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Order Statistics from Stationary Sequences

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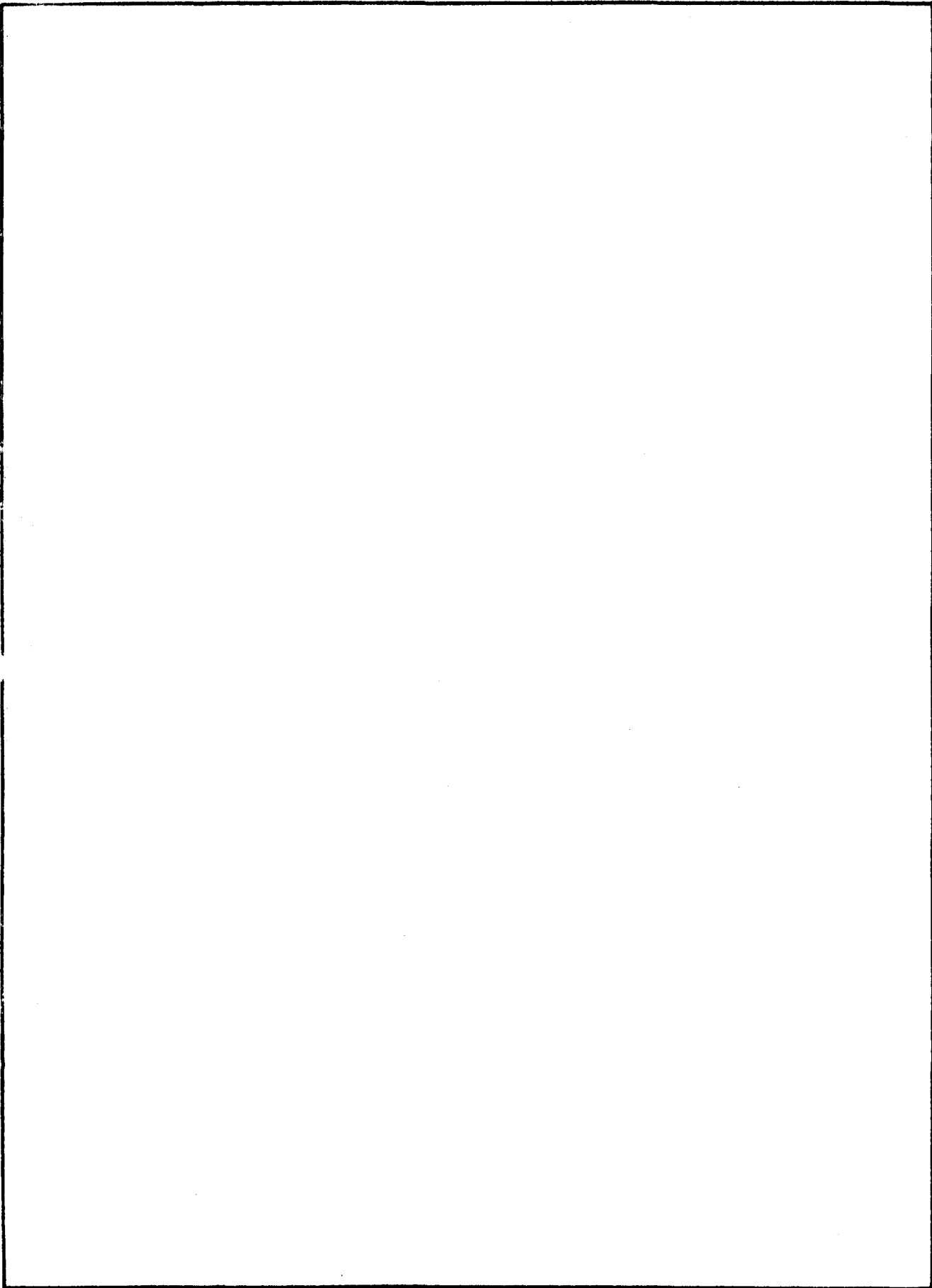
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On Limiting Distributions of Intermediate Order
Statistics from Stationary Sequences

