EXISTENCE AND EXPLICIT DETERMINATION OF OPTIMAL STOPPING TIMES

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Abstract

We consider large classes of continuous time optimal stopping problems for which we establish the existence and form of the optimal stopping times. These optimal times are then used to find approximate optimal solutions for a class of discrete time problems.

Introduction

This paper focuses on three problems in the theory of optimal stopping:

(1) The determination of a broad class of continuous time optimal stopping problems for which optimal stopping times exist. This problem is treated in Section (I).

(2) The explicit determination of existing optimal stopping times. This problem is treated in Section (II).

(3) The connection between weak convergence and optimal stopping, aspects of which are treated in Section (III).

A more detailed survey of the contents is given directly below. We remark here that we made no use of the "free-boundary" approach to optimal stopping which one finds, for instance, Van Moerbeke [ ] and Chernoff [ ].

Let $X = \{X_t,\ t \geq 0\}$ be a standard Markov process. Let $f\colon [0,\infty) \times R \to R$ be continuous. Let $P_{(t,x)}$ represent the conditional probability $P\{\cdot \mid X_t = x\}$. We define the optimal stopping problem for X and f as the problem of determining F, T_∞ where

$$F(t,x) = \sup_{T \geq t} \int f(T, X_T)\, dP_{(t,x)} = \int f(T_\infty, X_{T_\infty})\, dP_{(t,x)}$$

where T runs through stopping times and T_∞, if it exists, is a stopping time that realizes the supremum. We prove in Section (I) that for a broad class of processes X and continuous functions f the optimal time T_∞ exists and is the hitting time of the closed set $\Gamma_\infty = \{F = f\}$. Our principal result is Theorem (1.3). This result covers at least the following cases:

(A) X is standard Markov, $f(t,x) = c(t)x$ where c is continuous non-increasing and

$$\begin{cases} P_{(t,x)}\left\{\lim\limits_{s\to\infty} c(s)|X_s| = 0\right\} = 1, \quad \text{all} \quad (t,x) \\ \text{and} \\ \displaystyle\int \sup_{s\geq 0} c(t+s)|X_{t+s}|dP_{(t,x)} < \infty, \quad \text{all} \quad (t,x) \end{cases}$$

(B) X is standard Markov, $f(t,x) = x - at$ for $a > 0$, X has continuous paths, and

$$\begin{cases} P_{(t,x)}\left\{\lim\limits_{s\to\infty} X_s - as = -\infty\right\} = 1, \quad \text{all} \quad (t,x) \\ \displaystyle\int \sup_{s\geq 0}(X_{t+s} - a(t+s))^+ dP_{(t,x)} < \infty, \quad \text{all} \quad (t,x) \end{cases}$$

(C) X is a pure jump process with positive bounded jumps; $a > 0$, $b > 0$, $g\colon [0,\infty) \to R$ is continuous, and $f(t,x) = g(x) - ax - bt$ where

$$\begin{cases} P_{(t,x)}\left\{\lim\limits_{s\to\infty} g(X_s) - aX_s - bs = -\infty\right\} = 1, \quad \text{all} \quad (t,x) \\ \text{and} \\ \displaystyle\int \sup_{s\geq 0}(g(X_{t+s}) - aX_{t+s} - b(t+s))^+ dP_{(t,x)} < \infty, \quad \text{all} \quad (t,x) \end{cases}$$

In Section (II) we treat the subclass of (A) where $c(t) = {}^{-\lambda t}$, $\lambda > 0$. We show by elementary arguments that for a class of processes X which includes, in particular, martingales and processes with stationary independent increments, there exists a positive x_0 such that T_∞ is the hitting time of $[x_0, \infty)$ for the process X, i.e.

$$\Gamma_\infty = \{F = f\} = \{(t,x): \ x \geq x_0 \ \text{ or } \ t = \infty\}$$

If X is a diffusion the value x_0 can often be characterized as a solution to

$$xH(\lambda,x) = 1$$

where H is a smooth function related to Laplace transforms of first passage times. There results are treated in Theorem (2.1), (2.2), (2.3). Applications are given covering Brownian Motion, Ornstein-Uhlenbeck processes, and Poisson processes. Our solutions in these three cases agree with those obtained previously by Taylor [ ]; our methods are different.

In Section (III) we consider the relationships between weak convergence and optimal stopping. We have a sequence of processes, X_n, converging weakly a process X_∞. We define

$$F_n(t,x) = \sup_{T \geq t} \int f(T, X_n(t))dP_{(t,x)}, \quad n \leq \infty \ .$$

Suppose we can find T_∞, the optimal stopping time for the case $n = \infty$ above. Can we use T_∞ as an approximation to T_n, the optimal stopping time in those cases above where n is large? An affirmative answer is given in Theorems (3.2) and (3.3) under restrictions motivated by Theorem (3.1). The latter theorem may be loosely paraphrased as follows: The optimal hitting T_∞ defined by

$$F(t,x) = \sup_{T \geq t} \int c(T) X_T dP_{(t,x)} = \int c(T_\infty) X_{T_\infty} dP_{(t,x)}$$

where X is brownian motion and c is decreasing, is given by

$$T_\infty = \inf t \ni X_t \geq G(t)$$

where G is continuous, postive, non-decreasing. Similar results, subject to slight modifications, would hold for most processes X with stationary independent increments. Our methods show, in particular, that when $c(t) = \frac{1}{(A+Bt)^r}$, $r > \frac{1}{2}$, and when X is brownian motion, we have $G(t) = K\sqrt{\frac{A+Bt}{B}}$, k constant.

Our methods yield the square root form almost immediately; the determination of k requires other tools - see Walker [ ] or Shepp [ ].

(I) Some Optimal Stopping Theory

Let $X = \{X_t, t \geq 0\}$ be a real valued Markov process with the following properties

(1.1) X is strong Markov, right continuous with left limits and $P_x\{X_0 = x\} = 1$, all $x \in R$.

(1.2) X has the Feller property, i.e., for all bounded continuous f, $T^t f$ is continuous where

$$(T^t f)(x) = \int f(X_t)\,dP_x .$$

(1.3) X is extended quasi-left continuous, i.e., for all increasing sequences of stopping times, T_n, if $T_n \nearrow T_\infty$, then X_{T_∞} exists P_x a.e. and

$$X_{T_n} \to X_{T_\infty} \text{ a.e. } P_x \text{ a.e., all } x .$$

We remark that (1,1), (1.2) and the version of (1.3) restricted to $X_{T_n} \to X_{T_\infty}$ on $\{T_\infty < \infty\}$ defines a standard process; see Dynkin [ ].

Let $Z = \{(t, X_t), t \geq 0\}$ be the space-time version of X and let f be non-negative and continuous on $[0,\infty) \times R$. We will call f a *return function*. We assume f satisfies

(1.4) For all (t,x):

$$\int \sup_{s\geq 0} f(t+s, X_{t+s})\,dP_{(t,x)} < \infty$$

where, of course, the integral is taken as the canonical version of the conditional expectation, i.e.,

$$\int \sup_{s\geq 0} f(t+s, X_{t+s})\,dP_{(t,x)} = E\left\{\sup_{s\geq 0} f(t+s, X_{t+s}) \,|\, X_t = x\right\} .$$

Remarks

We assume throughout this section that $f(\infty, X_\infty)$ is well-defined $P_{(t,x)}$ a.e., i.e., f is continuous at infinity.

Statement of the Problem

To determine the *optimal payoff* F and the *optimal stopping time* T_∞ where

$$(1.5) \qquad F(t,x) = \sup_{T \geq t} \int f(T, X_T)\, dP_{(t,x)} = \int f(T_\infty, X_{T_\infty})\, dP_{(t,x)} ,$$

(1.6) Remarks

(I) The most interesting processes X will not satisfy (1.3), a hypothesis central to our development. However, a re-definition of X and f will often suffice to transform the problem stated in (1.5) into one where X satisfies (1.3). For example, if X is standard Brownian motion, and if $f(t,x) = e^{-\lambda t} x^+$, then the problem (1.5) is identical to that problem involving the process $\tilde{X}$ and the function $\tilde{f}$ where $\tilde{X}_t = e^{-\lambda t} X_t$ and $\tilde{f}(t,x) = x^+$, for now $\tilde{X}_t \to 0$ a.e. as $t \to \infty$ and so (1.3) obtains. This construction will be elaborated on in the next remark.

