ON SIMILARITY AND INDEPENDENCE PROPERTIES OF COMPOSITE EDF GOODNESS-OF-FIT TESTS*

C. P. Quesenberry

Institute of Statistics
Mimeograph Series No. 1149
December 1977

ON SIMILARITY AND INDEPENDENCE PROPERTIES OF COMPOSITE EDF GOODNESS-OF-FIT TESTS*

C. P. Quesenberry

North Carolina State University

ABSTRACT

Many test statistics for classical simple goodness-of-fit hypothesis testing problems are distance measures between the distribution function of the null hypothesis distribution and the empirical distribution function - sometimes called EDF tests. If a composite parametric null hypothesis is considered in place of the simple null hypothesis, then a test statistic can be obtained from each EDF test by replacing the known distribution function of the simple problem by the Rao-Blackwell estimating distribution function. In this note we use known results to show that these Rao-Blackwell-EDF test statistics have distributions that do not depend upon parameter values, and that these tests are independent of a complete sufficient statistic for the parame-

---

*Work supported by NSF Research Grant MCS76-82652.

ters.

1. TESTING PROBLEMS AND EDF STATISTICS

Let $X_1, \ldots, X_n$ denote a sample from a parent distribution with distribution function (df) F. The classical simple goodness-of-fit problem is to test H_o: $F = F_o$ *vs* H_a: $F \neq F_o$, for F_o a completely specified df. Denote by F_e the empirical df, *i.e.*, $F_e(x) = (1/n)\sum_{i=1}^{n} I_{(-\infty,x]}(X_i)$, where I_A is an indicator function of the set (or interval) A. The empirical distribution function, EDF, goodness-of-fit test statistics (*cf.* Stephens (1974)) are of the form $d(F_o,F_e)$, where d is a distance function that maps pairs of df's to nonegative real numbers. The Kolmagorov-Smirnov, Cramer von Mises, Anderson-Darling and Watson U^2 statistics are some well-known EDF statistics.

The composite goodness-of-fit problem is to test H_o: $F \in \mathcal{F}$ *vs* H_a: $F \notin \mathcal{F}$, where $\mathcal{F} = \{F_\theta : \theta \in \Omega\}$ is a parametric class of df's. A common approach to constructing a test for this composite testing problem is to replace F_o in an EDF statistic $d(F_o, F_e)$ by an estimating df of F such as a ML estimator $\hat{F}$, or a Rao-Blackwell estimator $\tilde{F}$, to obtain statistics $d(\hat{F}, F_e)$ or $d(\tilde{F}, F_e)$. It has been found empirically (*cf.* Stephens (1974), or Dyer (1974)) that some of these ML-EDF tests have better power properties than the simple EDF test, even when $F = F_o$ is known.

2. INVARIANCE PROPERTIES OF DF ESTIMATORS

David and Johnson (1948) showed that for a large collection of location-scale families $\mathcal{F}$ that the ML estimator $\hat{F}$ of F is invariant under location-scale transformations, and therefore that the distribution of $\hat{F}$ does not depend upon parameter values.

We next show that Rao-Blackwell, RB, estimating df's also have important invariance properties for a large collection of classes $\mathcal{F}$. In this we use terminology and notation from Quesenberry and Starbuck (1976), QS, section 2; which we now summarize.

Let $\mathcal{X}$ denote a Borel set of real numbers, $\mathcal{Q}$ the Borel subsets of $\mathcal{X}$; $X = (X_1,\ldots,X_n)$ a vector of i.i.d. continuous r.v.'s each distributed according to $P \in \mathcal{P} = \{P_\theta;\ \theta \in \Omega\}$, for $\theta = (\theta_1,\ldots,\theta_k)$ a k-dimensional parameter vector. Also, $(\mathcal{X}^n, \mathcal{Q}^n) = (\mathcal{X}\times \cdots \times \mathcal{X}, \mathcal{Q}\times \cdots\times \mathcal{Q})$ denotes the sample space, and T a sufficient statistic for θ.

Further, let $g:\mathcal{X} \to \mathcal{X}$ be a one-to-one onto transformation and g^n be the corresponding transformation of $\mathcal{X}^n$ onto $\mathcal{X}^n$ defined by $g^n(x_1,\ldots,x_n) = (g(x_1),\ldots,g(x_n))$. For each g^n, suppose there exists a function $\overline{g}:\Omega \to \Omega$ such that $P_{\overline{g}\theta}(X \in g^n A) = P_\theta(X \in A)$ for every $A \in \mathcal{Q}^n$. Let G denote a transformation group of g functions on $\mathcal{X}$, G^n the corresponding group on $\mathcal{X}^n$ and $\overline{G}$ the corresponding induced group on Ω. A transformation group on a space is said to be <u>transitive</u> if its maximal invariant is constant on the space.

Let $\tilde{F}(x) = F(x|T) = P(X_1 \le x|T)$ and we obtain immediately from Lemma 2.2 of QS the following special case.

<u>Lemma 1</u>. If G is a transformation group of strictly increasing functions on $\mathcal{X}$ that induces a transitive group $\overline{G}$ on Ω, then

$$F(x|T(x_1,\ldots,x_n)) = F(gx|T(gx_1,\ldots,gx_n)) \qquad (1)$$

a.s. $\mathcal{P}^n$ $\forall$ $g \in G$.

The function $\tilde{F}(x) = F(x|T)$ is the Rao-Blackwell estimating df of $F(x)$ based on the sufficient statistic T.

