

NONLINEAR THREE STAGE LEAST SQUARES :
A TEST OF RESTRICTIONS IN THE
PRESENCE OF A MAINTAINED HYPOTHESIS

by

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This technical report is to be regarded as Section 7 of Gallant (1977).

Section 7. Testing Restrictions

Consider testing

$$H: \rho = f(\beta) \text{ against } A: \rho \neq f(\beta)$$

where β is an s -vector with $s < r$ in the presence of the maintained hypothesis $\theta = g(\rho)$. Let $\hat{\Sigma}$ be any strongly consistent estimator of Σ rather than necessarily the estimator computed from two stage least squares residuals as above. Let $\tilde{\rho}$ denote the value of ρ which minimizes

$$S[g(\rho)] = (1/n) q'[g(\rho)](\hat{\Sigma}^{-1} \otimes P_Z)q[g(\rho)]$$

where $P_Z = Z(Z'Z)^{-1}Z'$ and let $\tilde{\beta}$ denote the value of β which minimizes

$$S\{g[f(\beta)]\} = (1/n)q'\{g[f(\beta)]\}(\hat{\Sigma}^{-1} \otimes P_Z)q\{g[f(\beta)]\} .$$

Let $\tilde{\theta} = g(\tilde{\rho})$ and let $\tilde{\theta} = g[f(\tilde{\beta})]$.

A test statistic is

$$\begin{aligned} T &= n(S\{g[f(\tilde{\beta})]\} - S[g(\tilde{\rho})]) \\ &= n[S(\tilde{\theta}) - S(\tilde{\theta})] \end{aligned}$$

which is characterized in Theorem 3, below, as

$$T = Y + a_n$$

where, assuming the errors $e_{\alpha t}$ normally distributed, Y is a chi-square random variable with $r-s$ degrees freedom and a_n converges in probability to zero.

The test is: reject H when T exceeds the upper $\alpha \times 100$ percentage point of a chi-square random variable with $r-s$ degrees freedom. This test is asymptotically level α .

Theorem 3. If the assumptions listed in Section 4 hold both with respect to $\theta = g(\rho)$ and with respect to $\theta = g[f(\beta)]$ and each equation in the system is identified then

$$T = Y + a_n$$

where a_n converges in probability to zero. If the errors e_t have the multivariate normal distribution then Y is a chi-square random variable with $r-s$ degrees freedom.

Proof: The choice of an alternative strongly consistent estimator of Σ does not change any details of the proof of Theorems 1 and 2.

As an intermediate step of Theorem 2 the equation

$$\begin{aligned} G'(\dot{\rho}) B(\dot{\theta}) [(1/n)(I \otimes Z') QG + (1/n)(I \otimes Z')H] \sqrt{n} (\dot{\rho} - \rho^*) \\ = -(1/\sqrt{n}) G'(\dot{\rho}) B(\dot{\theta}) U + \alpha_n \end{aligned}$$

is obtained; $\alpha_n = (\sqrt{n}/2) \nabla_{\rho} S(\dot{\theta})$ and converges almost surely to zero. The difference between the matrix

$$G'[(1/n) Q'(I \otimes Z)][\Sigma \otimes (1/n) Z'Z]^{-1}[(1/n)(I \otimes Z')Q]G$$

and the matrix premultiplying $\sqrt{n} (\dot{\rho} - \rho^*)$ converges almost surely to zero as a result of the remarks in the proof of Theorem 2, the almost sure convergence of $\hat{\Sigma}$ to Σ , and Lemma A.2. Also, $\sqrt{n}(\tilde{\rho} - \rho^*)$ is bounded in probability as a result of the conclusion of Theorem 2 and $\sqrt{n}(\tilde{\rho} - \rho^*) - \sqrt{n}(\dot{\rho} - \rho^*)$ converges almost surely to zero. In consequence, the equation may be rewritten as

$$(1/n)G'Q'(\Sigma^{-1} \otimes P_Z) QG\sqrt{n}(\tilde{\rho} - \rho^*) = -(1/\sqrt{n}) G'(\dot{\rho})B(\dot{\theta}) U + \alpha_n + \beta_n$$

where β_n converges in probability to zero. Analogous arguments yield

$$(1/n)G'Q'(\Sigma^{-1} \otimes P_Z) QG\sqrt{n}(\tilde{\rho} - \rho^*) = -(1/\sqrt{n})G'Q'(\Sigma^{-1} \otimes P_Z) q + \alpha_n + \beta_n + \gamma_n$$

where γ_n converges in probability to zero. Multiplication by

$(1/\sqrt{n})[(1/n) G'Q'(\Sigma^{-1} \otimes P_Z)QG]^{-1}$ yields

$$(\tilde{\rho} - \rho^*) = -[G'Q'(\Sigma^{-1} \otimes P_Z)QG]^{-1}G'Q'(\Sigma^{-1} \otimes P_Z)q + \delta_n$$

where $\sqrt{n} \delta_n$ converges in probability to zero.

Because $\nabla_{\rho} S[g(\tilde{\rho})] = 0$, the second order Taylor's expansion of $n S[g(\tilde{\rho})] - nS[g(\rho^*)]$ about $\tilde{\rho}$ has only the single term

$$n(\tilde{\rho} - \rho^*)' \nabla_{\rho}^2 S[g(\bar{\rho})](\tilde{\rho} - \rho^*)$$

where $\bar{\rho}$ is on the line segment joining $\tilde{\rho}$ to ρ^* .