(II) Let X be a Markov process which satisfies (1.1), (1.2) and where

$$(1.7) \quad \begin{cases} \text{(a)} & X_t = X_0 + Y_t + Z_t\,, \quad Y_0 = Z_0 = 0 \\ \text{(b)} & X_0 \text{ is square integrable} \\ \text{(c)} & Y = \{Y_t,\ t \geq 0\} \text{ is a martingale where,} \\ & \text{for all } t \geq 0,\ s \geq 0 \text{ and for some fixed } K > 0\text{:} \\ & E\{(Y_{t+s} - Y_t)^2 \mid X_t\} \leq Ks \\ \text{(d)} & |Z_t| \leq Kt, \text{ all } t \geq 0\ . \end{cases}$$

We list two classes of processes which have these properties;

Class (1)

$$X_t = X_0 + \int_0^t \sigma(X_s)\,dW_s + \int_0^t m(X_s)\,ds$$

where σ, m are bounded, W is Brownian Motion, X_0 is square integrable, independent of W.

Class (2)

$X = \{X_t,\ t \geq 0\}$ is pure jump, range the integers, having intensities $\{\lambda_x\}$ and transition probabilities $\{Q_x\}$ where, if T_x is the time for a jump from state x:

$$(1.9) \qquad \begin{cases} P_x\{T_x > t\} = e^{-\lambda_x t} \\ P_x\{X_{T_x} = x+y\} = Q_x\{y\} \\ \sup\limits_x \lambda_x < \infty \\ \sup\limits_x \int y^2 Q_x\{dy\} < \infty\ . \end{cases}$$

Our concern in later applications will be with determining F, T_∞ for processes of Class (1) or Class (2) subject to a *damping factor*, i.e., we consider

$$(1.10) \qquad F(t,x) = \sup_{T \geq t} \int C(T) X_T \, dP_{(t,x)} = \int C(T_\infty) X_{T_\infty} \, dP_{(t,x)}$$

where C decreases in t. In order to use our soon to be developed theory on such problems, where, as explained in the last remark, we'll rename our process as $\tilde{X} = \{C(t)X_t,\ t \geq 0\}$ and our return $\tilde{f}(t,x) = x^+$, we must determine that (1.3), (1.4) hold. We state a result — see Mucci, [p. 7] for a proof — which gives us what we need. Similar results occur in Walker [ ].

Proposition (1.1)

Let X satisfy (1,7) and let $C: [0,\infty) \to [0,\infty)$ be non-increasing.

Then

(A) If $Z_t \equiv 0$, all $t \geq 0$ and $\int_0^\infty C^2(t)dt < \infty$, then

$$(1.11) \qquad \begin{cases} P_{(t,x)}\left\{\lim_{s\to\infty} c(s)X_s = 0\right\} = 1 \quad \text{all} \quad (t,x) \\ \text{and} \\ \int \sup_{s\geq 0} c(t+s)|X_{t+s}|dP_{(t,x)} < \infty, \quad \text{all} \quad (t,x) \end{cases}$$

(B) For general Z, (1.11) continues to hold provided

$$(1.12) \qquad tc(t) \to 0 \quad \text{as} \quad t \to \infty .$$

Development of the Theory

Let (1.1) through (1.4) hold. We will determine the general form of F, T_∞ defined in (1.5).

(1.13) Definitions

(A) $G: [0,\infty) \times R \to [0,\infty)$ is called *excessive* if G is measurable and if, for all (t,x), all $s \geq 0$:

$$G(t,x) \geq (T^s G)(t,x) = \int G(t+s,X_{t+s})dP_{(t,x)}$$

and

$$G(t,x) = \lim_{s\downarrow 0} (T^s G)(t,x) .$$

(B) An excessive G is called an *excessive majorant* of f if $G \geq f$.

(C) An excessive majorant G of f is called the *least excessive majorant* if, for all excessive majorants H, of f, $H \geq G$.

Grigelionis-Shirgaev [ ] determine that f has a least excessive majorant, f_∞, defined by the recursion

$$f_0 = f,\ f_n = \sup_{t\geq 0} T^t f_{n-1},\ f_\infty = \uparrow f_n .$$

Taylor [ ] shows that the continuity of f and the Feller property (1.2) made f_∞ lower semi-continuous.

(1.14) <u>Definitions</u>

$$\Gamma_\infty = \{(t,x);\ f(t,x) = f_\infty(t,x)\}$$

$$T_\infty = \inf t \geq 0 \quad (t,X_t) \in \Gamma_\infty .$$

The lower semi-continuity of f_∞ makes Γ_∞ a closed set, and the hitting time, T_∞, is well-defined, since $f_\infty(\infty,X_\infty) = f(\infty,X_\infty)$, for we have

$$(1.15)\qquad \begin{cases} f_\infty(\infty,X_\infty) = \lim\limits_{s\to\infty} f_\infty(s,X_s) \quad \text{exists } P_{(t,x)} \text{ a.e.} \\ \text{and} \\ P_{(t,x)}\{f_\infty(\infty,X_\infty) = f(\infty,X_\infty)\} = 1 . \end{cases}$$

A proof of (1.15) can be adapted from Neveu [ ]. We begin with the observation that

$$(1.16)\qquad f_\infty(t,x) \leq \int \sup_{s\geq 0} f(t+s,X_{t+s})dP_{(t,x)}$$

This we can show inductively. Clearly

$$f_0(t,x) \leq \int \sup_{s\geq 0} f(t+s,X_{t+s})dP_{(t,x)} .$$

Assume this inequality obtains with f_n on the left. Then

$$f_{n+1}(t,x) = \sup_{s\geq 0} \int f_n(t+s,X_{t+s})dP_{(t,x)}$$

$$\leq \sup_{s\geq 0} \iint \sup_{r\geq 0} f(t+s+r,X_{t+s+r})dP_{(t+s,X_{t+s})}dP_{(t,x)}$$

$$= \sup_{s\geq 0} \int \sup_{r\geq s} f(t+r,X_{t+r})dP_{(t,x)}$$

$$\leq \int \sup_{s\geq 0} f(t+s,X_{t+s})dP_{(t,x)} \ .$$

Ineqality (1.16) now follows from $f_n \uparrow f_\infty$.

Next, fix $t < t_0 < r$, let $B_r = \sigma\{X_s,\ s \leq r\}$, and let $E_{(t,x)}\{\cdot|B_r\}$ be a conditional expectation relative to the measure $P_{(t,x)}$ and the field B_r. Then, using (1.16):

(1.16a)
$$f_\infty(r,X_r) \leq E_{(t,x)}\left\{\sup_{s\geq r} f(s,X_s)\,|B_r\right\}$$
$$\leq E_{(t,x)}\left\{\sup_{s\geq t_0} f(s,X_s)\,|B_r\right\} \ .$$

Letting $r \to \infty$, we have $P_{(t,x)}$ a.e.:

(1.16b) $$\overline{\lim_{r\to\infty}}\, f_\infty(r,X_r) \leq \overline{\lim_{r\to\infty}}\, E_{(t,x)}\left\{\sup_{s\geq t_0} f(s,X_s)\,|B_r\right\} = \sup_{s\geq t_0} f(s,X_s) \ .$$

Letting $t_0 \to \infty$ and using continuity of f and (1.3):

$$\overline{\lim_{r\to\infty}}\, f_\infty(r,X_r) \leq f(\infty,X_\infty) \qquad P_{(t,x)} \text{ a.e.}$$

Since $f_\infty(r,X_r) \geq f(r,X_r)$, the other direction is obvious.