by

Vernon Watts and M. R. Leadbetter*

1. Introduction

The problem of finding the asymptotic distribution of the maximum term from a stationary, dependent sequence of random variables (r.v.'s) has been extensively investigated in the literature. Of particular interest is when the dependence assumption made is that of strong mixing or related conditions, which formulate mathematically the concept of "approximate independence." To say that a sequence $\{X_n\}_{n \geq 1}$ on (Ω, \mathcal{F}, P) is strongly mixing of course means that there is a real sequence $g(\ell) \downarrow 0$ as $\ell \rightarrow \infty$ such that

$$|P(AB) - P(A)P(B)| \leq g(\ell)$$

for any events $A \in \sigma(X_1, \dots, X_m)$ and $B \in \sigma(X_{m+\ell+1}, X_{m+\ell+2}, \dots)$, for each $m \geq 1$, where $\sigma(\cdot)$ denotes the σ -field generated by the indicated r.v.'s. Loynes (1965) showed that under strong mixing and an additional restriction the maximum $X_1^{(n)} = \max\{X_1, \dots, X_n\}$, suitably normalized, has the same limiting distribution as if the sequence were actually independent and identically distributed (i.i.d), assuming this exists, and necessarily of one of the three classical types of extreme value limit laws. For stationary normal sequences Berman (1964) found two covariance conditions, one being implied by strong mixing, for which the distribution of the maximum converges to the double-exponential limit law, arising in the i.i.d. normal case. More recently, Leadbetter (1974) obtained the general result of Loynes under a substantially weaker "distributional mixing" assumption, and he showed that with either of Berman's covariance conditions the normal case may be placed into the general framework. Additionally, Leadbetter considered the related high-level exceedance problem for

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stationary sequences, as well as extensions of the results for the maximum to other extreme order statistics.

Our objective in this paper is to obtain analogous results for intermediate order statistics. For a given sequence of r.v.'s $\{X_n\}$, let $X_{k_n}^{(n)}$ denote the k_n -th largest of X_1, \dots, X_n . Let $\{k_n\}_{n \geq 1}$ be integers such that $1 \leq k_n \leq n$ for each n . Then if $k_n \rightarrow \infty$ but $k_n/n \rightarrow 0$, $\{X_{k_n}^{(n)}\}$ is called a sequence of intermediate order statistics and $\{k_n\}$ an intermediate rank sequence. Subject to the mild restriction that k_n increase monotonically, Wu (1966) found that when the $\{X_n\}$ are i.i.d. the only possible nondegenerate limit laws for the normalized sequence $\{a_n(X_{k_n}^{(n)} - b_n)\}$ are normal and lognormal. (For convenience here all limit laws will be assumed continuous.) We will establish general conditions under which the intermediate order statistic $X_{k_n}^{(n)}$ from a stationary, dependent sequence $\{X_n\}$ has the same asymptotic distribution as it would if the $\{X_n\}$ were i.i.d. These conditions parallel those known for the corresponding result in the extreme order statistic problem, a primary difference being that certain rapid "mixing" rates are assumed. Using our procedure it is convenient to deal directly with an appropriate level exceedance problem and to regard that of asymptotic distributions as a specialization. We also show that under a certain rapid "decay" rate of the covariance function, our general conditions are satisfied by a stationary normal sequence $\{X_n\}$; in this instance it is known (see Cheng (1965)) that the asymptotic distribution of $X_{k_n}^{(n)}$ under independence is itself normal.

2. The general stationary case.

First suppose that $\{X_n\}$ is an i.i.d. sequence of r.v.'s with marginal distribution function (d.f.) $F(x) = P(X_1 \leq x)$, and that $\{k_n\}$ is an intermediate rank sequence. Then for given real numbers $a_n > 0$, b_n , the relation

$$(2.1) \quad P(a_n(X_{k_n}^{(n)} - b_n) \leq x) \rightarrow G(x)$$

holds for x such that $0 < G(x) < 1$ if and only if

$$(2.2) \quad 1 - F(x/a_n + b_n) = k_n/n - u(x)\sqrt{k_n}/n + o(\sqrt{k_n}/n)$$

as $n \rightarrow \infty$, where $G(x) = \Phi(u(x))$, Φ being the standard normal d.f. (see, for example, Wu (1966)). Consider the basic identity

$$P(X_{k_n}^{(n)} \leq u_n) = P(W_n < k_n) ,$$

where $W_n = \sum_{i=1}^n I_{[X_i > u_n]}$ denotes the number of exceedances of the "level" u_n by X_1, \dots, X_n . For u_n satisfying

$$(2.3) \quad 1 - F(u_n) = k_n/n - u\sqrt{k_n}/n + o(\sqrt{k_n}/n)$$

for some real number u , it is easily seen upon suitable normalization of W_n and by applying the classical central limit theorem for row sums from triangular arrays of r.v.'s independent within each row that

$$(2.4) \quad P(W_n < k_n) \rightarrow \Phi(u)$$

as $n \rightarrow \infty$. Of course for an arbitrary d.f. F , it may not be possible to choose u_n 's satisfying (2.3), but at least this can be done for every real u if $X_{k_n}^{(n)}$ has an asymptotic distribution, since then (2.1) necessarily holds for all real x .

