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INVARIANCE THEOREMS FOR CENTERED SEQUENCES NORMED BY SUMS OF SQUARES

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ABSTRACT

Let $S_n = \sum_1^n f_k$, $L_n = \sum_1^n f_k^2$ where $|f_n| \le 1$, $E\{f_n | f_1, f_{n-1}\} = 0$ and $L_n \to \infty$ a.e. Then

$$\frac{S_n}{\sqrt{L_n}} \overset{d}{\to} n(0,1)$$

and

$$\overline{\lim_n} \frac{S_n}{\sqrt{2L_n \log \log L_n}} = 1 \quad \text{a.e.}$$

The methods leading to these results essentially involve strong inequalities of independent statistical interest. These inequalities are modifications of results due to Freedman.

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Introduction

Let (Ω, B, P) be a probability space supporting an increasing sequence of sigma-fields $\{B_n\}$ and a sequence of random variables $\{f_n\}$ where f_n is B_n-measurable, i.e., where $\{f_n, B_n\}$ is adapted. Assume $|f_n| \le 1$, $E\{f_n | B_{n-1}\} = 0$. Let

$$S_n = \sum_1^n f_k \; , \quad L_n = \sum_1^n f_k^2 \; .$$

Then, under the assumption that $L_n \to \infty$ a.e., we have

$$\text{(A)} \qquad \frac{S_n}{\sqrt{L_n}} \xrightarrow{d} n(0,1)$$

and

$$\text{(B)} \qquad \overline{\lim_n} \frac{S_n}{\sqrt{2 L_n \log \log L_n}} = 1 \quad \text{a.e.}$$

Result (A) is a special case of the following: Let $\alpha > 0$ and let X_α be the process whose paths are the linear interpolations of the points $(\alpha^2 L_n, \alpha S_n)$. Then

$$\text{(C)} \qquad X_\alpha \xrightarrow{d} W \quad \text{as} \quad \alpha \to 0$$

where W is Brownian Motion. I obtained these limit theorems by combining results from Freedman [ ] and Freedman [ ]. In Freedman [ ] one finds

$$\text{(D)} \qquad Y_\alpha \xrightarrow{d} W \quad \text{as} \quad \alpha \to 0$$

and

$$\text{(E)} \qquad \overline{\lim_n} \frac{S_n}{\sqrt{2 T_n \log \log T_n}} = 1 \quad \text{a.e.}$$

where $T_n = \sum_1^n E\{f_k^2|B_{k-1}\}$ and $T_n \to \infty$ a.e., and where Y_α is the process whose paths are linear interpolations of $(\alpha^2 T_n, \alpha S_n)$. In Freedman [ ] one finds

$$\text{(F)} \qquad \frac{C_n}{M_n} \to 1 \quad \text{a.e.}$$

where $C_n = \sum_1^n g_k$, $M_n = \sum_1^n E\{g_k|B_{k-1}\}$, $0 \le g_n \le 1$ and $M_n \to \infty$ a.e. My approach in deriving (B) and (C) from (D), (E), (F) consists in using f_k^2 for g_k. The body of the present paper is a combination and rearrangement of Freedman's papers, with statements of theorems and methods of proof modified either mildly or severely. My principal concern has not been the limit theorems but rather the inequalities which lead to the limit theorems, for these inequalities given considerable information concerning the distribution of partial sums in terms of the partial sums of squares and may therefor prove useful in statistical contexts where independence and knowledge of conditionals is not assumed. For example, in Section (IV) I prove that for $\theta \le 1/3$ and reasonably large N:

$$(*) \qquad P\left\{S_n \ge (1+\theta)\sqrt{2\, L_n \log\log L_n}\ , \quad \text{some} \quad L_n \ge N\right\} \le \frac{3}{\theta^{7/2}(\log N)^\theta}$$

Such inequalities are examples of the general form

$$P\{S_n \ge \phi(L_n)\ , \quad \text{some} \quad L_n \ge N\} \le \tilde{\phi}(N)$$

where $\phi, \tilde{\phi}$ are specified functions. The latter type of inequality is the modification called for, when independence and knowledge of conditionals is not assumed, of the inequality of type

$$P\{S_n \geq \phi(n) \ , \quad \text{some} \quad n \geq N\} \leq \tilde{\phi}(N)$$

where S_n is the sum of centered i.i.d.'s with variance 1.

Inequality (*) occurs in my proof that

$$\overline{\lim_n} \frac{S_n}{\sqrt{2 L_n \log \log L_n}} \leq 1 \ .$$

This proof differs considerably from Freedman's proof that

$$\overline{\lim_n} \frac{S_n}{\sqrt{2 T_n \log \log T_n}} \leq 1$$

and is an instance of a proof motivated by possible statistical applications, namely such as are provided in Darling, Robbins [ ]. Although statistical applications are not directly considered in the present paper, a path toward such applications is indicated by the inequalities and will be developed in a subsequent article.

The result (C) is not new — it appears in McLeish [ ], Rootzen [ ] and Drogin [ ], although these sources were unknown to me prior to completion of the draft of my paper. In fact, all these sources present a somewhat more general result. The methods used in these papers, however, are not appropriate for the determination of strong inequalities, and it is strong inequalities which form the object of the present paper. Result (B) appears to be new.

Finally, a first passage time is considered in the remarks at the close of Section (IV). The methods of this paper are shown, in this particular instance, to be useful for finding upper bounds on expectations of passage times. These methods will be extended in a later paper.

(I) Results for Positive Variables

Let $\{g_n, B_n, n \geq 1\}$ be an adapted sequence of non-negative random variables on (Ω, B, P). For convenience, set $B_0 = \{\phi, \Omega\}$. Assume $g_n \leq 1$ for all n. Set

$$C_n = \sum_1^n g_k$$

$$M_n = \sum_1^n E\{g_k | B_{k-1}\}$$

Set

$$F(\lambda) = 1-e^{-\lambda} , \quad \lambda \geq 0$$

$$G(\lambda) = e^{\lambda}-1 , \quad \lambda \geq 0 .$$

We define

$$Q_n = e^{F(\lambda)M_n - \lambda C_n}$$

$$R_n = e^{\lambda C_n - G(\lambda)M_n} .$$

Theorem (1.1). {Freedman [ ]}

$\{Q_n, B_n\}$ and $\{R_n, B_n\}$ are supermartingales.

Proof. We show that for $0 \leq g \leq 1$ and $A \subset B$, we have

$$E\{e^{-\lambda g} | A\} \leq e^{-F(\lambda)E\{g|A\}}$$

and

$$E\{e^{\lambda g} | A\} \leq e^{G(\lambda)E\{g|A\}} .$$

Since, for $x \geq 0$;

$$\frac{1-e^{-\lambda x}}{x} \text{ decreases in } x$$

while

$$\frac{e^{\lambda x}-1}{x} \text{ increases in } x ,$$

we have

$$e^{-\lambda g} \le 1 - F(\lambda) g$$

and

$$e^{\lambda g} \le 1 + G(\lambda) g .$$

Take conditionals on both sides and use the inequalities

$$1 - x \le e^{-x}$$

$$1 + x \le e^{x}$$

Q.E.D.