<u>Theorem (1.1)</u>

$$f_\infty(t,x) = \int f_\infty(T_\infty,X_{T_\infty})dP_{(t,x)}, \quad \text{all} \quad (t,x) \in [0,\infty) \times R \ ,$$

<u>Proof</u>: Dynkin [ ] shows the following. If $\varepsilon > 0$ and $\Gamma_\varepsilon = \{(t,x): f_\infty(t,x) \leq f(t,x) + \varepsilon\}$, then with T_ε the hitting time for Γ_ε and with f bounded:

$$f_\infty(t,x) = \int f_\infty(T_\varepsilon, X_{T_\varepsilon})dP_{(t,x)} .$$

It is clear that Γ_ε is closed, that $\Gamma_\varepsilon \downarrow \Gamma_\infty$ and that $T_\varepsilon \uparrow T_\infty$ so that by our quasi-left continuity assumptions $X_{T_\varepsilon} \to X_{T_\infty}$. Further,

$$f_\infty(t,x) \le \int f(T_\varepsilon, X_{T_\varepsilon})dP_{(t,x)} + \varepsilon$$

Letting $\varepsilon \to 0$ and using the quasi-left continuity of X, and continuity and boundedness of f, we have

$$f_\infty(t,x) \le \int f(T_\infty, X_{T_\infty})dP_{(t,x)} .$$

Since $f \le f_\infty$ and since f_∞ is excessive and bounded, we've established our result in the bounded case. We now consider the unbounded case, subject as usual to (1.4). Set, for each $a > 0$:

$$f_a = \min(f,a)$$

$$\tilde{f}_a = \text{least excessive majorant of } f_a$$

$$\Gamma_a = \{f_a = \tilde{f}_a\}$$

$$\tilde{\Gamma}_a = \{f \ge \tilde{f}_a\}$$

$$T_a = \text{hitting time for } \Gamma_a$$

$$N_a = \text{hitting time for } \tilde{\Gamma}_a .$$

Note that $\Gamma_a \subset \tilde{\Gamma}_a$, that $N_a \le T_a$ and the $b \ge a$ implies $\tilde{f}_b \ge \tilde{f}_a$ by properties of least excessive majorants. Let $\tilde{f}_\infty = \uparrow \tilde{f}_a$ as $a \to \infty$. Clearly $\tilde{f}_\infty$ is excessive and $\tilde{f}_\infty \ge f$, therefore $\tilde{f}_\infty \ge f_\infty$. On the other hand, $f_\infty \ge f_a$, hence $f_\infty \ge \tilde{f}_a$, from which $f_\infty = \tilde{f}_\infty$, so that

$N_a \nearrow T_\infty$, for if we set $N_\infty = \uparrow N_a$, then $N_\infty \le T_\infty$ while

$$f(N_\infty, X_{N_\infty}) = \lim_{a\uparrow\infty} f(N_a, X_{N_a}) \ge \underline{\lim}_{a\uparrow\infty} \tilde{f}_a(N_a, X_{N_a})$$

$$\ge \underline{\lim}_{a\uparrow\infty} \tilde{f}_b(N_a, X_{N_a}) \ge \tilde{f}_b(N_\infty, X_{N_\infty})$$

by lower semi-continuity of $\tilde{f}_b$. But then

$$f(N_\infty, X_{N_\infty}) \ge \lim_{b\uparrow\infty} \tilde{f}_b(N_\infty, X_{N_\infty}) = f_\infty(N_\infty, X_{N_\infty})$$

which implies $(N_\infty, X_{N_\infty}) \in \Gamma_\infty$, thus $N_\infty \ge T_\infty$.

Next,

$$\int f(N_a, X_{N_a}) dP_{(t,x)} \ge \int \tilde{f}_a(N_a, X_{N_a}) dP_{(t,x)}$$

$$\ge \int \tilde{f}_a(T_a, X_{T_a}) dP_{(t,x)}$$

since $\tilde{f}_a$ is excessive and $T_a \ge N_a$.

Since $\tilde{f}_a$ is bounded, we have by Dynkin's result,

$$\int \tilde{f}_a(T_a, X_{T_a}) dP_{(t,x)} = \tilde{f}_a(t,x) .$$

so that

$$\int f(N_a, X_{N_a}) dP_{(t,x)} \ge \tilde{f}_a(t,x) .$$

Now, (1.4) allows us to use Lebesgue dominated convergence

$$\int f(T_\infty, X_{T_\infty}) dP_{(t,x)} = \int \lim_{a\uparrow\infty} f(N_a, X_{N_a}) dP_{(t,)}$$

$$= \lim_{a\uparrow\infty} \int f(N_a, X_{N_a}) dP_{(t,x)} \ge \lim_{a\uparrow\infty} \tilde{f}_a(t,x)$$

$$= f_\infty(t,x) = \tilde{f}_\infty(t,x) .$$

Since $f_\infty \ge f$ and f_∞ is excessive, the result follows. Q.E.D.

Theorem (1.2)

$$F(t,x) = f_\infty(t,x) = \int f(T_\infty, X_{T_\infty})dP_{(t,x)}, \quad \text{all} \quad (t,x) \in [0,\infty) \times R \ .$$

Proof:

$$\begin{aligned} F(t,x) &= \sup_{T\geq t} \int f(T,X_T)dP_{(t,x)} \geq \int f(T_\infty, X_{T_\infty})dP_{(t,x)} \\ &= \int f_\infty(T_\infty, X_{T_\infty})dP_{(t,x)} = f_\infty(t,x) \\ &\geq \sup_{T\geq t} \int f_\infty(T,X_T)dP_{(t,x)} \\ &\geq \sup_{T\geq t} \int f(T,X_T)dP_{(t,x)} \\ &= F(t,x) \ . \end{aligned}$$

Q.E.D.

A Generalization

Consider the following situations:

(I) To determine F, T_∞ where $N = \{N_t, t \geq 0\}$ is pure jump, where g is continuous, non-negative, and where, for $A > 0$

$$F(t,x) = \sup_{T\geq t} \int (g(T,N_T) - AN_T)dP_{(t,x)}$$

(II) To determine F, T_∞ where $X = \{X_t, t \geq 0\}$ has continuous paths, where g is continuous, non-negative, and where for $A > 0$:

$$F(t,x) = \sup_{T\geq t} \int (g(T,X_T) - AT)dP_{(t,x)} \ .$$

In both cases we have an added cost factor which removes the integrand from the positive and, in a sense, bounded, type to which our previous theory applied. Since situations like (I), (II) are very natural, we will extend our results to include a broad class of such problems.

(1.17) Definition

We will call a process X which satisfies (1.1), (1.2), (1.3) *extended standard*. Dynkin [Markov Processes I, 104] defines as standard those processes X which satisfy (1.1), (1.2) and are quasi-left continuous. We will need a result from Dynkin:

(1.18) Result [Dynkin, Markov Processes I, Theorem 10.3]

Let X be a standard, G be open, T_G be the exit time from G. Then $\tilde{X} = \{X_{t\wedge T_G}, t \geq 0\}$ is standard.

The replacement of standard by extended standard leaves this result valid. We remark also that the statement of Theorem (10.3) in Dynkin is somewhat finer, involving measure theoretic conditions; however, Dynkin shows that there can always be met if X is standard.

(1.19) Definitions

Let $f\colon [0,\infty] \times [-\infty,\infty) \to [-\infty,\infty)$ be continuous. We say that f *decreases moderately with* X if there exist two increasing sequences of positive constants $\{a_n\}$, $\{b_n\}$ where $a_n \nearrow \infty$, $a_n < b_n < a_{n+1}$, and

(a) If $A_n = \{f > -a_n\}$, then $[0,\infty) \times (-\infty,\infty) \subset \bigcup_n A_n$

(b) If σ_n is the exit time from A_n, then $f(\sigma_n, X_{\sigma_n}) \geq -b_n$.

The idea behind this definition is that although $f(t,X_t) \to -\infty$ as $t \to \infty$ is possible, this convergence doesn't take place too rapidly.

Theorem (1.3). Let X be extended standard and let f decrease moderately with X. Suppose for all finite (t,x):

(A) $\int |f(t+s, X_{t+s})| dP_{(t,x)} < \infty$, all $s \geq 0$

(B) $\int \sup_{s\geq 0} f^+(t+s, X_{t+s})\,dP_{(t,x)} < \infty$.

