BIOMATHEMATICS TRAINING PROGRAM

DISTRIBUTION OF LIKELIHOOD RATIO TEST STATISTICS FOR NONSTATIONARY TIME SERIES

by

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ABSTRACT

Let the time series Y_t satisfy $Y_t = \alpha + \rho Y_{t-1} + e_t$, where $Y_1 = 0$ and $\{e_t\}_{t=1}^{n}$ is a sequence of normal independent $(0,\sigma^2)$ random variables. Let n observations $Y_1, Y_2, \ldots, Y_n$ from the time series be given. The likelihood ratio test of the hypothesis that $(\alpha,\rho) = (0,1)$ is investigated and a limit representation for the test statistic is presented. Percentage points for the limiting distribution and for finite sample distributions are estimated. The distribution of the least squares estimator of α is also discussed. A similar investigation is conducted for the model containing a time trend.

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1. Introduction

Let Y_t satisfy the model

$$Y_1 = 0$$

$$Y_t = \alpha + \rho Y_{t-1} + e_t , \qquad t = 2,3,\ldots,n, \tag{1.1}$$

where e_t is a sequence of normal independent random variables with mean 0 and variance σ^2 [$e_t \sim NID(0,\sigma^2)$]. The maximum likelihood estimators of ρ and α, conditional on Y_1 , are the least squares estimators

$$\hat{\rho}_\mu = \left[\sum_{t=2}^{n} (Y_{t-1} - \bar{y}_{(-1)})^2\right]^{-1} \sum_{t=2}^{n} (Y_t - \bar{y}_{(0)})(Y_{t-1} - \bar{y}_{(-1)})$$

$$\hat{\alpha}_\mu = \bar{y}_{(0)} - \hat{\rho}_\mu \bar{y}_{(-1)} , \tag{1.2}$$

where

$$\bar{y}_{(-1)} = (n-1)^{-1} \sum_{t=2}^{n} Y_{t-1}$$

$$\bar{y}_{(0)} = (n-1)^{-1} \sum_{t=2}^{n} Y_t .$$

In this article we investigate the limiting distribution of $\hat{\alpha}_\mu$ given that $(\alpha,\rho) = (0,1)$. We also study the limiting distribution of the likelihood ratio test of the hypothesis that $(\alpha,\rho) = (0,1)$.

An alternative model for Y_t is

$$Y_1 = 0$$

$$Y_t = \alpha + \beta(t-1-\tfrac{1}{2}n) + \rho Y_{t-1} + e_t \,, \qquad t = 2,3,\ldots,n, \qquad (1.3)$$

where, as before, $e_t \sim NID(0,\sigma^2)$. We study the least squares estimators of α and β of (1.3) under the assumption that $(\alpha,\beta,\rho) = (0,0,1)$. We also study the likelihood ratio test of the hypotheses $(\alpha,\beta,\rho) = (0,0,1)$.

Empirical distributions of the statistics are generated by Monte Carlo methods for finite sample sizes. Representations based on the results of Dickey (1976) are presented for the limiting distributions and the limiting distributions are simulated using these representations.

2. Distribution of normalized regression coefficients for finite n .

The statistic constructed by analogy to the regression "t-statistic" for the estimated α of model (1.1) is

$$\tau_{\alpha\mu} = S_{\alpha\mu}^{-1} \hat{\alpha}_\mu ,$$

$$= S_{\alpha\mu}^{-1} [\bar{e}_{(0)} - (\hat{\rho}_\mu - 1)(n - 1)^{-1} \sum_{t=1}^{n-1} (n - t)e_t] , \qquad (2.1)$$

where

$$S_{\alpha\mu}^2 = S_{e\mu}^2 [(n-1)^{-1} + \bar{y}_{(-1)}^2 \{\sum_{t=2}^{n} (Y_{t-1} - \bar{y}_{(-1)})^2\}^{-1}] \qquad (2.2)$$

$$S_{e\mu}^2 = (n-3)^{-1} \sum_{t=2}^{n} (Y_t - \hat{\alpha}_\mu - \hat{\rho}_\mu Y_{t-1})^2 \qquad (2.3)$$