<u>Corollary (1.1)</u>. Let $C_\infty = \sum_1^\infty g_n$, $M_\infty = \sum_1^\infty E\{g_n | B_{n-1}\}$. Then,

$$M_\infty = \infty \text{ a.e.} \iff C_\infty = \infty \text{ a.e.}$$

<u>Proof</u>. By optional stopping, for each proper stopping time τ:

$$1 \ge \int e^{F(\lambda)M_\tau - \lambda G_\tau}$$

and

$$1 \ge \int e^{\lambda G_\lambda - G(\lambda)M_\tau} .$$

Suppose $M_\infty = \infty$ a.e. Define

$$\tau = \text{first } n \quad M_n \ge a .$$

Note that τ is proper and $\tau \to \infty$ as $a \to \infty$. Fix $b > 0$. From the first inequality above we have

$$1 \ge \int_{\{C_\tau \le b\}} e^{F(\lambda)a - \lambda b}$$

so that

$$P\{C_\tau \le b\} \le e^{\lambda b - F(\lambda)a}$$

and letting $a \to \infty$ we have $P\{C_\infty \le b\} = 0$, all finite b, consequently $C_\infty = \infty$ a.e. The converse is established in the same manner. Q.E.D.

Corollary (1,2). Let $C_\infty = M_\infty = \infty$ a.e. Then for all fixed $t \ge 0$, $\theta \le \frac{1}{2}$:

(a) $$P\left\{\frac{M_n}{C_n} \ge 1+\theta \text{ , some } C_n \ge t\right\} \le e^{-\frac{\theta^2}{3} t}$$

(b) $$P\left\{\frac{M_n}{C_n} \le 1-\theta \text{ , some } C_n \ge t\right\} \le e^{-\frac{\theta^2}{2} t}$$

(c) $$P\left\{\frac{M_n}{C_n} \ge 1+\theta \text{ , some } M_n \ge t\right\} \le e^{-\frac{\theta^2}{8} t}$$

(d) $$P\left\{\frac{M_n}{C_n} \le 1-\theta \text{ , some } M_n \ge t\right\} \le e^{-\frac{\theta^2}{2} t}$$

Further, for all $a > 0$, $\varepsilon > 0$ and for all proper stopping times τ:

(e) $$P\{C_\tau \ge (1+\varepsilon)a,\ M_\tau \le a\} \le e^{\frac{-\varepsilon^2}{2(1+\varepsilon)} a}$$

(f) $$P\{C_\tau \le a,\ M_\tau \ge (1+\varepsilon)a\} \le e^{\frac{-\varepsilon^2}{2(1+\varepsilon)} a}$$

Proof.

(a) Let

$$\rho = \text{first } \eta \ni C_\eta \ge t$$

$$\sigma = \text{first } \eta \ni C_\eta \ge s \text{ , } s > t$$

$$\tau = \begin{cases} \text{first } \eta \in [\rho,\sigma] \text{ with } M_\eta \ge (1+\theta)C_\eta \\ \sigma \text{ if no such } \eta \text{ occurs} \end{cases} .$$

Then

$$1 \geq \int_{\{\tau<\eta\}} e^{F(\lambda)M_\tau - \lambda C_\tau} \geq \int_{\{\tau<\eta\}} e^{(F(\lambda)(1+\theta)-\lambda)C_\tau}$$

$$\geq e^{(F(\lambda)(1+\theta)-\lambda)t} \quad \text{provided} \quad \frac{F(\lambda)}{\lambda} \geq \frac{1}{1+\theta} .$$

Under such circumstances

$$P\{\tau > \eta\} \leq e^{(\lambda-(1+\theta)F(\lambda))t}$$

The right side minimizes for $\lambda = \ln(1+\theta)$, an allowable value for θ.

then using log expansions:

$$P\{\tau < \eta\} \leq e^{(-\frac{\theta^2}{2}+\frac{\theta^3}{3})t} \leq e^{-\frac{\theta^2}{3}t} \quad \text{for} \quad \theta \leq \tfrac{1}{2} .$$

Parts (b), (c), (d) are proven in the same fashion.

(e) For all $a > 0$, $b > 0$, $e^{\lambda(C_n-b)-G(\lambda)(M_n-a)}$ is a supermartingale so that for τ proper we have

$$e^{G(\lambda)a-\lambda b} \geq \int_{\{C_\tau \geq b, M_\tau \leq a\}} e^{\lambda(C_\tau-b)-G(\lambda)(M_\tau-a)}$$

$$\geq P\{C_\tau \geq b,\ M_\tau \leq a\}$$

Now let's minimize the left, letting $b = (1+\varepsilon)a$, and we have

$$P\{C_\tau \geq (1+\varepsilon)a,\ M_\tau \leq a\} \leq (1+\varepsilon)^{-(1+\varepsilon)a} e^{\varepsilon a}$$

$$= e^{\varepsilon a-(1+\varepsilon)a \log(1+\varepsilon)}$$

Now

$$(1+\varepsilon)\log(1+\varepsilon) \geq \varepsilon + \frac{\varepsilon^2}{2(1+\varepsilon)} ,$$

so that

$$P\{C_\tau \ge (1+\varepsilon)a,\ M_\tau \le a\} \le e^{-\frac{a\varepsilon^2}{2(1+\varepsilon)}}$$

(f) is proven in the same way as (e), using the supermartingale $e^{F(\lambda)(M_n-b)-\lambda(C_n-a)}$.

Corollary (1.3). Assume $M_\infty = C_\infty = \infty$ a.e. Then

$$\lim_{n\to\infty} \frac{M_n}{C_n} = 1 \quad \text{a.e.}$$

Proof: We show for all $\theta > 0$ that

$$P\left\{\frac{M_n}{C_n} \ge 1+\theta \quad \text{i.o.}\right\} = 0 .$$

Now the probability above is the same as

$$\lim_{t\to\infty} P\left\{\frac{M_n}{C_n} \ge 1+\theta, \quad \text{some} \quad C_n \ge t\right\} \le \lim_{t\to\infty} e^{-\frac{\theta^2}{3}t} = 0 .$$

In the same manner

$$P\left\{\frac{M_n}{C_n} \le 1-\theta \quad \text{i.o.}\right\} = 0 .$$

Q.E.D.

(II) Martingale Difference Sequences

We assume throughout that $\{f_n, B_n\}$ is adapted with $|f_n| \le 1$ and $E\{f_n|B_{n-1}\} = 0$. We define

$$S_n = \sum_1^n f_k$$

$$V_n = E\{f_n^2|B_{n-1}\}$$

$$T_n = \sum_1^n v_k^2$$

$$L_n = \sum_1^n f_k^2$$

In all that follows we assume that

$$T_\infty = L_\infty = \infty \quad \text{a.e.}$$

Let

$$K(\lambda) = e^\lambda - 1 - \lambda \ , \quad \lambda \ge 0$$

$$H(\lambda) = e^{-\lambda} - 1 + \lambda \ , \quad \lambda \ge 0 \ .$$

Theorem (2.1). {Freedman [ ]}

$$Q_n = e^{\lambda S_n - K(\lambda) T_n} \quad \text{is a supermartingale}$$

$$R_n = e^{\lambda S_n - H(\lambda) T_n} \quad \text{is a submartingale} \ .$$

Proof: It suffices that for f with $|f| \le 1$ and for a σ-field A with $E\{f|A\} = 0$, one has

(a) $$E\{e^{\lambda f}|A\} \le e^{K(\lambda)E\{f^2|A\}}$$

and

(b) $$E\{e^{\lambda f}|A\} \ge e^{H(\lambda)E\{f^2|A\}}$$

To establish (a), it suffices that

$$e^{\lambda x} \le 1 + \lambda x + K(\lambda)x^2 .$$

This inequality obtains for $x = 0$. To extend to all x, it suffices that the derivative on the left in λ is dominated by the derivative on the right in λ; this follows from inequalities established in Theorem (1.1).