(C) For all sequences τ_n of increasing stopping times, $t \leq \tau_n \nearrow \tau$, we have

$$\int f(\tau_n, X_{\tau_n})\,dP_{(t,x)} \to \int f(\tau, X_\tau)\,dP_{(t,x)} \; .$$

Let

$$F(t,x) = \sup_{T\geq t} \int f(T, X_T)\,dP_{(t,x)}$$

$$\Gamma_\infty = \{f = F\}$$

$$T_\infty = \text{hitting time of } \Gamma$$

Then Γ_∞ is closed, T_∞ is a well-defined, possibly extended hitting time, and

$$F(t,x) = \int f(T_\infty, X_{T_\infty})\,dP_{(t,x)}$$

<u>Proof</u>: Let $A_n = \{f > -a_n\}$ where, if σ_n is the exit time from A_n, then the closure B_n, of A_n union its exit points (σ_n, X_{σ_n}) satisfies $B_n \subset A_{n+1}$. By Dynkin's theorem $X^{(n)} = \{X_{t\wedge\sigma_n}\}$ is extended standard when confined to B_n. Since f restricted to B_n is bounded below by $-a_{n+1}$, we can use Theorem (1.2) as follows:

Let $(t,x) \in B_n$; let

$$F_n(t,x) = \sup_{T\in[t,\sigma_n]} \int f(T, X_T)\,dP_{(t,x)}$$

$$\Gamma_n = \{(t,x) \in B_n,\; f(t,x) = F_n(t,x)\}$$

$$T_n = \text{hitting time of } \Gamma_n \text{ for paths beginning in } B_n \; .$$

Then Γ_n is closed in B_n, T_n is a well-defined hitting time with $T_n \le \sigma_n$, F_n is lower semi-continuous on B_n, and is the least excessive majorant of f for the process $X^{(n)}$ restricted to B_n, and

$$F_n(t,x) = \int f(T_n, X_{T_n})dP_{(t,x)} \quad , \quad \text{all} \quad (t,x) \in B_n \; .$$

Note that F_n increases on A_{n_0}, all $n \ge n_0$, so that $F_\infty = \uparrow F_n$ is well-defined and lower semi-continuous on $[0,\infty) \times (-\infty,\infty)$. Now for any finite (t,x) and any stopping time T beginning at (t,x), (C) demands that

$$(1.20) \qquad \int f(T,X_T)dP_{(t,x)} = \lim_{n\to\infty} \int f(T_{\wedge\sigma_n}, X_{T\wedge\sigma_n})dP_{(t,x)}$$

But then we have $F_\infty = F$, for (1.20) shows $F \le F_\infty$ while the definition of F_n demands $F_\infty \le F$.

We want now to be able to define T_∞ as the hitting time of $\{F = f\}$, and since we want this definition to make sense at infinity, we'll establish that

$$(1.21) \qquad P_{(t,x)}\left\{\lim_{s\to\infty} F(s,X_s) = f(\infty,X_\infty)\right\} = 1 \;, \quad \text{all finite} \quad (t,x)$$

We refer the reader back to the proof of (1.15), noting here only that (1.16a) holds by reducing to F_n and then passing to limits, while (1.16b) holds from the fact that $\sup_{s\ge t_0} f(s,X_s)$ is bounded above by $\sup_{s\ge t_0} f^+(s,X_s)$ and below by $f(t_0,X_{t_0})$; then we use (A),(B). Thus, T_∞ is well-defined since $F(\infty,X_\infty)$ is well-defined.

Let's next consider the times T_n. Clearly, for $n \ge n_0$ and on paths starting in A_{n_0}, T_n is increasing in n, thus $T_n \nearrow T_*$, and our proof will be completed if we can show that $T_* = T_\infty$, for, by (C):

$$F(t,x) = \uparrow F_n(t,x) = \uparrow \int f(T_n, X_{T_n})\, dP_{(t,x)} = \int f(T_*, X_{T_*})\, dP_{(t,x)}$$

Fix finite $(t,x) \in A_{n_0}$. Now $F(t,x) > -\infty$, thus $P_{(t,x)}\{f(T_*, X_{T_*}) = -\infty\} = 0$, so that for almost all paths beginning at (t,x), there exists n, possibly dependent on the path, with $(T_*, X_{T_*}) \in A_n$. Then by extended quasi-left continuity, there exists m such that the entire path, $\{(s, X_s): s \in [t, T_*]\} \subset A_m$. But then

$$\begin{aligned} f(T_*, X_{T_*}) &= \lim_{n\to\infty} f(T_n, X_{T_n}) \geq \underline{\lim}_{n} F_n(T_n, X_{T_n}) \\ &\geq \underline{\lim}_{n} F_m(T_n, X_{T_n}) \geq F_m(T_*, X_{T_*}) \\ &\to F(T_*, X_{T_*}) \end{aligned}$$

where the last inequality depends on the lower semi-continuity of F_m and where the next to last inequality depends on the monotonicity of F_n in n on A_m. Thus, $T_* \geq T_\infty$. On the other hand, $T_n \leq T_{\infty \wedge \sigma n}$, so $T_* \leq T_\infty$. Q.E.D.

(1.22) Remarks on Discrete Time Optimal Stopping

The theory developed above holds for Markov processes that move on a discrete time lattice $\{kr,\ k = 0,1,2,\ldots\}$, $r > 0$ provided the discrete analogues of hypotheses involving transitions $p(t,x,dy)$ are used. In point of fact, the theory falls out much more simply and with fewer hypotheses and we will not develop it here.

(II) Applications

Let $X = \{X_t, t \geq 0\}$ be a standard process which is homogeneous in time, i.e., $P_{(t,x)} = P_x$. *This assumption will hold unless we specify otherwise.* Assume further that for all $\lambda > 0$:

$$(2.1) \quad \begin{cases} \text{(a)} & \int \sup_{s \geq 0} e^{-\lambda(t+s)} |X_{t+s}| dP_{(t,x)} < \infty \\ \text{(b)} & P_{(t,x)} \left\{ \lim_{s \to \infty} e^{-\lambda(t+s)} X_{t+s} = 0 \right\} = 1 \end{cases}$$

Then, we have from Theorem (1.3) in conjunction with Remarks (1.6) that there exists Γ_∞, a closed set, and its hitting time, T_∞ such that

$$(2.2) \quad F(t,x) = \sup_{T \geq t} \int e^{-\lambda T} X_T dP_{(t,x)} = \int e^{-\lambda T_\infty} X_{T_\infty} dP_{(t,x)}$$

It will be convenient to use a different notation for stopping times. Since T_∞ is a hitting time, we can restrict our stopping time considerations exclusively to hitting times, and given any closed Γ, we will let τ be the time it takes to enter Γ, starting from (t,x), and X_τ the increment in distance required. We then re-write (2.2) in the form

$$F(t,x) = \sup \int e^{-\lambda(t+\tau)} (x+X_\tau) dP_x = \int e^{-\lambda(t+\tau_\infty)} (x+x_{\tau_\infty}) dP_x$$

where τ_∞ is the time it takes to travel from (t,x) into Γ_∞.

Now Γ_∞ is specified by

$$(2.4) \quad (t,x) \in \Gamma_\infty \iff e^{-\lambda t} x \geq \sup_{\tau>0} \int e^{-\lambda(t+\tau)}(x+x_\tau)\,dP_x$$

which we write in the equivalent from

$$(2.5) \qquad (t,x) \in \Gamma_\infty \iff x \geq H(x) = \begin{cases} \dfrac{\sup_{\tau>0}\int e^{-\lambda\tau} x_\tau\,dP_x}{1-\int e^{-\lambda\tau}\,dP_x} & \text{if } x \neq 0 \\ \sup_{\tau>0}\int e^{-\lambda\tau} x_\tau\,dP_0 & \text{if } x = 0 \end{cases}$$

In most of the analysis which follows we will consider $x \neq 0$, the case $x = 0$ being either similar or irrelevant.

The description (2.5) demands that Γ_∞ have form

$$(2.6) \qquad \Gamma_\infty = \{(t,x): \ x \geq H(x) \ \text{ or } \ t = \infty\}$$

If we use τ_0 for the hitting time of Γ_∞ starting from (t,x), we have

$$(2.7) \qquad H(x) = \frac{\int e^{-\lambda\tau_0} X_{\tau_0}\,dP_x}{1 - \int e^{-\lambda\tau_0}\,dP_x} \qquad x \neq 0$$

(2.8) Remarks

We introduce some definitions and notation which will facilitate the statement and proof of the next theorem.

(I) Let $a < x < b$ be in the state space of X. Let τ be the exit time from (a,b), starting from x, and let x_τ be the distance travelled from x at exit time. If, along all paths where $\tau < \infty$ we have $x + x_\tau \varepsilon \{a,b\}$, we will say that X doesn't skip states.

(II) Two states $a < b$ in the state space of X will be called non-contiguous if there exists c in the state space of X and $c \varepsilon (a,b)$.