Now suppose $\{X_n\}$ is stationary with marginal d.f. F and that u_n satisfies (2.3) for some u . If under the appropriate normalization W_n has the limiting standard normal distribution, then we obtain (2.4). Therefore if (2.2) holds for some x and with $u(x) = u$, then so does (2.1). It is evident that the assumption that (2.2) hold with arbitrary u means equivalently that the intermediate order statistic $\hat{X}_{k_n}^{(n)}$ from the associated independent sequence $\{\hat{X}_n\}$,

the independent sequence having the same marginal d.f. as $\{X_n\}$, has the asymptotic distribution $G(x) = \Phi(u(x))$.

To establish (2.4) we present dependence conditions on $\{X_n\}$ under which W_n is asymptotically normal. The key to our approach is the following result of Dvoretzky (1972), a central limit theorem for row sums from triangular arrays of r.v.'s dependent in each row, and which generalizes the classical theorem. Other versions of this are available, but the one given here was selected as being most suitable for our purposes. We state the result in its original form, which is more general than what we actually require.

Lemma 2.1. Let $\{X_{n,i}\}_{n \geq 1, 1 \leq i \leq N_{n \rightarrow \infty}}$ be an array of r.v.'s with $EX_{n,i} = 0$ for all n, i , and let

$$g_n(k) = \sup_{1 \leq m < N_n - k} \sup_{\substack{A \in F_{n,m} \\ B \in G_{n,m+k+1}}} |P(AB) - P(A)P(B)|$$

where $F_{n,m} = \sigma\{X_{n,1}, \dots, X_{n,m}\}$ and $G_{n,m+k+1} = \sigma\{X_{n,m+k+1}, \dots, X_{n,N_n}\}$. Suppose there are integers $0 = j_n(0) < j_n(1) < \dots < j_n(r_n) = N_n$ such that, defining

$$Y_{n,i} = \sum_{i=j_n(r-1)+1}^{j_n(r)} X_{n,i}, \quad i=1, \dots, r_n,$$

we have

$$\lim_n \sum_{i \text{ odd}} EY_{n,i}^2 = 1,$$

$$\lim_n \sum_{i \text{ even}} EY_{n,i}^2 = 0,$$

and

$$\lim_n \sum_{i=1}^{r_n} E\{Y_{n,i}^2 I_{[|Y_{n,i}| > \epsilon]}\} = 0 \quad \text{for all } \epsilon > 0.$$

Then $\lim_n r_n g_n(j_n) = 0$ where $j_n = \min_{1 \leq r \leq r_n} \{j_n(r) - j_n(r-1)\}$ implies that $\sum_{i=1}^{N_n} X_{n,i}$ is asymptotically normal $(0,1)$.

Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of positive integers such that, as $n \rightarrow \infty$,

$$(2.5) \quad \alpha_n \rightarrow \infty, \beta_n \rightarrow \infty, \alpha_n = o(n), \text{ and } \beta_n = o(\alpha_n).$$

Then for each large n an integer ℓ_n is determined by the relation

$$n = (\alpha_n + \beta_n)\ell_n + R_n, \quad 0 \leq R_n < \alpha_n + 2\beta_n;$$

it is readily checked that

$$(2.6) \quad \ell_n \sim n\alpha_n^{-1}.$$

Now partition the first n positive integers into "intervals" $I_1^{(n)}, I_2^{(n)}, \dots, I_{2\ell_n}^{(n)}$, of respective alternating lengths $\alpha_n, \beta_n, \alpha_n, \beta_n, \dots, \alpha_n, R_n$. By interval we mean a finite set of consecutive integers, and the length of any interval is merely the number of integers it contains. For an interval I let W_I be the number of exceedances of the level u_n by $\{X_i: i \in I\}$, where for simplicity we delete explicit dependence upon n . Thus

$$W_n = \sum_{i=1}^{2\ell_n} W_{I_i^{(n)}}.$$

With this notation we now state our main dependence assumption, which we denote by $A(u_n)$. This is given in a form depending explicitly on the levels $\{u_n\}$ satisfying (2.3) for some real u , and for the asymptotic distribution problem the integer sequences assumed to exist may vary with different values of $u = u(x)$.

