On Tournaments Having
A Unique Hamiltonian Circuit*

by

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ABSTRACT

A tournament is a directed complete graph. A Hamiltonian circuit is a circuit which passes through each vertex of the graph once and only once. This note examines the family $\mathcal{T}_n$ of all nonisomorphic tournaments on n vertices which have a unique Hamiltonian circuit. We let $T_n = |\mathcal{T}_n|$. In [1] Douglas gives a graphical characterization of the family $\mathcal{T}_n$, and from this characterization obtains an involved formula for calculating the values T_n. In particular

n =	3	4	5	6	7	8	9	10
T_n =	1	1	3	8	21	55	144	377 .

This note presents a graphical construction of the family $\mathcal{T}_{n+1}$ from the family $\mathcal{T}_n$ and, using this construction, obtains a proof of the recurrence

$$T_{n+2} = 3\,T_{n+1} - T_n\,, \qquad (n \geq 4)\ .$$

1. Preliminaries.

The outdegree of a vertex v is the number of edges incident with v and directed away from it. The indegree of v is the number of incident edges directed into it.

THEOREM 1:

(i) If $T \in \mathcal{T}_n$, then T has at least one, and at most two, vertices with maximal outdegree, i.e., with outdegree n-2.

(ii) If $T \in \mathcal{T}_n$ has 2 vertices v_1, v_2 of maximal outdegree, then v_1, v_2 are successive vertices of the unique Hamiltonian circuit in T.

PROOF: Since $T \in \mathcal{T}_n$ has a Hamiltonian circuit, we first note that the maximal outdegree of any vertex is n-2.

(i) If $T \in \mathcal{T}_n$ then it follows from corollary 3 of Douglas [1] that at least one vertex must have outdegree n-2. Furthermore, if v_1, v_2 are vertices of maximal outdegree n-2, and v any other vertex, then the indegree of v must be at least 2. Consequently, the outdegree of v is at most $(n-1)-2 = n-3$.

(ii) Let v_1, v_2 be of maximal outdegree n-2. Without loss of generality, assume that the edge between v_1 and v_2 is $v_1 \longrightarrow v_2$. Then this is the only incoming edge for v_2. Hence it must be an edge of the Hamiltonian circuit. □

Thus, we can divide the family $\mathcal{T}_n$ into disjoint families according to whether a tournament $T \in \mathcal{T}_n$ has 1 or 2 vertices of maximal outdegree. We define

$$A_n = \{T \in \mathcal{T}_n: \text{ T has only 1 vertex of maximal outdegree}\}$$

$$B_n = \{T \in \mathcal{T}_n: \text{ T has 2 vertices of maximal outdegree}\} \quad ,$$

and

$$A_n = |\mathcal{A}_n| \quad , \quad B_n = |\mathcal{B}_n| \ .$$

Then clearly

$$\mathcal{T}_n = \mathcal{A}_n \cup \mathcal{B}_n$$

and

$$T_n = A_n + B_n \ .$$

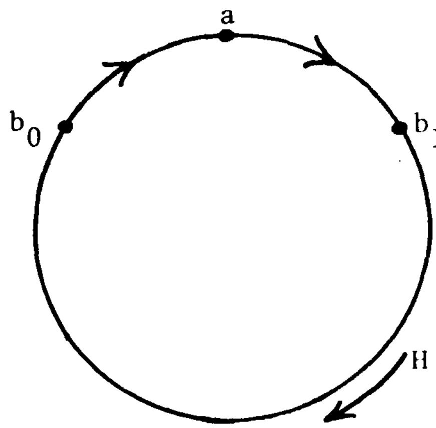
2. Construction.

For $n \geq 3$, if we are given the family $\mathcal{T}_n$ then we may use the following graphical construction to obtain the family $\mathcal{T}_{n+1}$. For each $T \in \mathcal{T}_n$ we will adjoin an (n+1)-st vertex of maximal outdegree (i.e., of outdegree n-1), while at the same time maintaining all edge orientations of T, thus obtaining a tournament on (n+1) vertices. This vertex may be adjoined at either 2 or 3 different positions (depending on whether $T \in \mathcal{A}_n$ or $T \in \mathcal{B}_n$) with the resulting tournaments being nonisomorphic. The resulting tournaments will, of necessity, be in $\mathcal{T}_{n+1}$.