<u>Example 1</u>. Let $\mathcal{P}$ be the class of univariate normal distributions with parameters μ and σ^2, and $\Omega = \{(\mu,\sigma^2)\colon -\infty < \mu < \infty,\ \sigma^2 > 0\}$. Then let $G = \{ax + b;\ a > 0,\ -\infty < b < \infty\}$, and $\overline{G}$ is given by $\overline{G} = \{\overline{g};\ \overline{g}(\mu, \sigma^2) = (a\mu + b,\ a^2\sigma^2)\}$. Here a sufficient statistic is given by $T = (\overline{X},\ s^2)$ for $\overline{X} = (X_1 + \ldots + X_n)/n$ and $s^2 = \sum_{i=1}^{n}(X_i - \overline{X})^2/n$. Then the conditional d.f. of (1) is given by (<u>cf</u>. Lieberman and Resnikoff (1955))

$$F(x|T) = G_{n-2}\{(n-2)^{\frac{1}{2}}(x-\overline{X})/[(n-1)s^2 - (x-\overline{X})^2]^{\frac{1}{2}}\}$$

$$\cdot I(x;\overline{X} - (n-1)^{\frac{1}{2}}s,\ \overline{X} + (n-1)^{\frac{1}{2}}s) + I(x;\overline{X} + (n-1)^{\frac{1}{2}}s,\ \infty) \qquad (2)$$

where $I(x;\ a,\ b) = I_{(a,b)}(x)$, and G_ν is the Student-t distribution function with ν degrees of freedom. It is easily verified directly that $F(x|T)$ of equation (2) has the invariance property of equation (1).

Next, we consider a different sufficient statistic. Let $T = (X_{(1)}, \ldots, X_{(n)})$ be the vector of order statistics, and otherwise the problem remain unchanged. Then

$$F_e(x) = \tilde{F}(x) = P(X_1 \leq x|T),$$

and the empirical df has the invariance property of (1), as is easily verified directly, also.

3. APPLICATION TO SIMILAR TESTS

Recall that a test φ is similar α on $F = \{F_\theta;\ \theta \in \Omega\}$ if $E_\theta(\varphi) = \alpha\ \forall\ \theta \in \Omega$. We have the following from sections 1 and 2.

Theorem 1. If G is a transformation group of strictly increasing functions an $\mathcal{X}$ that induces a transitive group G on Ω (or $\mathcal{J}$), then any RB - EDF test statistic $d(\tilde{F}, F_e)$ gives a similar test on $\mathcal{J}$.

Note particularly that this property does not depend upon the function d, but is due to the property given in (1) which is satisfied by all RB df's, and in particular by both $\tilde{F}$ and F_e. This property of $d(\tilde{F}, F_e)$ was demonstrated for the case of d a Kolmagorov-Smirnov distance and $\mathcal{J}$ either a two-parameter normal or scale parameter exponential by Srinivasan (1971). See Schafer, et al (1972), and Moore (1973) for some corrections and additions to Srinivasan's results.

In the foregoing the statistic T has only been taken to be a sufficient statistic for $\mathcal{J}$. If T is also a complete statistic, then $\tilde{F}$ above is well-known to be a minimum variance unbiased, MVU, estimator of F. For this case we have immediately an important independence result. The next theorem follows from Theorem 1 and a Theorem of Basu (1955, 1960).

Theorem 2. If the conditions for Theorem 1 are satisfied and T is complete and sufficient, then every MVU-EDF statistic $d(\tilde{F}, F_e)$

is independent of T.

The independence of the goodness-of-fit test statistic and the statistic that is complete and sufficient for the parameters under the null hypothesis is important for applications. From this independence it follows that a given sample can be used to make a goodness-of-fit test, and then to make inferences about the parameters, and the overall error rates assessed, under the null hypothesis. It should also be observed that for those classes of distributions for which the ML estimator $\hat{F}$ is also invariant under the characterizing group G of transformations (cf. David and Johnson (1948)), an independence theorem similar to Theorem 2 holds for ML-EDF statistics and the complete sufficient statistic T.

BIBLIOGRAPHY

Basu, D. (1955, 1960). On statistics independent of a complete sufficient statistic. Sankhyā 15, 337-80 (1955); 20, 223-6 (1960).

David, F. N. and Johnson, N. L. (1948). The probability integral transformation when parameters are estimated from the sample. Biometrika 35, 182-92.

Dyer, A. R. (1974). Comparison of tests for normality with a cautionary note. Biometrika 61, 185-89.

Lieberman, G. J. and Resnikoff, G. J. (1955). Sampling plans for inspection by variables. J. Amer. Statist. Assoc. 50, 457-516.

Moore, D. S. (1973). A note on Srinivasan's goodness-of-fit test. Biometrika 60, 209-11.

Quesenberry, C. P. and Starbuck, R. R. (1975). On optimal tests for separate hypotheses and conditional probability integral transformations. Comm. Statist. Theor. Meth. 6, 507-24.

Schafer, R. E., Finkelstein, M. J. and Collins, J. (1972). On a goodness-of-fit test for the exponential distribution with mean unknown. Biometrika 59, 222-4.

Srinivasan, R. (1970). An approach to testing the goodness-of-fit of incompletely specified distributions. Biometrika 57, 605-11.

Stephens, M. A. (1974). EDF statistics for goodness-of-fit and some comparisons. Jour. Am. Stat. Assn. 69, 730-7.