The proof of (b) is more involved; we sketch a variant of Freedman's method. Assuming regular conditionals, replacing $P\{\cdot|A\}$ by P, we show that

$$\int e^{\lambda f} dP \ge e^{H(\lambda)\int f^2 dP} .$$

We first assume that for $b > 0$

$$P\{f = b\} = p , \quad P\{f = -1\} = q$$

where $p + q = 1$, $pb - q = 0$. Some algebra then determines that it suffices that

$$1 + be^{-(1+b)\lambda} \ge (1+b)e^{(e^{-\lambda}-1)b} .$$

This obtains for $\lambda = 0$ and then for all λ from an examination of the derivatives in λ. Next, let $0 \le c \le 1$, $b \ge 0$ and assume

$$P\{f=b\} = p \ , \quad P\{f=-c\} = q \ ,$$

$$p+q = 1 \ , \quad pb - qc = 0 \ .$$

Then

$$\int e^{\lambda f} dP = \int e^{(\lambda c)\frac{f}{c}} dP \geq e^{\frac{H(\lambda c)}{c^2}\int f^2 dP}$$

$$\geq e^{H(\lambda)\int f^2 dP}$$

where this last inequality is obtained from $H(\lambda c) \geq c^2 H(\lambda)$ for $c \leq 1$ which is once more established by examining the case $\lambda = 0$, taking derivatives and using Theorem (1.1). The next cast involves f where

$$P\{f = a_k\} = p_k \qquad k \leq N$$

$$P\{f = -b_k\} = q_k \ , \quad q_k \leq M$$

where $1 \geq a_k \geq 0$, $1 \geq b_k \geq 0$, $\sum p_k a_k - \sum q_k b_k = 0$ and $\sum p_k + \sum q_k = 1$. It can be shown [see Breiman [ ]] that any such f can have its values and associated probabilities represented in the following manner

$$P\{f = \tilde{a}_k\} = \tilde{p}_k \qquad k \leq \tilde{N}$$

$$P\{f = -\tilde{b}_k\} = \tilde{q}_k \qquad k \leq \tilde{N}$$

where

$$\tilde{p}_k \tilde{a}_k - \tilde{q}_k \tilde{b}_k = 0 \ , \quad \text{all} \quad k \leq \tilde{N} \ .$$

We then have, suppressing the tildas:

$$\int e^{\lambda f} dP = \sum (p_k + q_k) E\{e^{\lambda f} | f = a_k \vee f = -b_k\}$$

$$\geq \sum (p_k + q_k) e^{H(\lambda) E\{f^2 | f = a_k \vee f = -b_k\}}$$

$$\geq e^{H(\lambda) \sum (p_k + q_k) E\{f^2 | f = a_k \vee f = -b_k\}}$$

$$= e^{H(\lambda) \int f^2 dP} ,$$

the last inequality following from the convexity of e^x. The final case involves arbitrary measurable f where $-1 \leq f \leq 1$. Let $\{A_n\}$ be an increasing sequence of finite σ-fields such that f is A_∞-measurable where $A_\infty = \vee A_n$. Then

$$\int e^{\lambda f} dP = \lim_{n \to \infty} \int e^{\lambda E\{f|A_n\}} dP$$

$$\geq \lim_{n \to \infty} e^{H(\lambda) \int E^2\{f|A_n\} dP} = e^{H(\lambda) \int f^2 dP} .$$

Q.E.D.

<u>Theorem (2.2)</u>. Let τ be any proper stopping time for which either

$$T_\eta \leq t \quad \text{for} \quad \eta \leq \tau$$

or

$$S_\eta \leq t \quad \text{for} \quad \eta \leq \tau$$

some fixed positive t. Then

(a) $$\int Q_\tau \leq 1$$

and

$$\int R_\tau \geq 1 .$$

Proof: (a) follows from the fact that $\{Q_\eta\}$ is a supermartingale, (b) will follow from the fact that τ is regular, i.e.,

$$\int R_\tau < \infty$$

and

$$\lim_{\eta\to\infty} \int_{\{\tau>\eta\}} R_\eta = 0 .$$

When $\eta \le \tau \Rightarrow T_\eta \le t$, we argue as follows

$$\int R_\tau \le \int Q_\tau \cdot e^{(K(\lambda)-H(\lambda))t} = e^{(K(\lambda)-H(\lambda))t} \int Q_\tau$$

$$\le e^{(K(\lambda)-H(\lambda))t} < \infty$$

and

$$\int_{\{\tau>\eta\}} R_\eta = \int_{\{\tau>\eta\}} e^{\lambda S_\eta - \frac{1}{2}K(2\lambda)T_\eta} \cdot e^{(\frac{1}{2}K(2\lambda)-H(\lambda))T_\eta}$$

$$\le e^{(\frac{1}{2}K(2\lambda)-H(\lambda))t} \sqrt{\int e^{(2\lambda)S_\eta - K(2\lambda)T_\eta}} \sqrt{P\{\tau>\eta\}}$$

$$\le e^{(\frac{1}{2}K(2\lambda)-H(\lambda))t} \sqrt{P\{\tau>\eta\}} ;$$

thus, τ is regular. The regularity of τ when $\eta \le \tau \Rightarrow S_\eta \le t$ is obvious. Q.E.D.

Theorem (2.3). {Freedman [ ] modified}

Let τ,β,ρ be proper stopping times where

(1) $$\beta \ge \rho \quad\text{and}\quad \sup_{\rho\le\eta\le\beta} S_\eta - S_\rho \in [a,b]$$

(2) $$\tau \ge \rho \quad\text{and}\quad T_\tau - T_\rho \in [a,b].$$

where $a < b$ are positive numbers.

Let H^{-1}, K^{-1} be the inverses of H, K respectively. Then

$$\text{(A)}\qquad e^{H(\lambda)a} \le E\left\{e^{\lambda(S_\tau - S_\rho)} \mid B_\rho\right\} \le e^{K(\lambda)b}$$

$$\text{(B)}\qquad e^{-H^{-1}(\lambda)b} \le E\left\{e^{-\lambda(T_\beta - T_\rho)} \mid B_\rho\right\} \le e^{-K^{-1}(\lambda)a}$$

Proof: Follows immediately from the last theorem when $\rho \equiv 0$. For the general case, replace f_n with $f_{\rho+n}$, B_n with $B_{\rho+n}$. Q.E.D.

Remarks (2.1). Theorem (2.3A) may be replaced by

$$e^{H(|\lambda|)a} \le E\left\{e^{\lambda(S_\tau - S_\rho)} \mid B_\rho\right\} \le e^{K(|\lambda|)b} ,$$

valid for all λ, whenever τ is a stopping time depending only on the sequence $\{T_\eta\}$, for the negative values for S_τ may be treated as positive values by interchanging $-f_n$ for f_n. This observation will be used in Section (III).

Corollary (2.1). Let τ, β, ρ be stopping times where

$$\text{(1)}\qquad \beta \ge \rho \quad\text{and}\quad \sup_{\rho \le \eta \le \beta} S_\eta - S_\rho \in [a,b]$$

$$\text{(2)}\qquad \tau \ge \rho \quad\text{and}\quad T_\tau - T_\rho \in [a,b] .$$

Then

$$\text{(A)}\qquad e^{\frac{\lambda^2 a}{2(1+\lambda)}} \le E\left\{e^{\lambda(S_\tau - S_\rho)} \mid B_\rho\right\} \le e^{K(\lambda)b}$$

$$e^{-\left(\sqrt{2\lambda+\lambda^2} + \lambda\right)b} \le E\left\{e^{-\lambda(T_\beta - T_\rho)} \mid B_\rho\right\} \le e^{-\sqrt{2\lambda}\, a}$$

Proof: It is easily seen that

$$\frac{\lambda^2}{2(1+\lambda)} \le H(\lambda) \le \frac{\lambda^2}{2} \quad \text{and} \quad \frac{\lambda^2}{2} \le K(\lambda)$$

For part B, invert the monotone functions $\frac{\lambda^2}{2}$ and $\frac{\lambda^2}{2(1+\lambda)}$. Q.E.D.