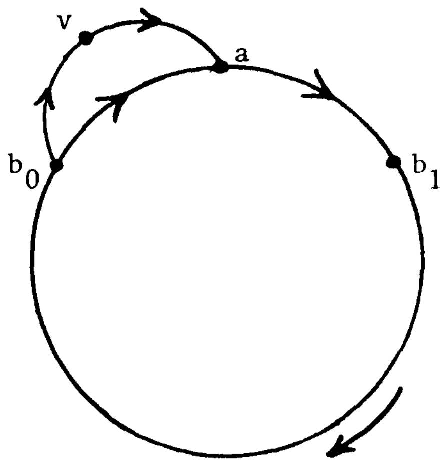
Conversely, we will show that every tournament in $\mathcal{T}_{n+1}$ can be gotten by using this construction. It will follow that $\mathcal{T}_{n+1}$ is precisely the family of all tournaments obtained by applying this construction to $\mathcal{T}_n$.

The graphical construction is as follows:

I. Suppose $T \in \mathcal{A}_n$. Let a be the vertex of maximal outdegree and the Hamiltonian circuit H be as shown.

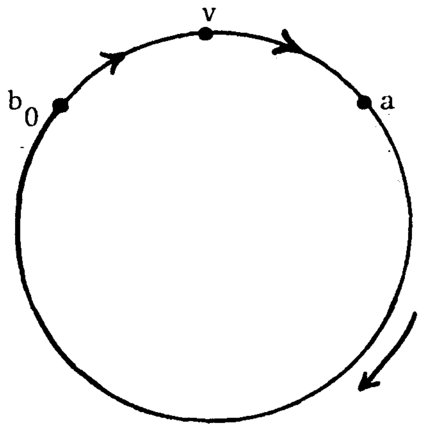


(i) Adjoin to T an (n+1)-st vertex $\underline{v}$ of maximal outdegree n-1 and whose only incoming edge is from b_0. (All other edges of T are retained with their same orientation.) We thus obtain a tournament T' on n+1 vertices containing



We wish to show that $T' \in \mathcal{T}_{n+1}$. Obviously T' has a Hamiltonian circuit; namely, the path (of H) from $\underline{a}$ to $\underline{b_0}$, along with the edges $b_0 \to v \to a$. Call this Hamiltonian circuit H'.

Furthermore, H' will be the only Hamiltonian circuit of T'. Indeed any Hamiltonian circuit of T' must contain the edges $b_0 \to v \to a$ since v has only one incoming edge, and $\underline{a}$ has only two incoming edges, one of which is from $\underline{b_0}$. Thus any Hamiltonian circuit of T' must look like



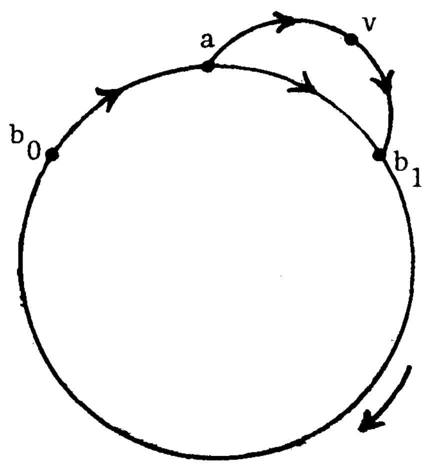
But then the path from a to b_0 along with the edge $b_0 \to a$ would constitute a Hamiltonian circuit of T, and by assumption there is only one such circuit, namely H. Thus the circuit H' is the only Hamiltonian circuit of T', and hence $T' \in \mathcal{T}_{n+1}$.

Note that $T' \in A_{n+1}$ since it has only one vertex of maximal outdegree, namely v.