$A(u_n)$: There are sequences of integers subject to (2.5) such that, as $n \rightarrow \infty$,

$$(2.7) \quad \alpha_n = o(\sqrt{k_n}),$$

for which

$$(2.8) \quad \alpha_n^{-1} \sum_{j=1}^{\alpha_n-1} (\alpha_n - j) \{P(X_1 > u_n, X_{j+1} > u_n) - (1 - F(u_n))^2\} = o(k_n/n)$$

and

$$(2.9) \quad \beta_n^{-1} \sum_{j=1}^{\beta_n-1} (\beta_n-j) \{P(X_1 > u_n, X_{j+1} > u_n) - (1-F(u_n))^2\} = o(k_n/n),$$

and such that

$$(2.10) \quad \frac{n}{\alpha_n} g_n \rightarrow 0,$$

where

$$g_n = \sup_{1 \leq m < 2\ell_n - 1} \sup_{\substack{A \in F_{n,m} \\ B \in G_{n,m+2}}} |P(AB) - P(A)P(B)|,$$

$$F_{n,m} = \sigma(W_{I_1^{(n)}}, \dots, W_{I_m^{(n)}}), \quad \text{and} \quad G_{n,m+2} = \sigma(W_{I_{m+2}^{(n)}}, \dots, W_{I_{2\ell_n}^{(n)}}).$$

We note that the intervals $I_m^{(n)}$ and $I_{m+2}^{(n)}$ are "separated" by at least β_n . It is then evident that a condition sufficient for (2.10), which is stronger in that more events must be considered, is

$$(2.11) \quad \frac{n}{\alpha_n} h_n(\beta_n) \rightarrow 0$$

as $n \rightarrow \infty$, where

$$h_n(\ell) = \sup_{1 \leq m < n-\ell} \sup_{\substack{A \in F'_{n,m} \\ B \in G'_{n,m+\ell+1}}} |P(AB) - P(A)P(B)|,$$

$$F'_{n,m} = \sigma\{I_{[X_1 > u_n]}, \dots, I_{[X_m > u_n]}\}, \quad \text{and} \quad G'_{n,m+\ell+1} = \sigma\{I_{[X_{m+\ell+1} > u_n]}, \dots, I_{[X_n > u_n]}\}.$$

Now, for fixed n, m let $\underline{i} \subseteq \{1, \dots, m\}$ and $\underline{j} \subseteq \{m+\beta_n+1, \dots, n\}$. Define events

$$(2.12) \quad A_{\underline{i}} = \{X_i > u_n, i \in \underline{i}; X_{i'} \leq u_n, i' \in \{1, \dots, m\} \setminus \underline{i}\},$$

$$B_{\underline{j}} = \{X_j > u_n, j \in \underline{j}; X_{j'} \leq u_n, j' \in \{m+\beta_n+1, \dots, n\} \setminus \underline{j}\}.$$

Then we can see that each $A \in F'_{n,m}$ is some disjoint union of at most 2^m events each of the form $A_{\underline{i}}$, and similarly each $B \in G'_{n,m+\beta_n+1}$ is a disjoint union of at most $2^{n-m-\beta_n}$ events of the form $B_{\underline{j}}$. Hence a condition sufficient for (2.11) is

$$(2.13) \quad 2^n \frac{n}{\alpha_n} h'_n \rightarrow 0,$$

where

$$h'_n = \sup_{1 \leq m < n - \beta_n} \sup_{\underline{i}, \underline{j}} |P(A_{\underline{i}} B_{\underline{j}}) - P(A_{\underline{i}})P(B_{\underline{j}})|.$$

One may show that the "distributional mixing" condition (2.13) is satisfied by strongly mixing sequences for which $g(\ell) \rightarrow 0$ at some particular exponential rate, but (2.13) and especially (2.10) are potentially much weaker, in that fewer pairs of events need to be "tested." Unfortunately, (2.10) may be hard to verify directly, as is the case for stationary normal sequences considered in Section 3, where with the presently known technical results pertaining to normal sequences we will be content to deal with (2.13).