Corollary (2.2). {Freedman [ ]}

Let ρ be a proper stopping time. Let $\tau \ge \rho$ be defined with

$$\tau = \sup \eta \ge \rho \ni T_\eta - T_\rho \le b \ .$$

Then

(A)
$$P\left\{\sup_{\rho\le\eta\le\tau} S_\eta - S_\rho \ge a \mid B_\rho\right\} \le e^{\frac{-a^2}{2(a+b)}}$$

and

$$P\left\{\sup_{\rho\le\eta\le\tau} |S_\eta - S_\rho| \ge a \mid B_\rho\right\} \le 2e^{\frac{-a^2}{2(a+b)}} \ .$$

Proof: It suffices to consider $\rho \equiv 0$ as in the last theorem. Further (2) follows from (1) by the substitution of $-f_k$ for f_k. Thus, it is sufficient to show that

$$P\left\{\sup_{\eta\le\tau} S_\eta \ge a\right\} \le e^{\frac{-a^2}{2(a+b)}}$$

where $\tau = \sup \eta \ni T_\eta \le b$. From the last theorem, with $\tau_* =$ first $\eta \le \tau$ with $S_\eta \ge a$, we have

$$\int e^{\lambda S_{\tau_*}} \le e^{K(\lambda)b} \ , \quad \text{from which}$$

$$P\left\{\sup_{\eta\le\tau} S_\eta \ge a\right\} \le e^{K(\lambda)b-\lambda(a)} \ .$$

The result now follows by minimization of the right side in λ. Q.E.D.

Corollary (2.3).

$$P\left\{\sup_{\eta} S_\eta = \infty\right\} = P\left\{\inf_{\eta} S_\eta = -\infty\right\} = 1 .$$

Proof: We prove the case $P\left\{\sup_{\eta} S_\eta = \infty\right\}$, the other case following from symmetry. Define

$$\tau_0 = \sup \eta \quad T_\eta \le b$$

$$\tau_1 = \inf \eta \quad S_\eta \ge a$$

$$\beta = \tau_0 \wedge \tau_1 .$$

Since β is proper and since $\sup_{\eta \le \beta} S_\eta \le a+1$, we can use Corollary (2.1) for the inequality

$$e^{-\left(\sqrt{2\lambda+\lambda^2}+\lambda\right)\cdot(a+1)} \le \int e^{-\lambda T_\beta}$$

$$\le P\left\{T_\beta < b\right\} + e^{-\lambda b} P\left\{T_\beta \ge b\right\} \le P\left\{T_\beta < b\right\} + e^{-\lambda b}.$$

Now let $b \to \infty$, and then set $\lambda = 0$. Q.E.D.

Remarks (2.2). This last result allows us to define proper stopping times

$$\tau = \inf \eta \quad S_\eta \ge a .$$

These stopping times play an important role in Freedman [ ]; in particular, Freedman's proof of the law of the iterated logarithm involves a complex sequence of computations employing such stopping times — see his Corollary 1.10 and its proof. Unfortunately I haven't been able to simplify his proof, I'll merely quote his result as my next corollary and borrow on it later in

Section IV. It is interesting, however, that a fairly simple computation yields an inequality which "almost" gives the upper half of the iterated logarithm; this inequality will be presented with proof as Corollary (2.5). In Section IV I'll sketch a proof of the "almost" LIL that it produces.

Corollary (2.4). {Freedman [ ]}

Given $\alpha < \frac{1}{3}$, $\frac{a}{b} < \frac{\alpha^2}{9}$, $\frac{a^2}{b} > \frac{16}{\alpha^2}\log\left(\frac{64}{\alpha^2}\right)$, then, for ρ proper, and for τ defined by $\tau = \text{last } \eta \geq \rho \ni T_\eta - T_\rho \leq b$, we have

$$P\left\{\sup_{\rho\leq\eta\leq\tau} S_\eta - S_\rho \geq a \mid B_\rho\right\} \geq \frac{1}{2}\, e^{-\frac{1}{2}(1+4\alpha)\frac{a^2}{b}} .$$

Corollary (2.5). Let ρ be proper, let

$$\tau = \text{last } \eta \geq \rho \ni T_\eta - T_\rho \leq b .$$

Then, for all α with $\alpha < \frac{1}{4}$, all $\frac{a^2}{b} \geq \frac{1}{\alpha}$, $\frac{a^3}{b^2} \leq \frac{1}{25}$:

$$P\left\{\sup_{\rho\leq\eta\leq\tau} S_\eta - S_\rho \geq a \mid B_\rho\right\} \geq \frac{1}{5}\, e^{-2(1+\alpha)\frac{a^2}{b}} .$$

Proof: We consider only the case $\rho \equiv 0$. We'll use Corollary ((2.1) B). Define

$$\tau = \inf \eta \ni S_\eta \geq a ,$$

Then

$$e^{-\left(\sqrt{2\lambda}+2\lambda\right)(a+1)} \leq \int e^{-\lambda T_\tau}$$

$$\leq P\left\{T_\tau \geq b\right\} + e^{-\lambda b} P\left\{T_\tau \geq b\right\}$$

so that

$$P\left\{T_\tau < b\right\} \geq e^{-(\sqrt{2\lambda}+2\lambda)(a+1)} - e^{-\lambda b} .$$

Let $\lambda = \frac{2a^2}{b^2}\theta^2$, θ temporarily unspecified, except for the requirement that $\theta \geq 1$. We have, since $\theta \geq 1$ and $\frac{a^2}{b} \geq 1$:

$$P\left\{T_\tau < b\right\} \geq e^{-2\frac{a^2}{b}\theta - 10\frac{a^3}{b^2}\theta^2} - e^{-2\frac{a^2}{b}\theta^2} .$$

Let $\theta = 1+\alpha$; we have

$$P\left\{T_\tau < b\right\} \geq e^{-2(1+\alpha)\frac{a^2}{b}}\left\{e^{-10\frac{a^3}{b^2}(1+\alpha)^2} - e^{-2\frac{a^2}{b}\alpha}\right\} .$$

Keeping $\frac{a^3}{b^2} \leq \frac{1}{25}$, $\frac{a^2}{b} \geq \frac{1}{\alpha}$, the bracketed expression dominates $\frac{1}{e} - \frac{1}{e^2} \geq \frac{1}{5}$ and we have our result. Q.E.D.

III. Invariance Results

For each $\alpha > 0$, let

$$Y_\alpha(t) = \alpha S_\tau + \alpha \frac{\left(\frac{t}{\alpha^2} - T_\tau\right)}{V_{\tau+1}^2} f_{\tau+1}$$

where

$$\tau = \sup \eta \ni T_\eta \le \frac{t}{\alpha^2}$$

and let

$$X_\alpha(t) = \alpha S_\sigma - \alpha \frac{\left\{L_\sigma - \frac{t}{\alpha^2}\right\}}{f_\sigma^2} f_\sigma$$

where

$$\sigma = \inf \eta \ni L_\eta \ge \frac{t}{\alpha^2} .$$

Note that X and Y are the linear interpolations of the sets of points $(\alpha^2 L_\eta, \alpha S_\eta)$ and $(\alpha^2 T_\eta, \alpha S_\eta)$ respectively.