We will denote this particular constructive technique (applied only to members of A_n) as C_1, and we write

$$T' = C_1(T) \ .$$

(ii) Similarly, if we adjoin to T an (n+1)-st vertex v of maximal outdegree and whose only incoming edge is from a

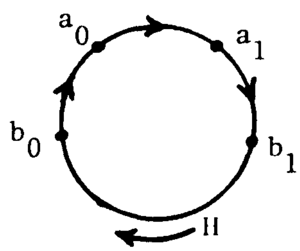


then we will obtain another tournament T' in $\mathcal{T}_{n+1}$. Note that in this case T' $\in \mathcal{B}_{n+1}$ since $\underline{a}$ and $\underline{v}$ both have maximal outdegree n-1.

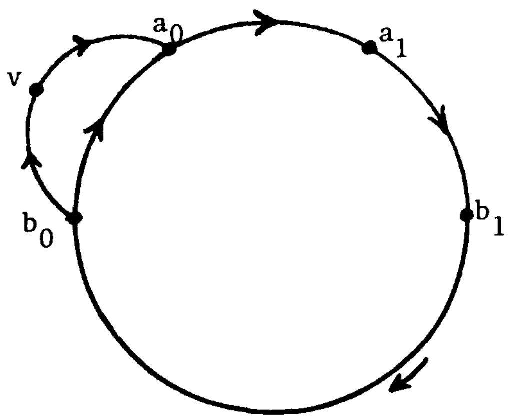
We denote this constructive technique (applied only to members of $\mathcal{A}_n$) as C_2 and write

$$T' = C_2(T).$$

II. Suppose T $\in \mathcal{B}_n$. Let $\underline{a_0}$, $\underline{a_1}$ be the vertices of maximal outdegree and the Hamiltonian circuit as shown.

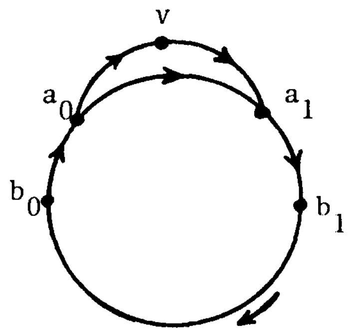


(i) Adjoin an (n+1)-st vertex $\underline{v}$ of maximal outdegree whose only incoming edge is from $\underline{b_0}$. We obtain a tournament T' on n+1 vertices containing



Again the fact that T' will have one and only one Hamiltonian circuit follows from the fact that the same property holds for T. Since $\underline{v}$ will be the only vertex of maximal degree, then T $\in \mathcal{A}_{n+1}$. We call this procedure C_3 and let T' = C_3(T).

(ii) Adjoin a vertex $\underline{v}$ of maximal outdegree n-1 whose only incoming edge is from a_0. The resulting tournament T' on n+1 vertices contains

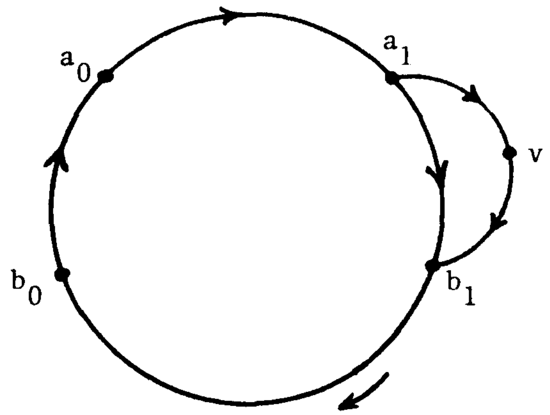


As before, T' has one and only one Hamiltonian circuit. Further, $T' \in \mathcal{B}_{n+1}$ since $\underline{v}$ and $\underline{a_0}$ have maximal outdegree. This technique we call C_4 and we write

$$T' = C_4(T)$$

in this case.

(iii) Adjoin a vertex $\underline{v}$ of maximal outdegree whose only incoming edge is from $\underline{a_1}$. This tournament T' contains



Again, $T' \in \mathcal{T}_{n+1}$ and in fact, $T' \in \mathcal{B}_{n+1}$ since $\underline{v}$ and $\underline{a_1}$ both have maximal outdegree.

This technique we denote as C_5 and thus

$$T' = C_5(T) \in \mathcal{B}_{n+1} \text{ for every } T \in \mathcal{B}_n.$$

This completes the graphical construction procedure.