$$\bar{e}_{(0)} = (n-1)^{-1} \sum_{t=2}^{n} e_t \quad .$$

Using the model

$$Y_0 = 0$$

$$Y_t = Y_{t-1} + e_t \qquad t = 2,3,\ldots,n,$$

the sampling distribution of $\tau_{\alpha\mu}$ was simulated. Because $\hat{\rho}_{\mu} - 1$ is a ratio of quadratic forms in $(Y_1, Y_2, \ldots, Y_n)$ it follows that the distribution of $\tau_{\alpha\mu}$ is symmetric. Therefore cells equidistant from zero were pooled to create a symmetric histogram. The histogram of $\tau_{\alpha\mu}$ for 50,000 samples with $n = 25$ is shown in Figure 1. Note that the distribution is bimodal. This is because $n(\hat{\rho}_{\mu} - 1)$ has a non-zero mode for positive values of $\sum_{t=1}^{n-1} (n-t)e_t$ as well as for negative values $\sum_{t=1}^{n-1} (n-t)e_t$.

Let $\underset{\sim}{X}$ denote the $(n-1) \times 3$ matrix whose i^{th} row is $(1, i-\frac{1}{2}n, Y_i)$ and let $\underset{\sim}{Y}' = (Y_2, Y_3, \ldots, Y_n)$. Then the least squares estimator of $\underset{\sim}{\theta} = (\alpha, \beta, \rho)'$ is

$$\hat{\underset{\sim}{\theta}} = (\hat{\alpha}_{\tau}, \hat{\beta}_{\tau}, \hat{\rho}_{\tau})' = (\underset{\sim}{X}'\underset{\sim}{X})^{-1} \underset{\sim}{X}'\underset{\sim}{Y} . \tag{2.4}$$

Let C_{ij} denote the ij^{th} element of $(\underset{\sim}{X}'\underset{\sim}{X})^{-1}$ and define

$$\tau_{\alpha\tau} = (C_{11} S_{e\tau}^2)^{-\frac{1}{2}} \hat{\alpha}_{\tau} , \tag{2.5}$$

$$\tau_{\beta\tau} = (C_{22} S_{e\tau}^2)^{-\frac{1}{2}} \hat{\beta}_{\tau} , \tag{2.6}$$

where

$$S^2_{e\tau} = (n-4)^{-1} \underset{\sim}{Y}' [\underset{\sim}{I} - \underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X})^{-1} \underset{\sim}{X}']\underset{\sim}{Y} \quad . \tag{2.7}$$

Figures 2 and 3 contain histograms for $\tau_{\alpha\tau}$ and $\tau_{\beta\tau}$ constructed from 50,000 samples of size $n = 25$. All of the distributions are symmetric and the histograms were constructed to be symmetric.

The distributions of the τ-statistics are distinctive in two respects; the distribution is bimodal and the "spread" of the distribution is much larger than that of Student's t-distribution. The estimation of percentiles for the distributions is discussed in Section 5.

3. Likelihood ratio tests

We construct the likelihood ratio tests for testing the null hypothesis that the true model is a random walk with zero drift. We consider first the test under the alternative (1.3). The logarithm of the likelihood function for a sample of n observations from model (1.3), conditional on Y_1, is

$$\log L = -\tfrac{1}{2}(n-1)\log(2\pi) - (n-1)\log\sigma$$

$$- (2\sigma^2)^{-1} \sum_{t=2}^{n} [Y_t - \alpha - \beta(t-1-\tfrac{1}{2}n) - \rho Y_{t-1}]^2 \quad . \qquad (3.1)$$

Under the null hypothesis, H_0: $(\alpha,\beta,\rho) = (0,0,1)$, the likelihood is maximized with respect to σ^2 to obtain

$$\hat{\sigma}_0^2 = (n-1)^{-1} \sum_{t=2}^{n} (Y_t - Y_{t-1})^2 \quad .$$

Under the alternative hypothesis the maximum of the likelihood occurs at $(\hat{\sigma}_1^2, \hat{\underset{\sim}{\theta}}{}')$, where $\hat{\underset{\sim}{\theta}} = (\hat{\alpha}_\tau, \hat{\beta}_\tau, \hat{\rho}_\tau)'$ was defined in (2.2), and

$$\hat{\sigma}_1^2 = (n-1)^{-1} \sum_{t=2}^{n} (Y_t - \hat{\alpha}_\tau - \hat{\beta}_\tau(t-1-\tfrac{1}{2}n) - \hat{\rho}_\tau Y_{t-1})^2 \quad .$$