Theorem (3.1). $Y_\alpha \xrightarrow{d} W$ and $X_\alpha \xrightarrow{d} W$ as $\alpha \to 0$ where W is Brownian Motion:

Proof: We must show that for all $0 \le t_1 < \quad < t_N$:

$$(Y_\alpha(t_1), \quad , Y_\alpha(t_N)) \xrightarrow{d} (W_{t_1}, \quad , W_{t_N})$$

$$(X_\alpha(t_1), \quad , X_\alpha(t_N)) \xrightarrow{d} (W_{t_1}, \quad , W_{t_N})$$

and that $\{X_\alpha, \alpha > 0\}$ and $\{Y_\alpha, \alpha > 0\}$ are tight. It is clear that the finite-dimensional convergence will follow if

$$(\alpha S_{\tau_1}, \quad ,\alpha S_{\tau_N}) \overset{d}{\rightarrow} (W_{t_1}, \quad ,W_{t_N})$$

and

$$(\alpha S_{\eta_1}, \quad ,\alpha S_{\eta_N}) \overset{d}{\rightarrow} (W_{t_1}, \quad ,W_{t_N})$$

where

$$\tau_k = \sup \eta \ni T_\eta \le \frac{t_k}{\alpha^2}$$

$$\sigma_k = \inf \eta \ni L_\eta \ge \frac{t_k}{\alpha^2} .$$

We have for all $\theta_1, \theta_2, \ldots, \theta_N$:

$$\int e^{\alpha \sum_1^N \theta S_{\tau_k}} \rightarrow \int e^{\sum \theta_k W_{t_k}} \quad \text{as} \quad \alpha \rightarrow 0 .$$

This follows from the estimates given in Theorem (2.3A):

$$e^{H\left(\alpha \sum_k^N \theta_j\right)\left(\frac{t_k - t_{k-1}}{\alpha^2} - 1\right)} \le E\left\{ e^{\alpha\left(\sum_k^N \theta_j\right)\left(S_{\tau_k} - S_{\tau_{k-1}}\right)} \middle| B_{\tau_k} \right\} \le e^{K\left(\alpha \sum_k^N \theta_j\right)\left(\frac{t_k - t_{k-1}}{\alpha^2} + 1\right)} .$$

Letting $\alpha \rightarrow 0$, the extreme left and right above converge to

$$e^{\frac{1}{2}\left(\sum_k^N \theta_j\right)^2 (t_k - t_{k-1})}$$

so that

$$\int e^{\alpha \sum_1^N \theta_k S_{\tau_k}} \rightarrow e^{\frac{1}{2} \sum_{k=1}^N \left\{ \sum_{j=k}^N \theta_j \right\}^2 (t_k - t_{k-1})} = \int e^{\sum \theta_k W_{t_k}} .$$

Next, to show $(\alpha S_{\eta_1}, \quad ,\alpha S_{\eta_N}) \overset{d}{\rightarrow} (W_{t_1}, \quad ,W_{t_N})$ it suffices that, for any $t > 0$, and for

$$\tau = \sup \eta \ni T_\eta \le \frac{t}{\alpha^2}$$

$$\sigma = \inf \eta \ni L_\eta \geq \frac{t}{\alpha^2}$$

we have $\alpha|S_\tau - S_\sigma| \xrightarrow{P} 0$ as $\alpha \to 0$.

We require some estimates. Let $\rho = \tau \vee \sigma$. Then, for all $\theta \leq \frac{1}{2}$, $\alpha^2 \leq \frac{1}{4}\theta t$:

$$P\{T_\rho \geq (1+2\theta)T_\tau\} \leq P\{T_\sigma \geq (1+2\theta)T_\tau\}$$

$$\leq P\left\{\frac{T_\sigma}{L_\sigma} \geq (1+2\theta)\ \frac{T_\tau}{L_\sigma}\right\}$$

$$\leq P\left\{\frac{T_\sigma}{L_\sigma} \geq 1+\theta\right\} \leq e^{-\frac{\theta^2}{3}\frac{t}{\alpha^2}}$$

from Corollary (1.2b). In the same manner one can show that for $\theta \leq \frac{1}{2}$, $\alpha^2 \leq \frac{1}{4}\theta t$:

$$P\left\{T_\rho \geq (1+2\theta)T_\sigma\right\} \leq e^{-\frac{\theta^2}{2}\frac{t}{\alpha^2}}$$

from Corollary (1.2b). Combining these estimates with Corollary (2.2B):

$$P\left\{|S_\rho - S_\tau| \geq \frac{\varepsilon}{\alpha}\right\} \leq P\left\{T_\rho \geq (1+2\theta)T_\tau\right\}$$

$$+ P\left\{T_\rho - T_\tau \leq 2\theta\ \frac{t}{\alpha^2},\ \ |S_\rho - S_\tau| \geq \frac{\varepsilon}{\alpha}\right\}$$

$$\leq e^{-\frac{\theta^2}{3}\frac{t}{\alpha^2}} + 2e^{-\frac{\varepsilon^2}{2(\varepsilon\alpha+2\theta t)}},$$

and the same procedure applied to $P\{|S_\rho - S_\sigma| \geq \frac{\varepsilon}{\alpha}\}$ leads to the estimate

$$P\left\{|S_\tau - S_\sigma| \geq \frac{\varepsilon}{\alpha}\right\} \leq 2e^{-\frac{\theta^2}{3}\frac{t}{\alpha^2}} + 4e^{-\frac{\varepsilon^2}{2(\varepsilon\alpha+2\theta t)}}$$

valid when $\varepsilon > 0$, $\theta \leq \frac{1}{2}$ and $\alpha^2 \leq \frac{1}{4}\theta t$.

Letting $\alpha \to 0$, we have

$$\lim_{\alpha\to 0} P\left\{|S_\tau - S_\sigma| \leq \frac{\varepsilon}{\alpha}\right\} \leq \inf_\theta 4e^{-\frac{\varepsilon^2}{4\theta t}} = 0 .$$

To complete the theorem we must prove tightness for $\{X_\alpha\}$ and $\{Y_\alpha\}$. We use the criterion given in Billingsley [1], (Theorem 8.3):

(*) Given $\varepsilon > 0$, $h > 0$, there exists $\delta_0 > 0$ and $\alpha_0 > 0$ such that

$$\sup_{\substack{t \\ \alpha\leq\alpha_0}} P\left\{\sup_{t\leq S\leq t+\delta_0} |Z_\alpha(S) - Z_\alpha(t)| \geq \varepsilon\right\} \leq \delta_0 h$$

where Z is X or Y. Note that the criterion above is specified in Billingsley for processes which live on $C[0,1]$ while our processes live on $C[0,\infty)$. Criterion (*) remains adequate for $C[0,\infty)$ – see Whitt [ ].

We begin with Y, defining, for $0 < t < s$:

$$\tau = \sup_\eta T_\eta \leq \frac{t}{\alpha^2} \quad \sigma = \sup_\eta T_\eta \leq \frac{S}{\alpha^2}$$

and reducing (*) to the equivalent form

$$\sup_{\substack{t \\ \alpha\leq\alpha_0}} \left\{\sup_{t\leq S\leq t+\delta_0} |S_\sigma - S_\tau| \geq \frac{\varepsilon}{\alpha}\right\} \leq h\delta_0 .$$

Using Corollary (2.2B), we estimate the left side with

$$P\left\{\sup_{t\leq S\leq t+\delta_0} |S_\sigma - S_\tau| \geq \frac{\varepsilon}{\alpha} \Big| B_\tau\right\} \leq 2e^{-\frac{\varepsilon^2}{2(\varepsilon\alpha+\delta_0+\alpha^2)}}$$

$$\leq 2e^{-\frac{\varepsilon^2}{\delta_0}} \leq h\delta_0$$

when $\varepsilon < 1$ and $\alpha_0 = \delta_0 < 1$ and δ_0 is small enough.