Notation: In the following, we let

$$\mathcal{C}_i(\mathcal{T}_n) = \{C_i(T): \; T \in \mathcal{A}_n\} \qquad \text{for } i = 1,2$$
$$\mathcal{C}_i(\mathcal{T}_n) = \{C_i(T): \; T \in \mathcal{B}_n\} \qquad \text{for } i = 3,4,5$$

and

$$\mathcal{C} = \bigcup_{i=1}^{5} \mathcal{C}_i(\mathcal{T}_n) \; . \tag{1}$$

In other words $\mathcal{C}$ is the family of all tournaments which can be constructed from members of $\mathcal{T}_n$.

3. Distinctness of Elements of $\mathcal{C}$.

We wish to show that all members of $\mathcal{C}$ are nonisomorphic. Specifically, we will show that each element of $\mathcal{T}_{n+1}$ can be constructed from one and only one member of $\mathcal{T}_n$, and this must be done using a uniquely determined procedure C_i. Since each member of $\mathcal{C}$ is in $\mathcal{T}_{n+1}$, and no two members of $\mathcal{T}_{n+1}$ are isomorphic, this will mean that no two different members of $\mathcal{C}$ (i.e., constructed by different procedures C_i or from different members of $\mathcal{T}_n$) can be isomorphic.

LEMMA 1: Let $T \in \mathcal{T}_{n+1}$. Then there exists a unique $S \in \mathcal{T}_n$ and a uniquely determined procedure C_i such that

$$T = C_i(S) .$$

PROOF: If $T \in A_{n+1}$, then T has only one vertex $\underline{v}$ of maximal outdegree, so $\underline{v}$ must have been the vertex added in the constructive process. Therefore, the member of T_n from which T was constructed must have been the subgraph S of T gotten by deleting the vertex $\underline{v}$. If $S \in A_n$, then the constructive procedure must have been C_1. If $S \in B_n$, it must have been C_3.

If $T \in B_{n+1}$ then let $\underline{v_1},\underline{v_2}$ be the vertices of maximal outdegree. By Theorem 1 (ii), $\underline{v_1}$ and $\underline{v_2}$ must be adjacent so, without loss of generality, assume that the edge $v_1 \to v_2$ is in T. From the description of the construction (and specifically procedures C_2, C_4, C_5) we see that $\underline{v_2}$ must have been the vertex which was added. Thus, again, the member of T_n from which T was constructed must have been the subgraph S of T gotten by deleting $\underline{v_2}$. If $S \in A_n$ then the constructive procedure must have been C_2. If $S \in B_n$ and its vertices of maximal outdegree $\underline{v_1}$, $\underline{v_3}$ are joined by the edge $v_1 \to v_3$ then the procedure had to be C_4; if they are joined by $v_3 \to v_1$ then the procedure had to be C_5. □

THEOREM 2: The members of C are nonisomorphic.

PROOF: Follows from lemma 1, since each member of C is also a member of T_{n+1}. □

4. Equivalence of C and T_{n+1}.

Each object we construct from T_n is an element of T_{n+1}. Since the constructed objects are nonisomorphic, then

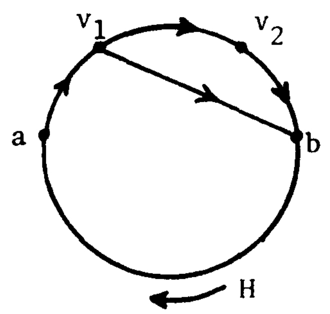
$$C \subseteq T_{n+1} .$$

We wish to show now that each member of $\mathcal{T}_{n+1}$ may be gotten by applying the construction to some member of $\mathcal{T}_n$.

<u>THEOREM 3</u>: $\mathcal{T}_{n+1} \subseteq \mathcal{C}$.

<u>PROOF</u>: Let $T \in \mathcal{T}_{n+1}$.