Theorem (2.1)

Let $X = \{X_t, t \geq 0\}$ be a standard process which satisfies (2.1) and which doesn't skip states. Suppose further that for all $0 \leq a < b \leq \infty$ where a and b are non-contiguous, there exists $x \varepsilon (a,b)$ where, if τ is the exit time from (a,b), starting at x:

$$(2.9) \qquad \begin{cases} P_x\{\tau < \infty\} = 1 \\ x \geq \int e^{-\lambda\tau}(x+x_\tau)\,dP_x \end{cases}$$

Then there exists a finite $x_0 \geq 0$ such that

$$\Gamma_\infty = \{(t,x):\ x \geq x_0 \ \text{ or } \ t = \infty\}$$

We have

$$F(t,x) = \begin{cases} e^{-\lambda t}x & \text{if } x \geq x_0 \\ e^{-\lambda t}x_0 \int e^{-\lambda\tau_0}\,dP_x & \text{if } x < x_0 \end{cases}$$

where τ_0 is the hitting time for x_0, starting from x.

Proof:

Note first that $x < 0$ implies $(t,x) \notin \Gamma_\infty$, for from (2.1), we can always do better than the negative value $e^{-\lambda t}x$ simply by continuing on to X_∞. Thus, if Γ_∞ is not of the proposed form, then (2.6) demands the existence of non-contiguous $0 \le a < b \le \infty$ with $a \ge H(a)$, $b \ge H(b)$ but $x < H(x)$, all $x \in (a,b)$. We note next that for all such x, the time, τ_0, to travel from (t,x) into Γ_∞ is the same as the time τ, to exit (a,b), starting at x, for X does not skip states. From (2.7), for $x \in (a,b)$

$$x < H(x) = \frac{\int e^{-\lambda\tau_0} x\tau_0 dP_x}{1 - \int e^{-\lambda\tau_0} dP_x}$$

On the other hand, for some $x \in (a,b)$, and using (2.9) and the equivalence $\tau = \tau_0$:

$$x \ge \int e^{-\lambda\tau_0} (x + x_{\tau_0}) dP_x$$

so that we reach a contradiction.

Thus $\Gamma_\infty = \{(t,x): \ x \ge x_0 \ \text{ or } \ t = \infty\}$ and it remains to show that x_0 is finite. If $x_0 = \infty$, then $\Gamma_\infty = \{(t,\infty): t \ge 0\}$ and using (2.1) we would have $F(t,x) = 0$, all finite t with $(t,x) \notin \Gamma_\infty$, leading to the contradiction $F(t,x) < e^{\lambda t}x$ for $(t,x) \notin \Gamma_\infty$, $x > 0$.

(2.10) Remarks

The class of processes X for which (2.9) holds is probably quite large. It includes, for instance, super martingales for which the first half of (2.9) holds, for we have, with $0 \le a < x < b$, and τ the exit time from (a,b):

$$x \ge \int (x+x_\tau) dP_x > \int e^{-\lambda\tau}(x+x_\tau) dP_x \quad .$$

<u>An Application to Diffusions</u>

Let $X = \{X_t, t \ge 0\}$ be a difusion.

For $0 \le x \le y < \infty$ we assume

(I) $P_x\{\tau_y < \infty\} = 1$, τ_y the hitting time for y

(II) $\int e^{-\lambda\tau} dP_x$ is continuous in x, all $\lambda > 0$

(III) $\frac{d}{dx} \int e^{-\lambda\tau_y} dP_x$ is continuous in x, all $x < y$, $\lambda > 0$

(IV) $H(\lambda,y) = \lim_{x \uparrow y} \frac{d}{dx} \int e^{-\lambda\tau_y} dP_x$ exists and is continuous in y

<u>Theorem (2.3)</u>

Let X be a diffusion satisfying (2.1), (2.9) and (I) through (IV) above. Assume $xH(\lambda,x) = 1$ has a unique non-negative solution, call it x_0. Then

(2.11) $$\Gamma_\infty = \{t,x): \ x \ge x_0 \ \text{ or } \ t = \infty\}$$

Further, $x_0 > 0$.

<u>Proof</u>:

First $x_0 > 0$, for if $(0,t) \varepsilon \Gamma_\infty$ then for all τ we would have

$$0 \ge \int e^{-\lambda\tau} X_\tau dP_0 \ ,$$

but using (I) with $\tau = \tau_y$, we would then have

$$0 \geq \int y e^{-\lambda\tau_y} dP_0 > 0 ,$$

a contradiction.

If $x < x_0$, and if τ_0 is the time to hit x_0, starting from x, then, from (2.5):

$$x < \frac{(x_0 - x)\int e^{-\lambda\tau_0} dP_x}{1 - \int e^{-\lambda\tau_0} dP_x} \tag{2.12}$$

Using the Mean Value Theorem on the denominator, there exists $z \in (x, x_0)$ with

$$1 - \int e^{-\lambda\tau_0} dP_x = (x_0 - x) \frac{d}{dz} \int e^{-\lambda\tau_0} dP_z ,$$

so that, (2.11) becomes

$$x < \frac{\int e^{-\lambda\tau_0} dP_x}{\frac{d}{dz}\int e^{-\lambda\tau_0} dP_z}$$

Now let $x \to x_0$ so that $z \to x_0$. Our hypothesis (III, IV) leads to

$$x_0 \leq \frac{1}{H(\lambda, x_0)}$$

On the other hand, if $x_0 \leq x \leq x_1$, and if τ_1 is the time to reach x_1, starting from x, then since the best policy at x is to stop at

x, we have

$$e^{-\lambda t}x > \int e^{-\lambda(t+\tau_1)} x_1 \, dP_x \ , \quad \text{i.e.,}$$

$$x \geq \frac{(x_1 - x)\int e^{-\lambda\tau_0} \, dP_x}{1 - \int e^{-\lambda\tau_1} dP_x}$$

Repeating our reasoning above, we have

$$x_1 \geq \frac{1}{H(\lambda, x_1)} \qquad \text{Q.E.D.}$$

(2.13) Example

The stationary Ornstein-Uhlenbeck process has the representation

$$X_t = e^{-at}\left\{X_0 + \sqrt{2ab} \int_0^t e^{as} \, dW_s\right\}$$

where W is standard brownian motion, X_0 is $n(o,b)$, and W and X_0 are independent. The transform

$$\psi(x, x_0) = \int e^{-\lambda\tau_0} dP_x \ , \quad x < x_0$$

satisfies

$$ab\psi'' - ax\psi' = \lambda\psi \quad , \qquad \psi(x_0) = 1 \ .$$

This equation is solved in Darling-Siegert [ ]. When $a = b = \lambda = 1$, the solution takes the relatively simple form

$$(x,x_0) = \frac{\int_0^\infty e^{xt - t^2/2}\,dt}{\int_0^\infty e^{x_0 t - t^2/2}\,dt}$$

The process X can be shown to satisfy the hypotheses of Theorem (2.2), consequently the optimal x_0 is the unique value for which

$$x_0 \lim_{x \uparrow x_0} \frac{d}{dx} \frac{\int_0^\infty e^{xt - t^2/2}\,dt}{\int_0^\infty e^{x_0 t - t^2/2}\,dt} = 1$$

Straightforward calculations reduce this to the equation

$$x_0^2 + \frac{x_0 e^{-\frac{1}{2}x_0^2}}{\int_{-x_0}^\infty e^{-\frac{1}{2}t^2}\,dt} = 1$$

for which the solution is $x_0 \sim .839$, agreeing with Taylor [ ].

<u>Theorem (2.3)</u>

Let $X = \{X_t,\ t \geq 0\}$ be a standard process satisfying (2.1) and having independent stationary increments. Assume for $x < y$ that $P_x\{\tau_y < \infty\} > 0$. Then there exists finite $x_0 > 0$ with $\Gamma_\infty = \{(t,x):\ x \geq x_0$ or $t = \infty\}$.

Proof:

Since $P_x\{\tau_y < \infty\} > 0$, we see that $(t,x) \varepsilon \Gamma_\infty \Rightarrow x > 0$. Thus $(t,x) \varepsilon \Gamma_\infty$ if and only if

$$(2.14) \qquad x \quad \sup_{\tau > 0} \frac{\int e^{-\lambda\tau} x_\tau \, dP_x}{1 - \int e^{-\lambda\tau} dP_x} .$$

The assumption that X has stationary independent increments implies that the right side above is a constant not dependent on x. Let this constant be x_0. As in the last theorem, x_0 is necessarily finite.