Now let's consider tightness for X, Define

$$\tau = \text{first } \eta \ni L_\eta \geq \frac{t}{\alpha^2}$$

$$\rho = \text{first } \eta \ni L_\eta \geq \frac{S}{\alpha^2}, \quad S > t .$$

It suffices that

$$P\left\{\sup_{t \leq s \leq t+\delta_0} |S_\rho - S_\tau| \geq \frac{\varepsilon}{\alpha} | B_\tau\right\} \leq \delta_0 h .$$

By the usual time translation, sending f_η into $f_{\tau+\eta}$, it suffices that

$$(**) \qquad P\left\{|S_\eta| \geq \frac{\varepsilon}{\alpha} \quad \text{some} \quad L_\eta \leq \frac{\delta_0}{\alpha^2} + 1\right\} \leq \delta_0 h .$$

The left side is bounded by the sum of the two terms

$$P\left\{|S_\eta| \geq \frac{\varepsilon}{\alpha}, \quad \text{some} \quad T_\eta \leq (1+2\theta)\left(\frac{\delta_0}{\alpha^2} + 1\right)\right\}$$

and

$$P\left\{L_\eta \leq \frac{\delta_0}{\alpha^2} + 1, \quad \text{some} \quad T_\eta \geq (1+2\theta)\left(\frac{\delta_0}{\alpha^2} + 1\right)\right\} .$$

Using Corollary (1.2C) and Corollary (2.2B), we see that the left side of (**) has bound

$$2e^{-\frac{\varepsilon^2}{12\delta_0}} + e^{-\frac{\varepsilon^2}{2\delta_0}}$$

upon taking $\varepsilon < 1$ and $\alpha_0 = \delta_0 < 1$. Letting $\theta = \frac{\varepsilon}{\sqrt{6}}$, we have the bound

$$3e^{-\frac{\varepsilon^2}{12\delta_0}} \leq \delta_0 h \quad \text{for small } \delta_0 .$$

Q.E.D.

Corollary (3.1).

$$\frac{S_\eta}{\sqrt{L_\eta}} \xrightarrow{d} W_1 .$$

Proof: Let $\alpha_m \searrow 0$, $\frac{\alpha_{m+1}}{\alpha_m} \to 1$, $\alpha_0 = 1$. Let τ_m = first $\eta \ni L_\eta \geq \frac{1}{\alpha_m^2}$

On a set of measure one, for each η, there exists m with

$$L_{\tau_m} \leq L_\eta < L_{\tau_{m+1}} .$$

Then

$$\frac{S_\eta}{\sqrt{L_\eta}} = \frac{S_\eta - S_{\tau_m}}{\sqrt{L_\eta}} + \frac{S_{\tau_m}}{\sqrt{L_{\tau_m}}}\sqrt{\frac{L_{\tau_m}}{L_\eta}} .$$

Now

$$\frac{L_{\tau_m}}{L_\eta} \to 1 \text{ a.e.}, \quad \frac{S_{\tau_m}}{\sqrt{L_{\tau_m}}} \xrightarrow{d} W_1 ,$$

so it suffices that we show

$$\alpha_{m+1}|S_\eta - S_{\tau_m}| \xrightarrow{P} 0 .$$

As usual:

$$P\left\{|S_\eta - S_{\tau_m}| \geq \frac{\varepsilon}{\alpha_{m+1}}\right\} = \int P\left\{|S_\eta - S_{\tau_m}| \geq \frac{\varepsilon}{\alpha_{m+1}} | B_{\tau_m}\right\}$$

and for large m the integral will be bounded by any universal bound for

$$P\left\{|S_\eta| \geq \frac{\varepsilon}{\alpha_{m+1}} , \quad \text{some} \quad L_\eta \leq \frac{2}{\alpha_{m+1}^2} - \frac{2}{\alpha_m^2}\right\} .$$

Now use Corollary (1.2) and Corollary (2.1) for appropriate θ and let $m \to \infty$. Q.E.D.

(IV) Iterated Logarithms

Theorem (4.1).

(1) For all $\theta \le \frac{1}{3}$, all N with $\frac{\log\log(N)}{N} \le \frac{\theta^2}{4}$, we have

$$P\left\{S_\eta \ge (1+2\theta)\sqrt{2T_\eta \log\log(T_\eta)}\ , \text{ some } T_\eta \ge N\right\} \le \frac{2}{\theta^{7/3}} \cdot \frac{1}{(\log(N))^\theta}\ .$$

(2) For all $\theta \le \frac{1}{4}$, all N with $\frac{\log\log N}{N} \le \frac{3}{16}\theta^2$, we have

$$P\left\{S_\eta \ge (1+4\theta)\sqrt{2L_\eta \log\log(L_\eta)}\ , \text{ some } L_\eta \ge N\right\} \le \frac{3}{\theta^{7/3}} \quad \frac{1}{(\log N)}\theta$$

In particular, for all $\theta > 0$:

(3) $$P\left\{S_\eta \ge (1+\theta)\sqrt{2 M_\eta \log\log M_\eta} \quad \text{i.o.}\right\} = 0$$

where $M_\eta = T_\eta$ or L_η.

Proof: Fix $0 < t_0 < t_1$ and define

$$\tau_0 = \text{first } \eta \ni T_\eta \ge t_0$$

$$\tau_1 = \text{last } \eta \ni T_\eta \le t_1$$

$$\tau = \begin{cases} \text{first } \eta \ni [\tau_0, \tau_1] \quad S_\eta \ge \phi(T_\eta) \\ \quad \tau_1 \text{ if no such } \eta \text{ occurs} \end{cases},$$

where

$$\phi(t) = (1+2\theta)\sqrt{2\, t \log\log(t)}\ .$$

Then

$$1 \ge \int_{\{S_\tau \ge \phi(T_\tau)\}} e^{\lambda S_\tau - K(\lambda)T_\tau} \ge \int_{\{S_\tau \ge \phi(T_\tau)\}} e^{\lambda\phi(T_\tau) - K(\lambda)T_\tau}\ ,$$

and this implies

$$P\{A_0\} \le \min_{\lambda} e^{K(\lambda)t_1-\lambda\phi(t_0)}$$

where

$$A_0 = \{S_\eta \ge \phi(T_\eta) \ , \quad \text{some} \quad T_\eta \in [t_0,t_1]\} \ .$$

The minimum over λ occurs for $\lambda = \ell n \left\{\frac{1+\phi(t_0)}{t_1}\right\}$. Using $\log(1+x) \ge x - \frac{x^2}{2}$ for $x \le 1$ and taking t_0, t_1 so that $\frac{\phi(t_0)}{t_1} \le \theta$, we have

$$P\{A_0\} \le e^{-\frac{(1-\theta)}{2}\frac{\phi^2(t_0)}{t_1}} \le \frac{1}{(\ell n\ t_0)^{1+\theta}}$$

if $\theta \le \frac{1}{3}$, and $t_1 = (1+\theta)t_0$. Repeating this argument for $t_0 = (1+\theta)^m$, $t_1 = (1+\theta)^{m+1}$, and denoting the set under consideration as A_m, we have

$$P\{A_m\} \le \frac{1}{(\ell n(1+\theta))^{1+\theta}} \quad \frac{1}{m^{1+\theta}}$$

provided $\theta \le \frac{1}{3}$ and $\frac{\log\log(1+\theta)^m}{(1+\theta)^m} \le \frac{\theta^2}{4}$. It is easily seen that the last constraint holds for all $m \ge m_0$ provided it holds for m_0. Thus, under such circumstances:

$$P\left\{\bigcup_{m_0}^{\infty} A_m\right\} \le \frac{1}{(\ell n(1+\theta))^{1+\theta}} \cdot \frac{1}{\theta} \cdot \frac{1}{(m_0-1)^{\theta}} \ .$$