Thus the likelihood ratio is

$$[\hat{\sigma}_0^{-1} \hat{\sigma}_1]^{n-1} \quad .$$

Using

$$(n-1)\hat{\sigma}_1^2 = \sum_{t=2}^{n}(Y_t - Y_{t-1})^2 - \sum_{t=2}^{n}[\hat{\alpha}_\tau + \hat{\beta}_\tau(t-1-\tfrac{1}{2}n) + (\hat{\rho}_\tau - 1)Y_{t-1}](Y_t - Y_{t-1})$$

we obtain

$$(\hat{\sigma}_1^2)^{-1}\hat{\sigma}_0^2 = 1 + [(n-1)\hat{\sigma}_1^2]^{-1}\sum_{t=2}^{n}[\hat{\alpha}_\tau + \hat{\beta}_\tau(t-1-\tfrac{n}{2}) + (\hat{\rho}_\tau - 1)Y_{t-1}](Y_t - Y_{t-1})$$

$$= 1+3(n-4)^{-1}\Phi_2 ,$$

where

$$\Phi_2 = (3S_{e\tau}^2)^{-1}[(n-1)\hat{\sigma}_0^2 - (n-4)S_{e\tau}^2]$$

and $S_{e\tau}^2$ was defined in (2.7). Thus the likelihood ratio test rejects H_0 for large values of Φ_2, where Φ_2 is the usual regression "F-test" of the hypothesis H_0: $(\alpha,\beta,\rho) = (0,0,1)$.

In a similar manner, it can be shown that the likelihood ratio statistic for testing H_0: $(\alpha,\rho) = (0,1)$ for the model (1.1) is

$$[1 + 2(n-3)^{-1}\Phi_1]^{\frac{1}{2}(n-1)} ,$$

where

$$\Phi_1 = (2S_{e\mu}^2)^{-1}[(n-1)\hat{\sigma}_0^2 - (n-3)S_{e\mu}^2]$$

and $S_{e\mu}^2$ was defined in (2.3).

The likelihood ratio test of the hypothesis $(\beta,\rho) = (0,1)$ against the alternative specified in model (1.3) is a monotone function of

$$\Phi_3 = (2S_{e\tau}^2)^{-1}\,[(n-1)\,\{\hat{\sigma}_0^2 - (\bar{y}_{(0)} - \bar{y}_{(-1)})^2\} - (n-4)S_{e\tau}^2] \ .$$

As with the other tests the statistic Φ_3 is the common "F-test" one would construct for the hypothesis. In this situation the null hypothesis is a random walk in which drift α is permitted. It is easily demonstrated that the distribution of the test statistic does not depend upon α .

4. Limiting distributions

The several statistics that we have discussed can all be expressed as functions of few sample statistics. Let

$$\Gamma_n = (n-1)^{-2} \sum_{t=2}^{n} \left(\sum_{j=1}^{t-1} e_j \right)^2 ,$$

$$T_n = (n-1)^{-\frac{1}{2}} \sum_{t=2}^{n} e_t = (n-1)^{-\frac{1}{2}} \bar{e}_{(0)}$$

$$W_n = (n-1)^{-\frac{3}{2}} \sum_{t=1}^{n-1} (n-t) e_t = (n-1)^{-\frac{1}{2}} \bar{Y}_{(-1)} ,$$

$$V_n = (n-1)^{-5/2} \sum_{t=1}^{n-1} (n-t)(t-1) e_t . \tag{4.1}$$

Then, for example,

$$(n-1)^{\frac{1}{2}} \hat{\alpha}_\mu = T_n - (\hat{\rho}_\mu - 1) W_n$$

and

$$(n-1)(\hat{\rho}_\mu - 1) = (\Gamma_n - W_n^2)^{-1} \{ \tfrac{1}{2} [(T_n + (n-1)^{-\frac{1}{2}} e_1)^2 - (n-1)^{-1} \sum_{t=1}^{n} e_t^2] - T_n W_n \}$$

$$= (\Gamma_n - W^2)^{-1} \{ \tfrac{1}{2}(T_n^2 - \sigma^2) - T_n W_n \} + O_p(n^{-\frac{1}{2}})$$