(2.15) Applications

(1) Let $X = \{X_t, t \geq 0\}$ be Brownian Motion with drift m variance σ^2. Then X has continuous paths and stationary independent increments, so that if τ_0 is the hitting time of x_0, starting from $x < x_0$, we have

$$\int e^{-\lambda\tau_0} dP_x = e^{-(x_0 - x)G(\lambda)}$$

where

$$G(\ ) = \frac{\sqrt{M^2 + 2\lambda\sigma^2} - M}{\sigma^2}$$

from which it follows immediately that

$$x_0 = \frac{\sigma^2}{\sqrt{M^2+2\lambda\sigma^2} - M}$$

and

$$F(t,x) = \begin{cases} e^{-\lambda t} x & \text{if} \quad x \le x_0 \\ x_0 e^{-\lambda t} e^{-(x_0-x)G(\lambda)} & \text{if} \quad x \ge x_0 \end{cases}$$

This agrees with Taylor [ ].

(2) Let $X = \{X_t, t \ge 0\}$ be a stationary independent increment pure jump process with state space the non-negative integers and with all jumps positive. More specifically, if T is generic for jump times, and if x, z are non-negative integers, with $z > 0$, then there exists fixed $a > 0$ and some fixed probability θ on the positive integers with

$$P_x\{T > t\} = e^{-at}$$

$$P_x\{X_T = x+z\} = \theta(z)$$

We will assume $\Sigma\, z^2\theta(z) < \infty$, from which it follows fairly easily that conditions (1.7) are met and Theorems (1.2), (2.3) are available. Letting τ_0 be the time to exit $[0, x_0)$ where x_0 is the optimal stopping value, we see that

$$x_0 - 1 < \frac{\int e^{-\lambda\tau_0} z_{\tau_0}\, dP_{x_0-1}}{1 - \int e^{-\lambda\tau_0} dP_{x_0-1}}$$

However, $\tau_0 = T$, the jump time from starting point $x_0 - 1$, so that we have

(2.16)
$$x_0 - 1 < \frac{\int e^{-\lambda T} z_T \, dP_{x_0-1}}{1 - \int e^{-\lambda T} dP_{x_0-1}}$$

Likewise, since x_0 is optimal:

(2.17)
$$x_0 \geq \frac{\int e^{-\lambda t} z_T dP_{x_0}}{1 - \int e^{-\lambda T} dP_{x_0}}$$

where again we are using T generically as the jump time from any given starting point. One calculates that the right side of (2.17) is $\frac{a}{\lambda} \Sigma \, z\theta(z)$, so that x_0 is the smallest integer with

(2.18)
$$x_0 \geq \frac{a}{\lambda} \Sigma \, z\theta(z)$$

For the case $\theta(1) = 1$, we have as optimal payoff:

(2.19)
$$F(t,x) = \begin{cases} e^{-\lambda t} x & \text{if } x \leq x_0 \\ x_0 \left(\frac{a}{a+\lambda}\right)^{x_0 - x} e^{-\lambda t} & \text{if } x < x_0 \end{cases}$$

and this agrees with Taylor [ ].

(III) Weak Convergence Related to Optimal Stopping

Let $X = \{X_t, t \geq 0\}$ be brownian motion with non-negative drift. We assume $P\{X_0 = 0\} = 1$. Let $c:[0,\infty) \to (0,\infty)$ be strictly decreasing, continuous. We want to determine Γ_∞, T_∞ where

$$(3.1) \qquad F(t,x) = \sup_{T \geq t} \int c(T) X_T \, dP_{(t,x)} = \int c(T_\infty) X_{T_\infty} \, dP_{(t,x)}$$

We will be able to use the characterization of Theorem (1.3) for Γ_∞ provided, for instance

$$(3.2) \qquad \begin{cases} P_{(0,0)}\left\{\lim\limits_{t\to\infty} c(t)X_t \text{ exists and is finite}\right\} = 1 \\ \int \sup\limits_{t\geq 0} c(t)|X_t| \, dP_{(0,0)} < \infty . \end{cases}$$

The restriction to $P_{(0,0)}$ in (3.2) is allowable since X has stationary independent increments. Conditions (3.2) will hold if, for instance, in analogy with Proposition (1.1), we have

$$(3.3) \qquad \lim_{t\uparrow\infty} tc(t) = M < \infty$$

We restrict c so that not only (3.3) holds but also

(3.4) c', the derivative of c, is continuous with strictly decreasing absolute value, and

$$\left|\frac{c'(t)}{c(t)}\right| \text{ is non-increasing in } t .$$

Theorem (3.1). Let X be brownian motion with non-negative drift; let c satisfy (3.3) and (3.4). Then Γ_∞, the optimal hitting set for the problem defined in (3.1) is given by

$$\Gamma_\infty = \{(t,x): x \geq G(t) \quad \text{or} \quad t = \infty\}$$

where

(I) G is non-decreasing

(II) For all $s > 0$, all $0 \leq r \leq t \leq s$:

$$G(t) - G(r) \leq \frac{M}{B(s)} (t-r)$$

where M is a universal constant and $B:[0,\infty) \to (0,\infty)$.

Proof: $(t,x) \in \Gamma_\infty$ if and only if

$$c(t)x \geq \sup_{\tau>0} \int c(t+\tau)(x+x_\tau)dP_{(t,x)} = F(t,x) . \tag{3.5}$$

Given any $x < y$, with T the time to travel from x to y, we have $P_x\{T < \infty\} = 1$, therefore $F(t,x) > 0$, hence $(t,x) \in \Gamma_\infty$ implies $x > 0$, and we can re-write (3.5) as

$$(t,x) \in \Gamma_\infty \iff x \geq \sup_{\tau>0} \frac{\int \frac{c(t+\tau)}{c(t)} x_\tau \, dP_{(t,x)}}{1 - \int \frac{c(t+\tau)}{c(t)} dP_{(t,x)}} . \tag{3.6}$$

Now we know that τ can be restricted to hitting times of closed sets, Γ, and that the fact that X has stationary independent increments gives τ, the hitting time of Γ, starting from (t,x), the same stochastic structure is τ', the hitting time of $\Gamma - (t,x)$, starting from $(0,0)$. This allows us to re-write (3.6) as

$$(t,x) \in \Gamma_\infty \iff x \geq G(t) = \sup_{\tau>0} \frac{\int \frac{c(t+\tau)}{c(t)} X_\tau dP_{(0,0)}}{1 - \int \frac{c(t+\tau)}{c(t)} dP_{(0,0)}} . \tag{3.7}$$

Further, (3.4) implies

$$(3.8) \qquad s \geq t \Rightarrow \frac{c(s+r)}{c(s)} \geq \frac{c(t+r)}{c(t)}$$

so that $G(t)$ is non-decreasing in t. It is also clear that $G(0) > 0$. Next, if $0 < x < G(t)$, the reasoning which led from (3.5) to (3.7) allows the following definition of G:

$$(3.9) \qquad G(t) = \frac{\int \frac{c(t+\tau_0)}{c(t)} X_{\tau_0} dP_{(0,0)}}{1 - \int \frac{c(t+\tau_0)}{c(t)} dP_{(0,0)}}$$

where τ_0 is the time to reach G, starting from (t,x). Fixing $x = \frac{G(0)}{2}$, we can re-define G with

$$(3.10) \qquad G(t) = \sup_{\tau \geq \tau_1} \frac{\int \frac{c(t+\tau)}{c(t)} X_\tau dP_{(0,0)}}{1 - \int \frac{c(t+\tau)}{c(t)} dP_{(0,0)}}$$

where τ_1 is the time to travel from zero to $\frac{G(0)}{2}$. That is, we are free to define G through stopping times which are strictly bounded away from zero so that the denominator above doesn't get too small. Let's now prove (II). We set

$$(3.11) \qquad G(t,\tau) = \frac{\int \frac{c(t+\tau)}{c(t)} X \, dP_{(0,0)}}{1 - \int \frac{c(t+\tau)}{c(t)} dP_{(0,0)}} .$$

We first get bounds on $G(s,\tau) - G(t,\tau)$, $s \geq t$.