Case I: If $T \in \mathcal{B}_{n+1}$ with vertices $\underline{v_1}, \underline{v_2}$ of maximal outdegree and Hamiltonian circuit H, then T contains the following structure

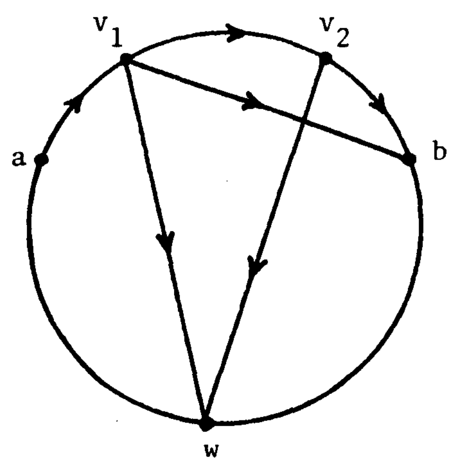


Consider the subgraph S gotten by deleting vertex $\underline{v_2}$ and all its incident edges. Then S is a tournament with at least one Hamiltonian circuit, namely

$$C = a, v_1, b, \ldots, a \ .$$

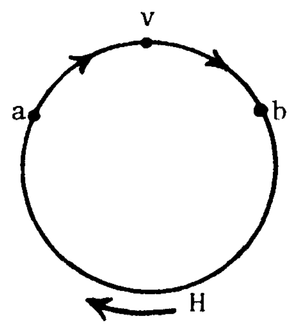
If we can show that C is the only Hamiltonian circuit of S, then it follows immediately that $S \in \mathcal{T}_n$ and that T can be constructed from S.

Suppose S has a second Hamiltonian circuit C_0 different from C.



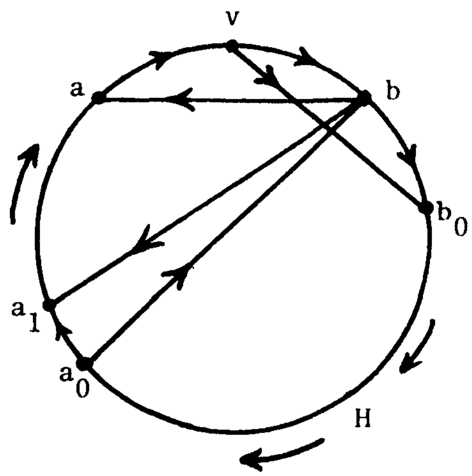
Then, for some vertex $\underline{w}$, $w \neq v_1$, in S we must have the edge $v_1 \to w$ in C_0. Now let C_0' be the circuit in T which is obtained by replacing the edge $v_1 \to w$ in C_0 by the two edges $v_1 \to v_2 \to w$. (Recall that the edge between v_2 and w must be directed towards w.) Further, it is clear that C_0 differing from C (in S) implies that C_0' will differ from H (in T). But this is not possible since H is unique. Thus our supposition that S has two Hamiltonian circuits must be false and the desired results follow.

Case II: Let $T \in A_{n+1}$ with $\underline{v}$ the vertex of maximum outdegree and H the unique Hamiltonian circuit.



Consider the subgraph S gotten by deleting $\underline{v}$ from T. Again S is a tournament. We must show that $S \in T_n$. Observe that if we knew that the edge $a \to b$ were in T then we would obviously have a Hamiltonian circuit in S; namely, the circuit gotten by replacing the edges $a \to v \to b$ in H by the edge $a \to b$.

In fact, we will prove that the edge $a \to b$ must be in T. Suppose otherwise, i.e., suppose that $b \to a$ is an edge of T.



Let $\underline{a_0}$ be the first vertex preceding $\underline{a}$ (in the Hamiltonian circuit H) for which the edge between $\underline{a_0}$ and $\underline{b}$ is directed towards $\underline{b}$. Thus if $a_0 \to a_1$ is an edge in H then the edge between $\underline{a_1}$ and $\underline{b}$ must be directed towards $\underline{a_1}$. Also if $b \to b_0$ is an edge of H then the edge between $\underline{v}$ and $\underline{b_0}$ is directed toward $\underline{b_0}$ since $\underline{v}$ has maximal outdegree. Therefore the circuit

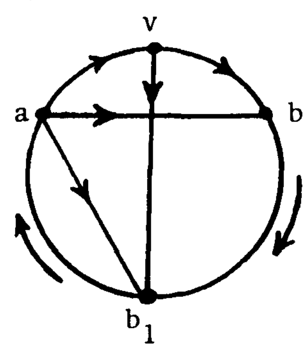
$$a_0 \to b \to a_1 \to \dots \to a \to v \to b_0 \to \dots \to a_0$$

(where the dots indicate we follow the circuit H) will be another Hamiltonian circuit in T, different from H. This is not possible, so we conclude that the edge $b \to a$ cannot be in T.

