{\sigma\gamma\ (\gamma-1)/2\}^2 |yz|^{\gamma-2} .$$

The model $M\delta$ $\quad \eta(s) = \sigma^2\, \Gamma(2-\delta)\{\pi\Gamma(\delta)\}^{-1}(s/2)^{2\delta-1}$, $0 < \delta \leq 1$, with

$$\mu(y,z) = \sigma^2(1-\delta)/(\pi r^{2\delta}) \quad .$$

For $M\alpha$ and $M\beta$ the compatibility between the expressions for η and μ follow from (3.1), (4.1),(5.1) and the identities (1), §8.1, and (7), §8.2 of Bateman (1954). The same thing for $M\gamma$ and $M\delta$ may be established by using the identity (iii), Theorem 6·2·5, of Okikiolu (1971).

Given $a,b > 0$ let $R(a,b)$ stand for the rectangle $-a/2 < y \leq a/2$, $-b/2 < z \leq b/2$, and write $V(a,b;-)$ for $\langle R(a,b), R(a,b)\rangle$ where the dash stands for one of the parameters α,β,γ, or δ. In the method used for fitting the above models it is $V(a,b;-)$ which plays the basic role, rather than the spectral measure ν, or some related measure such as η or μ. Rather lengthy integrations yield the following formulae:

$$V(a,b;\alpha) = (2\sigma^2/\pi)[V_\alpha(a,b) + V_\alpha(b,a) - \alpha\,(\alpha^2 + a^2 + b^2)^{1/2} - \alpha^2 + ab\,\tan^{-1}\{(ab)/(\alpha^4 + \alpha^2 a^2 + \alpha^2 b^2)^{1/2}\}] \quad (6.1)$$

where

$$V_\alpha(a,b) = \alpha\, a \sinh^{-1}\{a/(\alpha^2 + b^2)^{1/2}\} - \alpha\, a \sinh^{-1}(a/\alpha) + \alpha(\alpha^2 + a^2)^{1/2} \quad ;$$

$$V(a,b;\beta) = (4\sigma^2/\pi)[V_\beta(a,b) + V_\beta(b,a) - (3\beta/2)(a^2 + b^2)^{1/2} + (3\beta^2/2)\log\{1 + (a^2 + b^2)^{1/2}/\beta\}] \quad (6.2)$$

where

$$V_\beta(a,b) = \{b(a^2 - 2\beta^2)/(a^2 - \beta^2)^{1/2}\}\tan^{-1}[\{a^2 + b^2)^{1/2} - a\}(a^2 - \beta^2)^{1/2}/(ab + \beta b)]$$

$$+ 3\beta b/2 + \beta b \log[\{b + (a^2+b^2)^{1/2} - a\}/\{a+b - (a^2+b^2)^{1/2}\}]$$

$$-\beta\{(3\beta/2) + b\} \log \{1 + (b/\beta)\} \quad ;$$

$$V(a,b;\gamma) = \sigma^2(ab)^{\gamma} \quad ; \tag{6.3}$$

and

$$V(a,b;\delta) = (\sigma^2/\pi)[V_\delta(a,b) + V_\delta(b,a) + 2(a^{4-2\delta} + b^{4-2\delta})/(3-2\delta)$$

$$+ \{(a^2+b^2)^{2-\delta} - a^{4-2\delta} - b^{4-2\delta}\}/(2 - \delta)] \tag{6.4}$$

where

$$3\, V_\delta(a,b) = -\{2\, b^4 a^{3-2\delta}/(a^2+b^2)^{3/2}\}\, {}_2F_1\left(5/2 - \delta,\ 3/2; 5/2; b^2/(a^2 + b^2)\right)$$

and ${}_2F_1$ stands for the Gauss hypergeometric series (see p. 429 of Bateman (1954)) .

Concerning the appropriateness of such models, an initial problem is that of deciding whether a given one of them offers more than the simplest model of two-dimensional variation, namely

$$\langle R(a,b), R(a,b)\rangle = \sigma^2 a\, b. \tag{6.5}$$

We observe that this model is a particular case of each of the models considered, being realized with $\alpha = 0, \beta = 0, \gamma = 1$, and $\delta = 1$ respectively. Thus, in each case the estimated value of α, β, γ, or δ for a given set of data provides a measure of departure from this basic model.

Apart from allowing some measure of departure from the situation of orthogonal plot yields as represented by (6.5), there is little to justify the models $M\alpha$ and $M\beta$. Their inclusion here is due to their being isotropic and the fact that closed expressions for $V(a,b;-)$ can be written down. From (6.3) the model $M\gamma$ is seen to correspond to

the empirical law of agricultural variation devised by H. Fairfield Smith (1938). Whittle (1956) has shown that when plots are small or large and of fixed (rectangular) shape, $\langle R(a,b), R(a,b)\rangle$ varies in the limit as $(ab)^{\gamma}$, $1 \leq \gamma < 2$, for a large class of models. On these grounds M_γ may be expected to fit observed data fairly well, particularly when these data show that $\langle R(a,b), R(a,b)\rangle$ depends only on plot size and not on plot shape. For the purpose of comparison, M_δ has been included as the obvious isotropic equivalent of M_γ. It too has the property that $\langle R(a,b), R(a,b)\rangle$ varies as a power of $a\,b$ for a given plot shape, namely $(ab)^{2-\delta}$. However, for ab fixed, $V(a,b;\delta)$ has a local maximum when $a = b$; which suggests that M_δ might be an appropriate model in those cases where long narrow plots are observed to be less variable than square plots of the same area.