By straightforward calculations

$$\left|\int \frac{c(t+\tau)}{c(t)} X_\tau dP_{(0,0)} - \int \frac{c(s+\tau)}{c(s)} X_\tau dP_{(0,0)}\right|$$

$$\le \int \sup_{r\ge 0}\left|\frac{c(s+r)}{c(s)} - \frac{c(t+r)}{c(t)}\right| |X_r| dP_{(0,0)}$$

$$\le 2\left|\frac{c'(0)}{c(0)}\right| \cdot \frac{1}{c(s)} \cdot |t-s| \int \sup_{r\ge 0} c(r)|X_r| dP_{(0,0)}$$

$$= M \cdot \frac{|t-s|}{c(s)} \text{ for some universal } M > 0 \text{ .}$$

Likewise

$$\left|\int \frac{c(t+\tau)}{c(t)} dP_{(0,0)} - \int \frac{c(s+\tau)}{c(s)} dP_{(0,0)}\right| \le \frac{M}{c(s)} |t-s|$$

where we again use M for a generic universal constant, not necessarily the same from case to case.

But then, for all $\tau \ge \tau_1$ where τ_1 is as in (3.10): $G(s,\tau) - G(t,\tau) \le \frac{M}{c^2(s)} \cdot \frac{|s-t|}{B(\tau,s)}$ where $B(\tau,s) = \left(1 - \int \frac{c(s+\tau)}{c(s)} dP_{(0,0)}\right)^2$, and the latter quantity, with $\tau \ge \tau_1$, is bounded away from zero. Finally, given $\varepsilon > 0$, $t \le s \le s_0$ choose $\tau \ge \tau_1$ so that

$$G(s) \le G(s,\tau) + \varepsilon \text{ .}$$

We then have

$$G(s) \le G(t,\tau) + \varepsilon + \frac{M \cdot |s-t|}{c^2(s)B(\tau,s)}$$

$$\le G(t) + \varepsilon + \frac{M \cdot |s-t|}{B(s_0)}$$

where $B(s_0) = \inf_{\tau\ge\tau_1} c^2(s_0)B(\tau,s_0)$. Q.E.D.

(3.12) Remarks

(I) Let X be standard brownian motion; let $c(t) = \frac{1}{(A+Bt)^r}$, $r > \frac{1}{2}$, $A > 0$, $B > 0$. Then

$$G(t) = \sup_{\tau>0} (A+Bt)^r \frac{\int \frac{X_\tau}{(A+B(t+\tau))^r} dP_{(0,0)}}{1-(A+Bt)^r \int \frac{1}{(A+B(t+\tau))^r} dP_{(0,0)}} .$$

Using the fact that $X_r \overset{\mathcal{D}}{=} aX_{r/a^2}$ for X standard brownian motion, setting $a = \sqrt{\frac{A+Bt}{B}}$ and substituting in the integral above, we have

$$G(t) = K_r \cdot \sqrt{\frac{A+Bt}{B}}$$

where

$$K_r = \sup_{\tau>0} \frac{\int \frac{X_\tau}{(1+\tau)^r} dP_{(0,0)}}{1 - \int \frac{1}{(1+\tau)^r} dP_{(0,0)}} .$$

We see then that

$$\Gamma_\infty = \left\{(t,x): x \geq K_r \cdot \sqrt{\frac{A+Bt}{B}} \text{ or } t = \infty\right\} .$$

The constant K_r is determine by Walker [ ] and in special cases by Shepp [ ] and Taylor [ ]; we will not pursue this problem here.

(II) The last theorem, although restricted to Brownian motion with non-negative drift, can be extended, with minor changes in hypotheses and conclusions, to a large class of processes with stationary independent increments. Again, we will not develop these generalizations here.

Weak Convergence

For each n, let $X_n = \{X_n(t), t = m\phi(n), m = 0,1,2,\ldots\}$ be a discrete time Markov process whose time parameter lines on the lattice $\{m\phi(n)\}$ where $\phi(n+1)$ divides $\phi(n)$, and $\phi(n) \to 0$ as $n \to \infty$, for instance, we might have $\phi(n) = 2^{-n}$. We assume X_n is governed by a transition p_n, i.e.,

$$(3.13) \qquad P\{X_n(t+\phi(n)) \in dy \mid X_n(t) = x\} = p_n(x,dy) \ .$$

Let $f\colon [0,\infty) \times R \to [0,\infty)$ be continuous. We define

$$(3.14) \qquad F_n(t,x) = \sup_{T \geq t} \int f(T,X_T)\,dP^{(n)}_{(t,x)}$$

where it is to be understood that T runs through stopping times on the lattice $\{m\phi(n)\}$ and where $P^{(n)}_{(t,x)}$ is the probability on the process X_n under initial state (t,x). In order that extended times T be available we'll assume always that

$$(3.15) \qquad P^{(n)}_{(t,x)}\{f(\infty,X_n(\infty)) = 0\} = 1, \quad \text{all} \quad (t,x), \quad \text{all} \quad n \ .$$

For each n, extend the process X_n to $c[0,\infty)$ by the standard interpolation:

$$(3.16) \quad X_n(t) = \frac{((m+1)\phi(n)-t)}{\phi(n)} X_n(m\phi(n)) + \frac{(t-m\phi(n))}{\phi(n)} X_n((m+1)\phi(n))$$

for $t \in [m\phi(n),(m+1)\phi(n)]$. We will continue to use X_n as notation for the interpolated process and we'll also use $P^{(n)}_{(t,x)}$ for X_n conditioned on $X_n(t) = x$. Where no ambiguity arises we'll write $P_{(t,x)}$ for $P^{(n)}_{(t,x)}$. Now assume there exists X_∞, a standard continuous path Markov process with transition $p_\infty(t,x,dy)$ such that

$$X_n \xrightarrow{\mathcal{D}} X_\infty \quad \text{as} \quad n \to \infty$$

where the intended convergence is weak convergence on $C[0,\infty)$ under all initial (t,x) in the state space of X_n for large n, i.e., under all (t,x) where t is on the $\phi(n)$ lattice, all large n. For treatments of weak convergence see Billingsley [ ] and Whitt [ ].

We'll assume that $P_{(t,x)}\{f(\infty,X_\infty(\infty))=0\}=1$, all (t,x) and that T_∞ exists where

$$(3.17)\qquad F_\infty(t,x) = \sup_{T\geq t}\int f(T,X_\infty(T))dP_{(t,x)} = \int f(T_\infty,X_\infty(T_\infty))dP_{(t,x)}$$

and T_∞ is the hitting time of the closed set $\Gamma_\infty=\{f=F_\infty\}$, and F_∞ is the least excessive majorant under p_∞ of f. Our objective is to relate F_∞ to F_n through T_∞ in a manner to be made precise below. We begin with some assumptions.

Assumptions

(3.18) For all $n\leq\infty$ and for all (t,x) in the space-time range of X_n:

$$P_{(t,x)}\left\{\lim_{s\to\infty} f(s,X_n(s))=0\right\}=1 .$$

Thus, we interpret $f(\infty,X_n(\infty))=0$, as already indicated above.

(3.19) $f(t,x)$ increases in x, decreases in t, and is uniformly continuous in t.

(3.20) There exists $G\in C[0,\infty)$ non-decreasing in t with

$$\Gamma_\infty=\{(t,x):\ x\geq G(t)\ \text{ or }\ t=\infty\} .$$

(3.21) For all (t,x):

$$P_{(t,x)}\left\{\bigcap_{\varepsilon>0}\ \bigcup_{0<s<\varepsilon}\left\{X_\infty(T_\infty+s)>G(T_\infty+s)\right\}\right\}=1$$

conditional on $\{T_\infty<\infty\}$.

(3.22) For all large n, F_∞ is excessive for discrete time X_n, i.e.,

$$F_\infty(m\phi(n),x) \geq \int F_\infty((m+1)\phi(n),y)p_n(x,dy) \ .$$

(3.23) <u>Comments on the assumptions</u>

(I) The determination of Γ_∞ though non-decreasing, continuous G is motivated by Theorem (3.1).

(II) Assumption (3.21) prevents X_∞ from moving away from Γ_∞ once it hits Γ_∞. This is a rather weak assumption, true, for instance, when X_∞ is brownian motion and G has less than infinite slope everywhere — a proof is easily established by the local law of the iterated logarithm.

Let's now define T_n as the approximate hitting time of Γ_∞ for the discrete time process X_n:

$$T_n = \begin{cases} \text{minimal} \quad m\phi(n) \ni X_n(t) \in \Gamma_\infty \quad \text{for} \quad t \in ((m-1)\phi(n), m\phi(n) \\ \quad \infty \quad \text{if no such} \quad m \quad \text{exists} \ . \end{cases}$$

Our main result is that T_n is approximately optimal for the problem defined in (3.14).