Now

$$\nabla_{\rho}^2 S[g(\bar{\rho})] = (2/n) G'(\bar{\rho}) Q'(\bar{\theta})(\hat{\Sigma}^{-1} \otimes P_Z) Q(\bar{\theta}) G(\bar{\rho}) + 2 D_n$$

where D_n has typical element

$$d_{ijn} = [(1/n)(\partial^2/\partial \rho_i \partial \rho_j) q'[g(\bar{\rho})](I \otimes Z)][\hat{\Sigma} \otimes (1/n)Z'Z]^{-1}[(1/n)(I \otimes Z')q(\bar{\theta})]$$

Now $\bar{\theta} = q(\bar{\rho})$ converges almost surely to θ^* and, by Lemma A.1, the last term in brackets converges almost surely to

$$\lim_{n \rightarrow \infty} (1/n) (I \otimes Z') q(\theta^*)$$

which is the zero vector, almost surely, by the strong law of large numbers.

The remaining terms in brackets converge almost surely to finite matrices of finite order by similar arguments whence d_{ijn} converges almost surely to zero.

Similar arguments imply that $(1/n) G'(\bar{\rho}) Q'(\bar{\theta})(\hat{\Sigma}^{-1} \otimes P_Z) Q(\bar{\theta}) G(\bar{\rho})$ converges almost surely to $G'QG$. Now $\sqrt{n}(\tilde{\rho} - \rho^*)$ is bounded in probability and the difference

$$\nabla_{\rho}^2 S[g(\bar{\rho})] - (2/n) G'Q'(\Sigma^{-1} \otimes P_Z) QG$$

converges almost surely to zero whence

$$n S[g(\tilde{\rho})] - n S[g(\rho^*)] = (\tilde{\rho} - \rho^*)' G' Q' (\Sigma^{-1} \otimes P_Z) Q G (\tilde{\rho} - \rho^*) + \epsilon_n$$

where ϵ_n converges in probability to zero.

Substitute the above expression for $(\tilde{\rho} - \rho^*)$ to obtain

$$\begin{aligned} n S[g(\tilde{\rho})] - n S[g(\rho^*)] &= q' (\Sigma^{-1} \otimes P_Z) Q G [G' Q' (\Sigma^{-1} \otimes P_Z) Q G]^{-1} G' Q' (\Sigma^{-1} \otimes P_Z) q \\ &\quad - 2 \delta_n' G' Q' (\Sigma^{-1} \otimes P_Z) q + \delta_n' G' Q' (\Sigma^{-1} \otimes P_Z) Q G \delta_n + \epsilon_n \end{aligned}$$

The second term may be written as

$$-2 \sqrt{n} \delta_n' G' [(1/n) Q' (I \otimes Z)] [\Sigma \otimes (1/n) Z' Z]^{-1} [(1/\sqrt{n}) (I \otimes Z') q]$$

which converges in probability to zero since $\sqrt{n} \delta_n$ converges in probability to zero and $[(1/\sqrt{n}) (I \otimes Z') q]$ is bounded in probability. Similarly the third term

$$\sqrt{n} \delta_n G' [(1/n) Q' (I \otimes Z)] [\Sigma \otimes (1/n) Z' Z]^{-1} [(1/n) (I \otimes Z') Q] G \sqrt{n} \delta$$

converges in probability to zero whence

$$\begin{aligned} n S[g(\tilde{\rho})] - n S[g(\rho^*)] &= q' (\Sigma^{-1} \otimes P_Z) Q G [G' Q' (\Sigma^{-1} \otimes P_Z) Q G]^{-1} G' Q' (\Sigma^{-1} \otimes P_Z) q + \zeta_n \end{aligned}$$

where ζ_n converges in probability to zero. Then we may write

$$n S[g(\tilde{\rho})] - n S[g(\rho^*)] = z' X (X' X)^{-1} X' z + \xi_n$$

where

$$z = (\Sigma \otimes Z'Z)^{-\frac{1}{2}} (I \otimes Z')q \quad ,$$

$$X = (\Sigma \otimes Z'Z)^{-\frac{1}{2}} (I \otimes Z')Q \quad .$$

To obtain a similar result for $\tilde{\beta}$ note that

$$(\partial/\partial\beta)g[f(\beta)] = G[f(\beta)] F(\beta)$$

where $F(\beta) = (\partial/\partial\beta) f(\beta)$ and that the exact same arguments as above when applied to $\tilde{\beta}$ instead of $\hat{\beta}$ yield

$$\begin{aligned} n S\{g[f(\tilde{\beta})]\} - n S\{g[f(\beta^*)]\} \\ = z'XF(F'X'XF)^{-1}F'X'z + \eta_n \end{aligned}$$

where η_n converges in probability to zero and $F = F(\beta^*)$. Then

$$T = z'[X(X'X)^{-1}X' - XF(F'X'XF)^{-1}F'X']z + \zeta_n + \eta_n \quad .$$

Now if q has the multivariate normal distribution with mean zero and variance-covariance matrix $(\Sigma \otimes I)$ then z has the multivariate normal distribution with mean zero and the identity as the variance-covariance matrix. The matrix within brackets is idempotent with rank $r-s$ whence

$$Y = z'[X(X'X)^{-1}X' - XF(F'X'XF)^{-1}F'X']z$$

is a chi-square random variable with $r-s$ degrees freedom.

Reference

Gallant, A. Ronald (1977) "Three-Stage Least Squares Estimation for a System of Simultaneous, Nonlinear, Implicit Equations." Journal of Econometrics 5, pp. 71-88; Institute of Statistics Mimeo Series No. 1032, Raleigh: North Carolina State University; or H.G.B. Alexander Research Foundation Series, Chicago: University of Chicago.