In this section, however, we work with the assumption $A(u_n)$ as stated, and we now suppose for the given stationary sequence $\{X_n\}$ and levels $\{u_n\}$ satisfying (2.3), that integer sequences $\{\alpha_n\}$ and $\{\beta_n\}$ for which $A(u_n)$ holds have been chosen. Then we define a triangular array of r.v.'s $\{X_{n,i}\}$ for n large and $1 \leq i \leq 2\ell_n$, by

$$X_{n,i} = \frac{W_{I_i(n)} - EW_{I_i(n)}}{\sqrt{\ell_n} \sigma_n},$$

where

$$(2.14) \quad \sigma_n^2 = \text{Var } W_{I_1(n)} = \alpha_n F(u_n)(1-F(u_n)) + 2 \sum_{j=1}^{\alpha_n-1} (\alpha_n-j) \{P(X_1 > u_n, X_{j+1} > u_n) - (1-F(u_n))^2\}.$$

The following lemma shows that $\sum_{i=1}^{2\ell_n} X_{n,i}$ is the appropriate normalization of W_n .

This result will constitute the major portion of establishing (2.4).

Lemma 2.2. Under $A(u_n)$, $\sum_{i=1}^{2\ell_n} X_{n,i}$ is asymptotically normal $(0,1)$.

Proof: Clearly $EX_{n,i} = 0$ for all n,i . Upon showing that

$$(2.15) \quad \lim_n \sum_{i \text{ odd}} EX_{n,i}^2 = 1$$

$$(2.16) \quad \lim_n \sum_{i \text{ even}} EX_{n,i}^2 = 0,$$

and

$$(2.17) \quad \lim_n \sum_{i=1}^{2\ell_n} E(X_{n,i}^2 I_{[|X_{n,i}| > \epsilon]}) = 0 \text{ for every } \epsilon > 0,$$

then along with (2.10) we will have fulfilled the requirements of Lemma 2.1 and the conclusion will follow. Here we are taking $N_n = r_n = 2\ell_n$ and $j_n(r) = r$, $0 \leq r \leq 2\ell_n$.

First we have that

$$\sum_{i \text{ odd}} EX_{n,i}^2 = \sum_{i \text{ odd}} \frac{\text{Var } W_{I_i}^{(n)}}{\ell_n \sigma_n^2} = 1$$

for each n by stationarity, so that (2.15) obviously holds.

For convenience we write $p_n = 1 - F(u_n)$. Then from (2.8) and the obvious relation $p_n \sim k_n/n$ we have

$$(2.18) \quad \sigma_n^2 \sim \alpha_n k_n/n$$

and moreover, if $\ell_n > 1$,

$$\text{Var } W_{I_2}^{(n)} = \beta_n p_n (1-p_n) + 2 \sum_{j=1}^{\beta_n-1} (\beta_n - j) \{P(X_1 > u_n, X_{j+1} > u_n) - p_n^2\} = o(\sigma_n^2),$$

by (2.5) and (2.9). Thus

$$(2.19) \quad \sum_{\substack{i \text{ even} \\ i < 2\ell_n}} EX_{n,i}^2 = \frac{(\ell_n - 1) \text{Var } W_{I_2}^{(n)}}{\ell_n \sigma_n^2} < \frac{\text{Var } W_{I_2}^{(n)}}{\sigma_n^2} \rightarrow 0.$$

Also, $EX_{n,2\ell_n}^2 = O((\alpha_n + 2\beta_n)/k_n)$, which tends to zero by (2.7). This combined with (2.19) gives (2.16).

Finally, $|X_{n,i}|$ is bounded by

$$\frac{\alpha_n + 2\beta_n}{\sqrt{\ell_n} \sigma_n} = O(\alpha_n / \sqrt{k_n}),$$

so for given $\varepsilon > 0$, $P(|X_{n,i}| > \varepsilon) = 0$ for all large n , which leads to (2.17). \square

Remark: If $\{\beta_n\}$ satisfies $\beta_n k_n = o(n)$, then (2.9) automatically follows from (2.8) (see Watts (1977)). However, for the derivation of a covariance condition sufficient for $A(u_n)$ in the normal case, the requirement that $\beta_n k_n = o(n)$ is unnecessarily restrictive, whereas (2.8) and (2.9) are verified with the same calculation.