7. FITTING THE MODELS

Given positive integers a, b, r, c consider a rectangular region of length ra and breadth cb divided into rc contiguous rectangular plots P_{jk} each of length a and breadth b, and let y_{jk} be some response measurement for the plot P_{jk}, where $j = 1,\ldots,r$ and $k = 1,\ldots,c$. The data from many uniformity trials (see Smith (1938)) are of this form. To model such data we suppose y_{jk} to be an observation on a random variable Y_{jk} having the form

$$Y_{jk} = \theta + W(P_{jk})$$

where $\{W(A), A \in \mathcal{N}\}$ is a random measure with $\langle A\rangle = 0$ for all $A \in \mathcal{N}$.

To infer something about $\langle A,A\rangle$, $A \in \mathcal{N}$, from such data, an

obvious statistic to look at is

$$s^2 = \{\sum_{j=1}^{r}\sum_{k=1}^{c} y_{jk}^2 - (\sum_{j=1}^{r}\sum_{k=1}^{c} y_{jk})^2/(rc)\}/\{ab(rc-1)\}.$$

This is the observed value of a random variable, S^2 say, whose expectation satisfies

$$E(S^2) = \{rc\, V(a,b;-) - (rc)^{-1} V(ra,cb;-)\}/\{ab(rc-1)\} .$$

Clearly, it is possible for the observations y_{jk} to be added together in several different ways corresponding to different values of a, b, r and c, with a corresponding s^2 calculated for each of these combinations. Thus, suppose there are m such combinations a_j, b_j, r_j, c_j, $j = 1,\ldots, m$ of the variables a, b, r, and c for which this corrected sum of squares is calculated, and let the resulting values be denoted by s_j^2, $j = 1,\ldots,m$. Then any unknown parameters in the specification of $V(a,b;-)$ could be estimated by those values which, in some sense, make s_j^2 close to $E(S_j^2)$ for $j = 1,\ldots,m$. The criterion adopted here is the minimization of

$$\phi = \sum_{j=1}^{m} \{s_j^2 - E(S_j^2)\}\,(r_j c_j - 1)/(s_j^2)^2 .$$

The main objective in devising this technique was to get estimates without expending too much computer time, using a procedure that is directly applicable to the results of many uniformity trials as they are reported in the literature, i.e., effectively as so many combi-

nations of the quantities a_j, b_j, r_j, c_j, and s_j^2. Also, through ϕ, the fit of competing models may be compared.

Applied to the 45 data combinations given in Smith's paper this technique estimates α, β, γ, and δ by 0·50, 0·46, 1·256, and 0·730, the corresponding variance components being estimated as 12,739, 14,353, 3,596, and 6,014, with ϕ- values of 103·5, 97·6, 38·8, and 80·8 respectively. The estimate 1·256 for γ may be compared with the value 1·251 obtained by Smith. The clear superiority of M_γ over the three isotropic models, particularly M_δ calls for some comment. In the first place, one might attribute the apparent lack of isotropy to the usual row-by-row wheat-planting technique adopted. A more important consideration perhaps, is the scale of this particular uniformity trial. With a and b expressed in terms of length-units of six inches, the total area occupied by the trial was 5 x 12 = 60 square yards. It is doubtful if any model of variation based on a trial of this size could be a reliable guide to the variation encountered in the usual run of wheat experiments. What kind of experimental data would allow reliable model-building is, of course, a good question.

In Smith's paper the statistic

$$t^2 = s^2/(ab)$$

was used rather than s^2. Table 1 sets out nine of the combinations a, b, r, c, t^2, together with $E(T^2)$ for each of the fitted models. The main purpose of the table is to indicate how all three isotropic models predict that long narrow plots will be less variable -- as measured by $E(T^2)$ - than square plots of the same area.

Table 1. Fitted values of $E(T^2)$ under $M\alpha$, $M\beta$, $M\gamma$, and $M\delta$ for Smith's data.

	a	1	3	1	2	6	9	18	2	12
	b	6	2	36	18	6	4	2	72	12
	r	30	10	30	15	5	3	1	15	2
	c	12	36	2	4	12	18	36	1	6
	t^2	863	976	276	272	241	214	275	63	96
$E(T^2)$	$M\alpha$	861	1051	153	209	249	243	208	52	72
	$M\beta$	864	1040	154	210	254	247	208	52	75
	$M\gamma$	939	939	242	242	242	242	239	83	82
	$M\delta$	873	995	161	215	265	256	212	53	92

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