<u>Theorem (3.2)</u>. Let $X_n \overset{\mathcal{D}}{\to} X_\infty$ under all initial (t,x) in the space-time range of X_n, large n. Assume (3.18) through (3.22). Then, for all such (t,x):

$$F_\infty(t,x) = \lim_{n\to\infty} F_n(t,x) = \lim_{n\to\infty} \int f(T_n, X_n(T_n))dP_{(t,x)} \ .$$

<u>Proof</u>: By Shorokhod's theorem — see Billingsley [ ] — there exists a probability triple (Ω,B,P) supporting random elements $Y_n\colon \Omega \to C[0,\infty)$, $n \leq \infty$ such that

$$Y_n \overset{\mathcal{D}}{=} X_n\ , \quad n \le \infty \tag{3.25}$$

$$Y_n(\omega) \to Y_\infty(\omega) \quad \text{all} \quad \omega \in \Omega\ .$$

Since the matric on $C[0,\infty)$ is uniform convergence on all closed intervals $[0,t_0]$, it is easily seen from assumptions (3.20) and (3.21) that if V_n is the hitting time of $G(t)$ for the continuous time process X_n and if S_n is the corresponding hitting time for Y_n, then

$$V_n \overset{\mathcal{D}}{=} S_n\ , \quad \text{all} \quad n \le \infty \tag{3.26}$$

and

$$S_n \to S_\infty \quad \text{on} \quad \{S_\infty > \infty\}\ . \tag{3.27}$$

Clearly, (3.21) implies (3.27) by making hitting times $C[0,\infty)$ - continuous where they occur. But then, by continuity of f and Y_n, we have

$$f(S_n,Y_n(S_n)) \to f(S_\infty,Y_\infty(S_\infty))\ , \quad \text{all} \quad \omega \in \Omega\ . \tag{3.28}$$

Here we use (3.18) on $\{S_\infty = \infty\}$. Now (3.26) demands that

$$\int f(V_n,X_n(V_n))dP_{(t,x)} = \int f(S_n,Y_n(S_n))dP\ , \quad \text{all} \quad n \le \infty \tag{3.29}$$

and then Fatou's lemma applied to the right and interpreted for the left yields (note $V_\infty = T_\infty$):

$$\int f(T_\infty,X_\infty(T_\infty))dP_{(t,x)} \le \underset{n}{\underline{\lim}} \int f(V_n,X_n(V_n))dP_{(t,x)}\ . \tag{3.30}$$

Now we want to replace V_n by T_n so that we can relate behaviour of discrete time X_n to continuous time X_∞. Since $|T_n - V_n| \le \phi(n)$, and $\phi(n) \to 0$, the uniform continuity of f in t demands that for large n

$$f(V_n,X_n(V_n)) \le f(T_n,X_n(V_n)) + \varepsilon\ . \tag{3.31}$$

Further, since G is non-decreasing and V_n is the hitting time for G, and f is increasing in x:

$$f(T_n, X_n(V_n)) \le f(T_n, X_n(T_n)) \; . \tag{3.32}$$

Consequently:

$$\int f(T_\infty, X_\infty(T_\infty))\, dP_{(t,x)} \le \underline{\lim}_{n} \int f(T_n, X_n(T_n))\, dP_{(t,x)} \; . \tag{3.33}$$

Finally, assumption (3.22) and the properties of least excessive majorants gives $F_\infty \ge F_n$ so that, using (3.33)

$$\begin{aligned} F_\infty(t,x) &= \int f(T_\infty, X_\infty(T_\infty))\, dP_{(t,x)} \le \underline{\lim}_{n} \int f(T_n, X_n(T_n))\, dP_{(t,x)} \\ &\le \underline{\lim}_{n} F_n(t,x) \le F_\infty(t,x) \; . \end{aligned} \tag{3.34}$$

Q.E.D.

A Related Result for $D[0,\infty)$

Let $\{X_n\}$ be a discrete time sequence of integer-valued processes where X_n has time lattice $\{m\phi(n)\}$, and where X_n has transitions governed by p_n where, for all integer x,y:

$$P\{X_n((m+1)\phi(n)) = y \mid X_n(m\phi(n)) = x\} = p_n(x,y) \; .$$

Let the continuous time version of X_n be defined by

$$X_n(t) = X_n(m\phi(n)) \quad \text{if} \quad t \in [m\phi(n), (m+1)\phi(n)) \; .$$

Each X_n is a pure-jump process with jumps occurring only at time values $m\phi(n)$. Let X_∞ be a pure-jump continuous time process which lives on the integers, hence is a random element in $D[0,\infty)$, and assume that X_∞ is non-explosive, i.e., X_∞ has at most finitely many jumps in finite time.

(3.35) Note: We'll assume also that the probability under all intial

(t,x) that X_∞ jumps at some fixed $s>t$ is zero.

Now suppose $X_n \xrightarrow{\mathcal{D}} X_\infty$ on $D[0,\infty)$; this type of weak convergence is treated in Whitt [ ] and Lindvall [ ]. We want then to relate F_n to F_∞ where, as usual

$$F_n(t,x) = \sup_{T\geq t} \int f(T,X_n(T))dP_{(t,x)} , \quad n \leq \infty .$$

Assumptions

(3.36) $f(t,x)$ is increasing in x, decreasing in t, continuous in t.

(3.37) $P_{(t,x)}\left\{\lim_{s\to\infty} f(s,x_n(s)) = 0\right\} = 1$, all initial (t,x).

(3.38) There exists non-decreasing G $C[0,)$, with

$$\Gamma_\infty = \{(t,x):\ x\geq G(t) \quad \text{or} \quad t = \infty\}$$

and we again interpret $T_\infty = \infty$ on paths where $X_\infty(t) < G(t)$, all t.

(3.39) For all large n, F_∞ is excessive for discrete time X_n.

Theorem (3.3)

If X_n, X_∞ above satisfy $X_n \xrightarrow{\mathcal{D}} X_\infty$ on $D[0,)$, and if (3.35) through (3.39) hold, then if T_n = minimal $m\phi(n)$ where $X_n(m\phi(n)) \geq G(t)$, we have

$$F_\infty(t,x) = \lim_{n\to\infty} F_n(t,x) = \lim_{n\to\infty} \int f(T_n,X_n(T_n))dP_{(t,x)} .$$

Proof: We remark, before entering into particulars, that $X_n \xrightarrow{\mathcal{D}} X_\infty$ is intended as weak convergence under all allowable intial (t,x). Also, note (3.35) constitutes an assumption which substitutes for (3.21) of the last theorem in that (3.35) implies that for small s, $X_\infty(T_\infty+s) \geq G(T_\infty+s)$ on $\{T_\infty < \infty\}$.

Just as before, we replace X_n be Y_n, $n \leq \infty$ where $X_n \overset{\mathcal{D}}{=} Y_n$ and $Y_n \to Y_\infty$ in the $D[0,\infty)$ metric for each path. Since all paths are integer valued, and since Y_∞ executes at most finitely many jumps in finite time, we see that for a particular path ω, $Y_n(\omega) \to Y_\infty(\omega)$ if and only if Y_n eventually makes exactly the same jumps as Y_∞, i.e., $Y_n(s_m^n) = Y_\infty(s_m^\infty)$ where s_m^n is the time of the m-th jump and further $s_m^n \to s_m^\infty$. Since G is continuous and since $Y_n(\omega)$ looks like $Y_\infty(\omega)$ except for a slight distortion of the time axis, we have $S_n \to S_\infty$ on $\{S_\infty < \infty\}$ where S_n, S_∞ are defined as in the last theorem. Also, since $Y_n(S_n) = Y_\infty(S_\infty)$ for large n on $\{S_\infty < \infty\}$, and since f is continuous in t, we have, given (3.37), that $f(S_n,Y_n(S_n))) \to f(S_\infty,Y_\infty(S_\infty))$. Continuing the logic and notation of the previous theorem, noting that $T_n = V_n$, we see that

$$\int f(T_\infty,X_\infty(T_\infty))dP_{(t,x)} \leq \underline{\lim_n} \int f(T_n,X_n(T_n))dP_{(t,x)}$$

and that

$F_\infty \geq F_n$, so that all conclusions follow. Q.E.D.

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