We now give the main result.

Theorem 2.3. Let $\{X_n\}$ be a stationary sequence of r.v.'s with marginal d.f. F and $\{k_n\}$ an intermediate rank sequence. Suppose the level u_n satisfies (2.3) for some real u and let W_n be the number of exceedances of u_n by X_1, \dots, X_n . If $A(u_n)$ holds, then $P(W_n < k_n) \rightarrow \Phi(u)$ as $n \rightarrow \infty$.

Proof: For large n we have

$$P(W_n < k_n) = P\left(\sum_{i=1}^{2\ell_n} X_{n,i} < \frac{k_n - np_n}{\sqrt{\ell_n} \sigma_n}\right),$$

where $p_n = 1 - F(u_n)$. From (2.3), (2.6), and (2.18) we see that

$$\frac{k_n - np_n}{\sqrt{\ell_n} \sigma_n} \rightarrow u,$$

and the conclusion follows by Lemma 2.2. \square

From this result we may state the following theorem giving sufficient conditions for $X_{k_n}^{(n)}$ to have an asymptotic distribution, which is the same as

if the $\{X_n\}$ were i.i.d.

Theorem 2.4. Let $\{X_n\}$ be stationary and suppose for some constants $a_n > 0$, b_n ,

$$P(a_n(X_{k_n}^{(n)} - b_n) \leq x) \rightarrow G(x) = \Phi(u(x))$$

as $n \rightarrow \infty$ for all x , where $\{\hat{X}_n\}$ is the independent sequence associated with $\{X_n\}$.

If $A(u_n)$ is satisfied for $u_n = x/a_n + b_n$ with $u = u(x)$ for all x for which $u(x)$ is finite, or equivalently, for which $0 < G(x) < 1$, then also

$$P(a_n(X_{k_n}^{(n)} - b_n) \leq x) \rightarrow G(x)$$

for all real x .

Proof: The conclusion for x such that $0 < G(x) < 1$ follows from the previous theorem and is easily established as well for all x , using the continuity of G .

□

3. The normal case.

In this section we develop a covariance condition under which $A(u_n)$ holds for stationary normal sequences. As previously indicated, in doing this we work with (2.10) replaced by (2.13), from which will arise the requirement of convergence of covariances at an exponential rate. For simplicity we deal exclusively with standard (that is, zero mean, unit variance) normal sequences, since results obtained for this case can be easily transformed to other normal sequences.

Thus suppose $\{X_n\}$ is stationary with marginal d.f. $\Phi(x)$, and let $r_n = EX_1 X_{n+1}$, $n \geq 1$. Let $\{k_n\}$ be an intermediate rank sequence and for a given real x , suppose the level u_n satisfies (2.3) with $u = u(x) = x$, or upon relabeling,

$$(3.1) \quad 1 - \Phi(u_n) = k_n/n - x\sqrt{k_n}/n + o(\sqrt{k_n}/n) .$$

One such u_n is $u_n = x/a_n + b_n$, where $\Phi(b_n) = 1 - k_n/n$ and $a_n = n\Phi'(b_n)/\sqrt{k_n}$ (see Cheng (1965)). Again let W_n be the number of exceedances of u_n by X_1, \dots, X_n . Then we seek a condition on $\{r_n\}$ under which $A(u_n)$ holds.

We assume that $r_n \rightarrow 0$ as $n \rightarrow \infty$, which is essential for the required "approximate independence" between X_i and X_j when $|j-i|$ is large, and it is convenient to define the quantities

$$\delta = \sup_{n \geq 1} |r_n| \quad \text{and} \quad \delta_n = \sup_{m \geq n} |r_m| .$$

As has been indicated by Berman (1964), the assumption $r_n \rightarrow 0$ implies $\delta < 1$. Also, it is clear that for each $n \geq 1$ there is an integer $m_n \geq n$ such that $\delta_n = |r_{m_n}|$, so if $t_n \downarrow 0$ and $r_n = O(t_n)$, then since $\delta_n t_n^{-1} \leq |r_{m_n}| t_{m_n}^{-1}$, it follows that also $\delta_n = O(t_n)$.