Thus, S has at least one Hamiltonian circuit:

$$C: \; a \to b \to b_0 \to \dots \to a \; .$$

Suppose S has a second Hamiltonian circuit C_0, different from C.



Let $a \to b_1$ be the edge of C_0 which is directed away from $\underline{a}$ ($\underline{b_1}$ may equal $\underline{b}$). If, in C_0, we replace $a \to b_1$ by the edges $a \to v \to b_1$ then we will have a Hamiltonian circuit in T which is different from H. This is impossible, so we conclude that we cannot have a second Hamiltonian circuit in S, i.e.,

$$S \in \mathcal{T}_n .$$

Thus if $T \in \mathcal{A}_{n+1}$ then T can be gotten by applying the construction to $S \in \mathcal{T}_n$. □

THEOREM 4: $\mathcal{C} = \mathcal{T}_{n+1}$.

In words, by applying the construction of section 2 to $\mathcal{T}_n$, we obtain $\mathcal{T}_{n+1}$ and each member of $\mathcal{T}_{n+1}$ is constructed in a uniquely determined fashion.

PROOF: Follows from (1), Lemma 1, and Theorem 3. □

5. Recurrence.

From the description of the construction we see that

$$\mathcal{A}_{n+1} = \mathcal{C}_1(\mathcal{T}_n) \uplus \mathcal{C}_3(\mathcal{T}_n)$$
$$\mathcal{B}_{n+1} = \mathcal{C}_2(\mathcal{T}_n) \uplus \mathcal{C}_4(\mathcal{T}_n) \uplus \mathcal{C}_5(\mathcal{T}_n)$$

where the unions are disjoint, and by Theorem 2

$$\begin{aligned} A_{n+1} &= |\mathcal{A}_{n+1}| = |\mathcal{C}_1(\mathcal{T}_n)| + |\mathcal{C}_3(\mathcal{T}_n)| \\ &= |\mathcal{A}_n| + |\mathcal{B}_n| \\ &= A_n + B_n \end{aligned} \qquad (2)$$

and

$$\begin{aligned} B_{n+1} &= |\mathcal{B}_{n+1}| = |C_2(T_n)| + |C_4(T_n)| + |C_5(T_n)| \\ &= |A_n| + |\mathcal{B}_n| + |\mathcal{B}_n| \\ &= A_n + 2B_n \qquad (3) \\ &= A_{n+1} + B_n \qquad (4) \end{aligned}$$

where the least equality follows by applying (2).

<u>THEOREM 5</u>: The sequences $\{A_n\}$ and $\{B_n\}$ both satisfy the recurrence

$$x_{n+2} = 3\, x_{n+1} - x_n, \qquad (n \geq 4)\ .$$

<u>PROOF</u>: From (2) and (4),

$$\begin{aligned} A_{n+2} - A_{n+1} &= B_{n+1} \\ &= A_{n+1} + B_n \\ &= A_{n+1} + (A_{n+1} - A_n) \\ &= 2\, A_{n+1} - A_n \end{aligned}$$

so that

$$A_{n+2} = 3\, A_{n+1} - A_n\ .$$

Also from (3) and (4)

$$\begin{aligned} B_{n+2} &= A_{n+1} + 2\, B_{n+1} \\ &= (B_{n+1} - B_n) + 2\, B_{n+1} \\ &= 3\, B_{n+1} - B_n\ . \end{aligned}$$

□

THEOREM 6: The sequence $\{T_n\}$ which counts the number of nonisomorphic tournaments on n vertices, which have a unique Hamiltonian circuit, satisfies

$$T_{n+2} = 3\, T_{n+1} - T_n \qquad (n \geq 4)\ .$$

PROOF: Follows from Theorem 5 since $T_n = A_n + B_n$ for every n. □

Bibliography

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