Given that $\sigma^2 = 1$, Dickey (1976) has shown that $[\Gamma_n, T_n, W_n, V_n, n(\hat{\rho}_\mu - 1)]$

converges in distribution to $(\Gamma, T, W, V, \delta)$, where

$$\Gamma = \sum_{i=1}^{\infty} \gamma_i^2 Z_i^2 ,$$

$$T = \sum_{i=1}^{\infty} 2^{\frac{1}{2}} \gamma_i Z_i ,$$

$$W = \sum_{i=1}^{\infty} 2^{\frac{1}{2}} \gamma_i^2 Z_i ,$$

$$V = \sum_{i=1}^{\infty} (2^{\frac{3}{2}} \gamma_i^3 - 2^{\frac{1}{2}} \gamma_i^2) Z_i ,$$

$$\gamma_i = \frac{2}{(2i-1)\pi} (-1)^{i+1} ,$$

$$\delta = (\Gamma - W^2)^{-1} [\tfrac{1}{2}(T^2 - 1) - TW] \tag{4.2}$$

and $\{Z_i\}_{i=1}^{\infty}$ is a sequence of normal independent $(0,1)$ random variables. It follows that

$$n^{\frac{1}{2}} \sigma^{-1} \hat{\alpha}_{\mu} \xrightarrow{\mathcal{L}} T - \delta W ,$$

under the assumption that $(\alpha, \rho) = (0,1)$.

In a similar manner

$$\tau_{\alpha\mu} \xrightarrow{\mathcal{L}} (T - \delta W)(\Gamma - W^2)^{\frac{1}{2}} \Gamma^{-\frac{1}{2}} , \tag{4.3}$$

because $S_{e\mu}^2$ converges in probability to σ^2 .

For model (1.3) with the assumption that $(\alpha, \beta, \rho) = (0, 0, 1)$ and that $\sigma^2 = 1$, we have

$$\underset{\sim}{X}'\underset{\sim}{X} = (n-1)\begin{pmatrix} 1 & 0 & (n-1)^{\frac{1}{2}} W_n \\ 0 & 12^{-1} n(n-2) & \frac{1}{2}(n-1)^{\frac{3}{2}} V_n \\ (n-1)^{\frac{1}{2}} W_n & \frac{1}{2}(n-1)^{\frac{3}{2}} V_n & (n-1)\Gamma_n \end{pmatrix} .$$

Letting

$$\underset{\sim}{D}_n = \text{diag}[(n-1)^{\frac{1}{2}} , (n-1)^{\frac{3}{2}} , n-1] ,$$

$$\underset{\sim}{A} = \begin{pmatrix} 1 & 0 & W \\ 0 & \frac{1}{12} & \frac{1}{2}V \\ W & \frac{1}{2}V & \Gamma \end{pmatrix} ,$$

we obtain

$$\underset{\sim}{D}_n^{-1} \underset{\sim}{X}'\underset{\sim}{X} \underset{\sim}{D}_n^{-1} \xrightarrow{\mathcal{L}} \underset{\sim}{A} . \tag{4.4}$$

Under our assumptions, $\hat{\underset{\sim}{\theta}}$ is estimating $\underset{\sim}{\theta} = (0,0,1)'$ and we have

$$\underset{\sim}{D}_n^{-1} (\underset{\sim}{X}'\underset{\sim}{Y} - \underset{\sim}{X}'\underset{\sim}{X} \underset{\sim}{\theta}) \xrightarrow{\mathcal{L}} \underset{\sim}{f} , \tag{4.5}$$

where $\underset{\sim}{f} = (T, \frac{1}{2}T - W, \frac{1}{2}(T^2 - \sigma^2))'$. The matrix is invertible with probability 1 and it is readily verified that

$$\underset{\sim}{A}^{-1} = Q^{-1} \begin{pmatrix} Q + W^2 & 6\,VW & -W \\ 6VW & 12Q + 36V^2 & -6V \\ -W & -6V & 1 \end{pmatrix} , \tag{4.6}$$

where $Q = \Gamma - W^2 - 3V^2$. Thus

$$\underset{\sim}{D}_n(\underset{\sim}{X}'\underset{\sim}{X})^{-1}\underset{\sim}{D}_n \xrightarrow{\mathcal{L}} \underset{\sim}{A}^{-1} \tag{4.7}$$

and

$$\underset{\sim}{D}_n(\hat{\underset{\sim}{\theta}} - \underset{\sim}{\theta}) \xrightarrow{\mathcal{L}} \underset{\sim}{A}^{-1}\underset{\sim}{f} \; . \tag{4.8}$$

The third element of $\underset{\sim}{A}^{-1}\underset{\sim}{f}$ is the limit random variable for $n(\hat{\rho}_\tau - 1)$ as given in Dickey (1976).