We begin the development with some useful technical results pertaining to normal sequences. First, we have for u_n defined by (3.1),

$$(k_n/n) \sim 1 - \Phi(u_n) \sim \frac{1}{\sqrt{2\pi} u_n} e^{-u_n^2/2} .$$

Taking logarithms gives $u_n \sim \sqrt{2 \log n/k_n}$, so that

$$(3.2) \quad e^{-u_n^2} \sim 4\pi(k_n/n)^2 \log n/k_n .$$

Also required is the following result, various forms of which have been used for consideration of the maximum term from normal sequences by a number of authors, including Berman (1964) and Leadbetter (1974). The proof makes use of a now standard technique and will be omitted.

Lemma 3.1. Let $\{X_n\}$ be a standard stationary normal sequence with covariances $r_n \rightarrow 0$. Let $I, I', J,$ and J' be four mutually disjoint sets of distinct positive integers. Then

$$(3.3) \quad \begin{aligned} & |P(X_i > u, i \in I; X_i \leq u, i' \in I'; X_j > u, j \in J; X_j \leq u, j' \in J') \\ & \quad - P(X_i > u, i \in I; X_i \leq u, i' \in I')P(X_j > u, j \in J; X_j \leq u, j' \in J')| \\ & \leq K \sum_{i,j} |\rho_{i,j}| e^{-u^2/(1+|\rho_{i,j}|)} \end{aligned}$$

for some finite constant K depending only on δ , where $\rho_{i,j} = r_{|j-i|}$ for $i \in I \cup I', j \in J \cup J'$, and the summation is taken over all such possible pairs i, j . In particular for $i < j$ and any real u ,

$$(3.4) \quad |P(X_i > u, X_j > u) - (1 - \Phi(u))^2| \leq K |r_{j-i}| e^{-u^2/(1+|r_{j-i}|)}.$$

Finally, we require a measure of how rapidly $k_n \rightarrow \infty$, which we formulate precisely as follows. Define $\theta = \theta(\{k_n\})$ by

$$\theta = \inf\{\theta_1 : k_n = O(n^{\theta_1})\}.$$

Then it is evident that $0 \leq \theta \leq 1$, and moreover

$$(3.5) \quad k_n = o(n^{\theta+\epsilon}) \quad \text{for every } \epsilon > 0.$$

As part of our covariance condition we will restrict what intermediate rank sequences can be considered, in terms of values of θ .

Having the preceding results and notation we now suppose for the stationary normal sequence $\{X_n\}$, that the covariances $\{r_n\}$ satisfy

$$(3.6) \quad r_n = O(e^{-n^\rho}) \quad \text{for some } \rho > 2.$$

Let $\{k_n\}$ be subject to the restriction $2/\rho < \theta < 1$. Then we may choose sequences of integers $\{\alpha_n\}$ and $\{\beta_n\}$ for which $\alpha_n \sim n^\lambda$ and $\beta_n \sim n^\mu$, where

$$(3.7) \quad 0 < \frac{1}{\rho} < \mu < \lambda < \frac{\theta}{2}.$$

(We note this choice of sequences does not depend on x .) It is clear the relation (2.7) holds, and as the next lemma shows, so do (2.8) and (2.9).

Lemma 3.2. Suppose $\{r_n\}$ satisfies (3.6) and $\{k_n\}$ is such that $\theta < 1$. Then for $\{\alpha_n\}$ and $\{\beta_n\}$ as chosen above, (2.8) and (2.9) hold for u_n given by (3.1), for any real x . In fact, for each such u_n we have

$$\frac{n}{k_n} \sum_{j=1}^{[\sqrt{k_n}]} |P(X_1 > u_n, X_{j+1} > u_n) - (1 - \Phi(u_n))^2| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

($[\cdot]$ denotes the greatest integer function.)

Proof: Let γ be such that $0 < \gamma < (1 - \delta)(1 - \theta)/(1 + \delta)$. Then by (3.4)

$$\frac{n}{k_n} \sum_{j=1}^{[n^\gamma]} |P(X_1 > u_n, X_{j+1} > u_n) - (1 - \Phi(u_n))^2| \leq K \frac{n}{k_n} \sum_{j=1}^{[n^\gamma]} e^{-u_n^2/(1+|r_j|)},$$

and by (3.2) the right side does not exceed

$$K \frac{n^{1+\gamma}}{k_n} e^{-u_n^2/(1+\delta)} \leq K_1 (\log n) n^{\gamma - (1-\delta)(1-\theta)/(1+\delta)} \left(\frac{k_n}{n^\theta} \right)^{(1-\delta)/(1+\delta)}$$