Since $m_0 \ge 2 \Rightarrow m_0-1 \ge m_0 \log(1+\theta)$, we have

$$P\left\{S_\eta \ge \phi(T_\eta) \ , \quad \text{some} \quad T_\eta \ge (1+\theta)^{m_0}\right\} \le \frac{1}{(\ell n(1+\theta))^{1+2\theta}} \cdot \frac{1}{\theta} \cdot \frac{1}{m_0^{\theta}} \ .$$

Letting $N = (1+\theta)^{m_0}$ with $\frac{\log\log(N)}{N} \le \frac{\theta^2}{4}$, we have

$$P\left\{S_\eta \geq \phi(T_\eta) \ , \quad \text{some} \quad T_\eta \geq N\right\} \leq \frac{1}{\theta} \cdot \frac{1}{(\ell n(1+\theta))^{1+\theta}} \cdot \frac{1}{(\ell n N)^\theta}$$

$$\leq \frac{1}{\theta} \cdot \frac{1}{(\ell n(1+\theta))^{4/3}} \quad \frac{1}{(\ell n N)^\theta}$$

$$\leq \frac{1}{\theta^{7/3}} \cdot \frac{1}{(\ell n N)^\theta} \ .$$

Next, letting $\varepsilon \leq \frac{1}{4}$ and M temporarily unspecified:

$$P\left\{S_\eta \geq (1+2M\theta)\sqrt{2L_\eta \log\log L_\eta} \ , \quad \text{some} \quad L_\eta \geq N\right\}$$

$$\leq P\left\{\frac{T_\eta}{L_\eta} \notin [1-\varepsilon, 1+\varepsilon] \ , \quad \text{some} \quad L_\eta \geq N\right\}$$

$$+ P\left\{S_\eta \geq (1+2M\theta)\sqrt{2(1-\varepsilon)T_\eta \log\log (1-\varepsilon)T_\eta} \ , \quad \text{some} \quad T_\eta \geq (1+\varepsilon)N\right\} .$$

We now use Corollary (1.2b) and (1.2c) and the fact that if $\log T \geq 3.2$ and if $\varepsilon \leq \frac{1}{4}$, then $\log\log (1-\varepsilon)T \geq (1-\varepsilon) \log\log T$ so that

$$P\left\{\frac{T_\eta}{L_\eta} \notin [1-\varepsilon, 1+\varepsilon] \ , \quad \text{some} \quad L_\eta \geq N\right\} \leq 2e^{-\frac{\varepsilon^2}{8} N}$$

and

$$P\left\{S_\eta \geq (1+2M\theta)\sqrt{2(1-\varepsilon)T_\eta \log\log (1-\varepsilon)T_\eta} \ , \quad \text{some} \quad T_\eta \geq (1-\varepsilon)N\right\}$$

$$\leq P\left\{S_\eta \geq (1+2M\theta)(1-\varepsilon)\sqrt{2T_\eta \log\log T_\eta} \ , \quad \text{some} \quad T_\eta \geq (1-\varepsilon)N\right\}$$

for $\varepsilon = \theta$, $M = 2$, $\theta \leq \frac{1}{4}$ we have the last term dominated by

$$P\left\{S_\eta \geq (1+2\theta)\sqrt{2T_\eta \log\log (T_\eta)} \ , \quad \text{some} \quad T_\eta \geq (1-\theta)N\right\}$$

$$\leq \frac{2}{\theta^{7/3}} \frac{1}{(\log(1-\theta)N)^\theta} \quad \text{if} \quad \frac{\log\log (1-\theta)N}{(1-\theta)N} \leq \frac{\theta^2}{4} \ ,$$

true if

$$\frac{\log\log N}{N} \le \frac{3}{16}\theta^2 .$$

Thus,

$$P\left\{S_\eta \ge (1+4\theta)\sqrt{2L_\eta \log\log (L_\eta)} \ , \quad \text{some} \quad L_\eta \ge N\right\}$$

$$\le 2e^{-\frac{\theta^2}{8}N} + \frac{2}{\theta^{7/3}} \frac{1}{(\log(1-\theta)N)^\theta}$$

$$\le 2e^{-\frac{\theta^2}{8}N} + \frac{2}{\theta^{7/3}} \frac{1}{(1-\theta)^\theta} \frac{1}{(\log (N))^\theta} \le \frac{3}{\theta^{7/3}} \frac{1}{(\log N)^\theta}$$

under the restrictions on N. Q.E.D.

Theorem (4.2). For all $\theta > 0$:

$$P\left\{S_\eta \ge (1-\theta)\sqrt{2M_\eta \log\log (M_\eta)} \quad \text{i.o.}\right\} = 1$$

where $M_\eta = T_\eta$ or L_η.

Proof: We prove the result for $M_\eta = L_\eta$, modelling the proof on Freedman's proof for $M_\eta = T_\eta$.

Let

$$\tau_1 = \text{first} \quad \eta \ni L_\eta \ge (1+\varepsilon)r^k \ , \quad r > 1 .$$

$$A_k = \left\{S_\eta - S_{\tau_k} \ge \left(1 - \frac{1}{\sqrt{r}}\right)\phi(r^{k+1}) \ , \quad \text{some} \quad n \in [\tau_k, \tau_{k+1})\right\}$$

where

$$\phi(t) = \sqrt{2t \log\log (t)}$$

and where r,ε are chosen so that for a given $\delta > 0$:

$$\theta = \frac{\left(1 - \frac{1}{\sqrt{r}}\right)^2}{\left(1 - \frac{1}{r}\right)}(1+4\delta) < 1 .$$

We estimate $P\{A_k | B_{\tau_k}\}$ by finding a universal bound for

$$P\left\{S_\eta \geq \left(1 - \frac{1}{\sqrt{r}}\right)\phi(r^{k+1}) \ , \quad \text{some} \quad L_\eta \leq (1+\varepsilon)(r^{k+1}-r^k)\right\} .$$

For arbitrary positive $a,b,$ we have

$$P\left\{S_\eta \geq a \ , \quad \text{some} \quad T_\eta \leq b\right\} \leq P\left\{L_\eta \geq (1+\varepsilon)b \ , \quad \text{some} \quad T_\eta \leq b\right\}$$

$$+ P\left\{S_\eta \geq a \ , \quad \text{some} \quad L_\eta \leq (1+\varepsilon)b\right\} .$$

Assuming $a = \left(1 - \frac{1}{\sqrt{r}}\right)\phi(r^{k+1})$, $b = r^{k+1} - r^k$ and that k is large enough for Corollary (2.4) to hold, and using Theorem (1.2), we have

$$P\left\{A_k | B_{\tau_k}\right\} \geq \frac{1}{2(\log r)^\theta} \cdot \frac{1}{(k+1)^\theta} - e^{-\frac{r^k(r-1)\varepsilon^2}{2(1+\varepsilon)}} ;$$

thus we see that for all $\varepsilon > 0$, all $r > 1$, $\delta > 0$ for which $\theta < 1$, we have

$$\sum_k P\left\{A_k | B_{\tau_k}\right\} = \infty \quad \text{a.e.}$$

Using Corollary (1.1) with $g_\eta = I_{A_\eta}$, we see that $\sum I_{A_k} = \infty$ a.e. so that, for infinitely many k:

$$S_\eta \geq S_{\tau_k} + \left(1 - \frac{1}{\sqrt{r}}\right)\phi(r^{k+1}) \ , \quad \text{some} \quad \eta \in [\tau_k, \tau_{k+1}) \ .$$