Using (4.6) and the fact that S_{eT}^2 converges in probability σ^2, we obtain

$$\tau_{\alpha T} \xrightarrow{\mathcal{L}} Q^{-\frac{1}{2}} (Q + W^2)^{\frac{1}{2}} (1,0,0)\underset{\sim}{A}^{-1} \underset{\sim}{f}$$

and

$$\tau_{\beta T} \xrightarrow{\mathcal{L}} Q^{-\frac{1}{2}} (12Q + 36V^2)^{\frac{1}{2}} (0,1,0)\underset{\sim}{A}^{-1} \underset{\sim}{f} \ . \tag{4.9}$$

Likewise

$$\Phi_1 \xrightarrow{\mathcal{L}} 2^{-1} \{T^2 + \delta^2(\Gamma - W^2)\} , \tag{4.10}$$

$$\Phi_2 \xrightarrow{\mathcal{L}} 3^{-1} \underset{\sim}{f}'\underset{\sim}{A}^{-1} \underset{\sim}{f} = 3^{-1}[T^2 + 12(\tfrac{1}{2}T - W)^2 + \tau_{\tau}^2], \tag{4.11}$$

and

$$\Phi_3 \xrightarrow{\mathcal{L}} 2^{-1} (\underset{\sim}{f}' \underset{\sim}{A}^{-1} \underset{\sim}{f} - T^2) = 2^{-1}[12(\tfrac{1}{2}T - W)^2 + \tau_{\tau}^2] \tag{4.12}$$

where τ_τ is the limit random variable for the regression "t-statistic" for $\rho = 1$ in model (1.3).

5. Simulation

The distributions of the statistics for finite samples were simulated for time series generated by the model with $Y_0 = 0$ and $Y_t = Y_{t-1} + e_t$, $t = 1,2,\ldots,n$ for $n = 25$, 50, 100, 250, and 500. For each time series 50,000 samples of size n were generated and the statistics computed for those samples. Three replicates of 50,000 were generated for $n = 25$, two for $n = 50$, 100, and 250, and one for $n = 500$. The simulation of the limit case was conducted using Dickey's (1976) procedure. Three replicates of 50,000 were generated for the limit case.

For each of the nine estimators and for each sample size, the 0.01, 0.025, 0.05, 0.10, 0.90, 0.95, 0.975, and 0.99 percentage points of the distributions were calculated. These empirical percentiles were then plotted against n . Based on the plots, regression functions of the form $P = \alpha + \beta n^{\gamma}$ were fitted to the percentiles of the empirical distributions. The regression smoothed percentiles are given in Tables 1 through 9 .

David (1970 section 2.5) gives a method for constructing distribution free confidence intervals for the percentiles of a distribution based on empirical percentiles. We used the half length of a 68.26% confidence interval as an estimated standard error (based on the fact that 0.6826 is the probability that a normal random variable will differ from its mean by no more than one standard deviation). In Tables 1 through 8 the number in the row labeled "s.e." is the largest of the standard errors constructed for $n = 25$ and for the limit case. These standard errors provide an upper bound for the standard errors of the regression smoothed percentiles.

Since several observations on each percentile were available for n = 25, 50, 100, 250, and for the limit case, regression F-tests for lack of fit for the smoothing regressions were computed. Of the 48 lack of fit statistics computed, 13 were significant at the 0.25 level, 3 at the 0.05 level, and none at the 0.01 level.

6. Distributions for higher order processes

In this section we demonstrate that the test statistics investigated in the previous sections can be applied in higher order autoregressive processes.

Consider data generated by the model

$$Y_0 = 0$$

$$Y_t = Y_{t-1} + Z_t \ , \tag{6.1}$$

where

$$Z_t = \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \dots + \theta_p Z_{t-p} + e_t$$

is a stationary autoregressive process and the e_t are $\mathrm{NID}(0,\sigma^2)$. The model can also be written

$$Y_t = \rho Y_{t-1} + \sum_{i=1}^{p} \theta_i (Y_{t-i} - Y_{t-1-i}) + e_t \ ,$$

where $\rho = 1$ and $Z_t = Y_t - Y_{t-1}$. To simplify the presentation we assume, without loss of generality, $\sigma^2 = 1$.