for some constant K_1 . Then from (3.5) and the choice of γ , this last expression tends to zero. The proof is complete if it is possible to choose γ such that $[\sqrt{k_n}] \leq n^\gamma$ for all large n . Otherwise, if $[\sqrt{k_n}] > n^\gamma$ we have

$$\begin{aligned} \frac{n}{k_n} \sum_{j=[n^\gamma]+1}^{[\sqrt{k_n}]} |P(X_1 > u_n, X_{j+1} > u_n) - (1 - \Phi(u_n))^2| &\leq K \frac{n}{\sqrt{k_n}} \delta_{[n^\gamma]} e^{-u_n^2/(1+\delta)} [n^\gamma] \\ &\leq K_1 \sqrt{k_n} (\log n) \delta_{[n^\gamma]}, \end{aligned}$$

which tends to zero since $\delta_{[n^\gamma]} = O(e^{-n^{\gamma\rho_1}})$ for every $\rho_1 < \rho$. The conclusion then follows upon combining the two summations. \square

We now show that (2.13) holds with the above choices of $\{\alpha_n\}$ and $\{\beta_n\}$. Fix n and m with $1 \leq m < n < \beta_n$ and let $A_{\underline{i}}$ and $B_{\underline{j}}$ be events of the form (2.12). Then by (3.3) we have

$$|P(A_{\underline{i}}B_{\underline{j}}) - P(A_{\underline{i}})P(B_{\underline{j}})| \leq K \sum_{i,j} |r_{j-i}| e^{-u_n^2/(1+|r_{j-i}|)},$$

where the summation is taken over all pairs i, j with $1 \leq i \leq m$ and $m + \beta_n + 1 \leq j \leq n$. The right side above does not exceed

$$Kn^2 \delta_{\beta_n} e^{-u_n^2/(1+\delta_{\beta_n})} < K_1(\log n) nk_n \delta_{\beta_n}.$$

But $\delta_{\beta_n} = O(e^{-n^{\mu_{\rho_1}}})$ for every $\rho_1 < \rho$, and by (3.7), we may suppose $\mu_{\rho_1} > 1$. Then taking the supremum over all choices of indices $\underline{i}, \underline{j}$, and all m , $1 \leq m < n - \beta_n$, we see that (2.13) holds. Summarizing, we have proved the following theorem.

Theorem 3.3. Suppose $\{X_n\}$ is a standard stationary normal sequence with covariances $r_n = EX_1X_{n+1}$, $n \geq 1$, satisfying $r_n = O(e^{-n^\rho})$ for some $\rho > 2$. Let $\{k_n\}$ be an intermediate rank sequence for which $2/\rho < \theta < 1$, and suppose the level u_n satisfies (3.1) for some real x . Then $A(u_n)$ holds, so by Theorem 2.3,

$$P(W_n < k_n) \rightarrow \Phi(x)$$

as $n \rightarrow \infty$, where W_n is the number of exceedances of u_n by X_1, \dots, X_n . Moreover, it follows that

$$P(a_n(X_{k_n}^{(n)} - b_n) \leq x) \rightarrow \Phi(x)$$

as $n \rightarrow \infty$ for all real x , where a_n and b_n are given by $\Phi(b_n) = 1 - k_n/n$ and

$$a_n = n\phi'(b_n)/\sqrt{k_n}.$$

Finally, we mention that the conclusion of Theorem 2.3 has been obtained under an alternative set of conditions by Watts (1977). There a constructive procedure is given for actually calculating an approximation to the value of $P(W_n < k_n)$ which has the same limit. Moreover it is shown that for stationary normal sequences the exponential convergence rate for the covariances again arises. However, the precise rate required is less severe than that indicated in Theorem 3.3, for consideration of intermediate rank sequences $\{k_n\}$ for which $\theta = \theta(\{k_n\})$ is "small," or close to zero. Specifically, if $r_n = O(e^{-n^\rho})$ for some $\rho > 0$ and if θ satisfies the two restrictions $\theta < 2/5(1 + 1/\rho)$ and $\theta < (1-\delta)/2$, where $\delta = \sup_{n \geq 1} |r_n|$, then the conclusion of Theorem 3.3 holds.

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