Now Theorem (4.1), with $-S_\eta$ in place of S_η demands that, eventually

$$S_{\tau_k} \geq -(1+\varepsilon)\phi((1+\varepsilon)r^k + 1) \ ,$$

thus for fixed $\varepsilon > 0$,

$$S_{\tau_k} \geq -(1+2\varepsilon)\phi(r^k)$$

eventually, consequently, for infinitely many k;

$$S_\eta \geq -(1+2\varepsilon)\phi(r^k) + \left(1 - \frac{1}{\sqrt{r}}\right)\phi(r^{k+1}) \ , \quad \text{some} \quad \eta \in [\tau_k, \tau_{k+1}) \ ,$$

from which

$$S_\eta \geq -\frac{(1+2\varepsilon)}{\sqrt{r}}\phi(r^{k+1}) + \left(1 - \frac{1}{\sqrt{r}}\right)\phi(r^{k+1}) \ , \quad \text{some} \quad \eta \in [\tau_k, \tau_{k+1})$$

from which, for infinitely many η:

$$S_\eta \geq \left(1 - \frac{(2+2\varepsilon)}{\sqrt{r}}\right)\phi((1-\varepsilon)L_\eta) \ .$$

Now $\phi((1-\varepsilon)L_\eta) \geq (1-2\varepsilon)\phi(L_\eta)$ for small ε; thus for large r, small ε

$$S_\eta \geq (1-6\varepsilon)\phi(L_\eta) \quad \text{i.o.} \ .$$ Q.E.D.

<u>Remarks (4.1)</u>. Theorem (4.2) depends strongly on the inequality given in Corollary (2.4). As mentioned earlier, the simpler result Corollary (2.5) almost yields the L.I.L. That is, using Corollary (2.5) and imitating the proof of Theorem (4.2), it is easy to show that

$$P\left\{S_\eta \geq (1-\theta)\sqrt{\frac{M_\eta}{2}\log\log M_\eta} \quad \text{i.o.}\right\} = 1$$

for $M_\eta = T_\eta$ or L_η.

Remarks on First Passage Times

Suppose $\{f_\eta, B_\eta\}$ is no longer centered but satisfies instead

$$(A)\begin{cases} E\{f_\eta|B_{\eta-1}\} \geq aE\{f_\eta^2|B_{\eta-1}\}\ , & \text{some } a \in (0,1] \\ \sum_1^\infty \sigma^2\{f_\eta|B_{\eta-1}\} = \infty \text{ a.e.} & \end{cases}$$

where $\sigma^2\{f_\eta|B_{\eta-1}\} = E\{f_\eta^2|B_{\eta-1}\} - E^2\{f_\eta|B_{\eta-1}\}$.

Fix $t_0 \geq 6$ and define

$$\phi(t) = \begin{cases} (1+\theta)\sqrt{2\, t_0 \log\log t_0} & \text{if } t \leq t_0 \\ (1+\theta)\sqrt{2\, t \log\log t} & \text{if } t \geq t_0\ . \end{cases}$$

We want to consider

$$\tau = \text{first } \eta \ni S_\eta \geq \phi(T_\eta)$$

where $\theta \leq \frac{1}{3}$. We will consider $\int T_\tau$; in the spirit of Darling-Robbins [ ]. The time τ is proper; we define

$$g_\eta = \frac{1}{2}(f_\eta - E\{f_\eta|B_{\eta-1}\})$$

$$\tilde{S}_\eta = \sum_1^n g_k$$

$$\tilde{T}_\eta = \sum_1^n E\{g_k^2|B_{k-1}\} = \frac{1}{4}\sum_1^n \sigma^2\{f_k|B_{k-1}\}\ .$$

Using the second part of (A) we have

$$\tilde{S}_\eta \geq (1-\theta)\sqrt{2\,\tilde{T}_\eta \log\log \tilde{T}_\eta} \quad \text{i.o.}$$

Then using the first part of (A) we conclude $\overline{\lim_\eta} \frac{S_\eta}{T_\eta} \geq a$, from which it follows that τ is proper. Thus, with $-\tilde{S}_\eta$ in place of $\tilde{S}_\eta$, we have

$$1 \geq \int_{\{T_\tau \geq t\}} e^{\lambda(-\tilde{S}'_\tau) - K(\lambda)\tilde{T}_\tau}$$

$$\geq \int_{\{T_\tau \; t\}} e^{\frac{\lambda}{2}(aT_\tau - (\phi(T_\tau)+1)) - \frac{K(\lambda)}{4} T_\tau}$$

$$= \int_{\{T_\tau \geq t\}} e^{\frac{\lambda}{2} T_\tau \left\{a - \frac{K(\lambda)}{2} - \frac{(\phi(T_\tau)+1)}{T_\tau}\right\}}$$

We note that $\frac{\log \log t}{t}$ is decreasing for $t \geq 6$, so if we assume $t \geq t_0 \geq 6$ and if we further assume

(C) $$0 \leq \frac{K(\lambda)}{\lambda} \leq 2a - 2\,\frac{(\phi(t)+1)}{t} ,$$

then

$$P\{T_\tau \geq t\} \leq e^{\left(\frac{\phi(t)+1}{2} - at\right)\lambda + \frac{K(\lambda)}{4} t} .$$

A straightforward calculation shows that the right side minimizes for

$$K'(\lambda) = 2a - 2\,\frac{(\phi(t)+1)}{t}$$

or

$$\lambda = \ell n\left\{1 + 2a - 2\,\frac{(\phi(t)+1)}{t}\right\} .$$

Since $K(\lambda) \leq \lambda K'(\lambda)$, this value satisfies the constraint (C) and is therefore available.

Using $(1+x)\log(1+x) \geq x + \frac{x^2}{2(1+x)}$, all $x \geq 0$, we have

$$P\{T_\tau \geq t\} \leq e^{-\frac{t}{2(1+2a)}\left(a - \frac{(\phi(t)+1)}{t}\right)^2} .$$

Now our assumption that $t \geq t_0 \geq 6$ implies that $\frac{\phi(t)+1}{t}$ is decreasing in t with a maximal initial value $\frac{\phi(t_0)+1}{t_0}$. Choose t_1 as the unique t for which

$$\frac{\phi(t_1)+1}{t_1} = ba ,$$

some $b < 1$ where

$$\frac{\phi(t_0)+1}{t_0} > ba .$$

Then

$$t \geq t_1 \Rightarrow P\{T_\tau \geq t\} \leq e^{-\frac{a^2}{2(1+2a)}(1-b)^2 t} .$$

Let $c = \frac{(1-b)^2}{2(1+2a)}$, we have

$$\int T_\tau \leq t_1 + \int_{t_1}^{\infty} e^{-ca^2 t}\, dt$$

$$\leq t_1 + \frac{1}{ca^2} e^{-ca^2 t_1} \leq t_1 + \frac{1}{ca^2} .$$

The defining property of t_1 demands that

$$\frac{\sqrt{2 t_1 \log\log t_1}}{t_1} \geq \frac{b}{2} a ,$$

i.e., that

$$\log\log t_1 \geq \frac{b^2 a^2}{8} t_1 .$$

A crude bound is

$$t_1 \leq \frac{6e^2}{a^2 b^2} \log \frac{e}{ab} ,$$

yielding

$$\int T_\tau \leq \frac{6}{a^2} \left\{ \frac{1}{(1-b)^2} + \frac{e^2}{b^2} \log \frac{e}{b} + \frac{e^2}{b^2} \log \frac{1}{a} \right\} .$$

For example, when $t_0 = 6$ and when $\theta = \frac{1}{3}$ we have

$$\frac{\phi(t_0)+1}{t_0} > .75 \geq .75a ,$$

so we can take $b = .75$ and we have

$$\int T_\tau \leq \frac{1}{a^2}\left\{14 \log \frac{1}{a} + 200\right\} .$$

Note that the method used in finding such upper bounds will work for all ϕ for which $\frac{\phi(t)}{t}$ is eventually decreasing and has limit zero as $t \to \infty$.

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