Consider the regression equation

$$Y_{t+p} = \alpha + \beta \left[t - \tfrac{1}{2}(n-p+1)\right] + \rho \, Y_{t+p-1} + \sum_{i=1}^{p} \theta_i Z_{t+p-i} + e_t \ ,$$

$t = 1,2,\dots,n-p$. Let $\underset{\sim}{H}_n$ denote the $(p+3) \times (p+3)$ sums of squares and products matrix needed to compute the regression, let $\underset{\sim}{M}_n$ denote the square roots of the diagonal elements of $\underset{\sim}{H}_n$, let $\underset{\sim}{\gamma}_n' = (\alpha, \beta, \rho, \theta_1, \theta_2, \ldots, \theta_p)$ and let $\hat{\underset{\sim}{\gamma}}_n'$ denote the least squares estimator of $\underset{\sim}{\gamma}_n'$. Then

$$M_n(\hat{\underset{\sim}{\gamma}}_n - \underset{\sim}{\gamma}_n) = [\underset{\sim}{M}_n^{-1} \underset{\sim}{H}_n \underset{\sim}{M}_n^{-1}]^{-1} M_n^{-1} \underset{\sim}{g}_n \quad ,$$

where

$$\underset{\sim}{g}_n' = \sum_{t=1}^{n-p} (1,\ t,\ Y_{t+p-1},\ Z_{t+p-1},\ \ldots,\ Z_t)'\ e_t \ .$$

Fuller (1976, p. 374) has demonstrated that $n^{-\frac{1}{2}} Y_t$ is converging to $n^{-\frac{1}{2}}(1 - \sum_{i=1}^{p} \theta_i)^{-1} \sum_{j=1}^{t} e_j$ as t increases. By the results of Fuller, we have,

$$n^{-1} \sum_{t=2}^{n} Z_t = O_p(n^{-\frac{1}{2}})$$

$$\sum_{t=2}^{n} Y_{t-1}^2 = O_p(n^2)$$

$$n^{-2} \sum_{t=1}^{n} [t - \tfrac{1}{2}(n+p-1)]Z_{t+p-j} = O_p(n^{-\frac{1}{2}})$$

$$\sum_{t=2}^{n} Y_{t-1} Z_{t-1} = O_p(n) \ .$$

Therefore

$$\text{plim } \underset{\sim}{M}_n^{-1} \underset{\sim}{H}_n \underset{\sim}{M}_n^{-1} = \text{Block diag}(\underset{\sim}{H}_{11}, \underset{\sim}{H}_{22}) ,$$

where

$$\underset{\sim}{H}_{11} = \begin{pmatrix} 1 & 0 & \Gamma^{-\frac{1}{2}} W \\ 0 & 1 & \Gamma^{-\frac{1}{2}} 3^{\frac{1}{2}} V \\ \Gamma^{-\frac{1}{2}} W & \Gamma^{-\frac{1}{2}} 3^{\frac{1}{2}} V & 1 \end{pmatrix}$$

$\underset{\sim}{H}_{22}$ is the $p \times p$ correlation matrix of the process Z_t, and Γ, W, and V were defined in (4.2). It follows that the limiting distribution of the vector composed of the first three elements of $\underset{\sim}{M}_n(\hat{\underset{\sim}{\gamma}}_n - \underset{\sim}{\gamma}_n)$ is the same as the limiting distribution of

$$[n^{\frac{1}{2}} \hat{\alpha}_\tau , n^{\frac{3}{2}} \hat{\beta}_\tau , \left(\sum_{t=2}^{n} Y_{t-1}^2 \right)^{\frac{1}{2}} (\hat{\rho}_\tau - 1)]$$

discussed in section 4.

7. Example

Friedman and Schwartz (1963) give yearly observations of the velocity of money from 1869 through 1960 ($n = 92$). Gould and Nelson (1974) conclude that the logarithms of the observations are consistent with the model $X_t = X_{t-1} + e_t$, where $e_t \sim NID(0, \sigma^2)$. To illustrate the use of Tables 1 through 9 in hypothesis testing, we fit two models to the data. Below we list the models with the fitted coefficients, the standard errors of the coefficients, the regression error mean square and the regression Φ statistic. Note that using $X_t - X_{t-1}$ on the left side of the regression equation yields an estimator of $\rho - 1$ for the regression coefficient of X_{t-1}. The regression statistics are:

$$\begin{aligned} X_t - X_{t-1} &= \underset{(0.017)}{0.016} - \underset{(0.058)}{0.034} X_{t-1} \\ s^2_{e\mu} &= 0.0050, \; \Phi_1 = 2.99 \end{aligned} \tag{7.1}$$

$$\begin{aligned} X_t - X_{t-1} &= \underset{(0.047)}{0.086} - \underset{(0.0008)}{0.0013}(t-47) - \underset{(0.058)}{0.120} X_{t-1} \\ s^2_{e\tau} &= 0.0049, \; \Phi_2 = 2.77 \end{aligned} \tag{7.2}$$

For regression equation (7.1) the test statistic is $\Phi_1 = 2.99$ which is smaller than the tabular 0.90 value of 3.88 (obtained by interpolating between $n = 50$ and $n = 100$ in Table 7). Therefore the hypothesis of a random walk with zero drift is accepted at the 0.10 level when tested against the alternative (1.1).

For regression equation (7.2) the statistic for testing H_0: $(\alpha, \beta, \rho) = (0, 0, 1)$ is $\Phi_2 = 2.77$. Comparing this to the value 4.19 obtained by interpolation from Table 8 the hypothesis of a random walk is accepted at the 0.10 level when tested against the alternative (1.3).

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Table 1. Empirical Distribution of $n^{\frac{1}{2}} \sigma^{-1} \hat{\alpha}_\mu$
(Symmetric Distribution)

Sample size n	Probability of a smaller value			
	0.90	0.95	0.975	0.99
25	4.32	5.77	7.17	8.92
50	4.40	5.88	7.31	9.15
100	4.42	5.93	7.39	9.27
250	4.43	5.96	7.45	9.34
500	4.43	5.97	7.47	9.37
∞	4.43	5.99	7.50	9.40
s.e.	0.01	0.02	0.02	0.03

Table 2. Empirical Distribution of $\tau_{\alpha\mu}$
(Symmetric Distribution)

Sample size n	Probability of a smaller value			
	0.90	0.95	0.975	0.99
25	2.20	2.61	2.97	3.41
50	2.18	2.56	2.89	3.28
100	2.17	2.54	2.86	3.22
250	2.16	2.53	2.84	3.19
500	2.16	2.52	2.83	3.18
∞	2.16	2.52	2.83	3.18
s.e.	0.003	0.004	0.006	0.008

Table 3. Empirical Distribution of $n^{\frac{1}{2}} \sigma^{-1} \hat{\alpha}_T$ (Symmetric Distribution)

Sample size n	Probability of a smaller value			
	0.90	0.95	0.975	0.99
25	7.63	10.49	13.41	17.26
50	7.95	11.07	14.22	18.37
100	8.11	11.34	14.62	19.00
250	8.20	11.50	14.87	19.43
500	8.23	11.55	14.95	19.60
∞	8.26	11.59	15.04	19.82
s.e.	0.02	0.03	0.05	0.07

Table 4. Empirical Distribution of $\tau_{\alpha\tau}$
(Symmetric Distribution)

Sample size n	Probability of a smaller value			
	0.90	0.95	0.975	0.99
25	2.77	3.20	3.59	4.05
50	2.75	3.14	3.47	3.87
100	2.73	3.11	3.42	3.78
250	2.73	3.09	3.39	3.74
500	2.72	3.08	3.38	3.72
∞	2.72	3.08	3.38	3.71
s.e.	0.004	0.005	0.007	0.008

Table 5. Empirical Distribution of $n^{\frac{1}{2}} \sigma^{-1} \hat{\beta}_T$
(Symmetric Distribution)

Sample size n	Probability of a smaller value			
	0.90	0.95	0.975	0.99
25	19.73	19.06	24.34	31.47
50	14.28	19.98	25.81	33.74
100	14.57	20.48	26.60	34.98
250	14.76	20.81	27.11	35.80
500	14.83	20.93	27.29	36.11
∞	14.90	21.06	27.50	36.48
s.e.	0.04	0.06	0.10	0.14

Table 6. Empirical Distribution of $\tau_{\beta\tau}$
(Symmetric Distribution)

Sample size n	Probability of a smaller value			
	0.90	0.95	0.975	0.99
25	2.39	2.85	3.25	3.74
50	2.38	2.81	3.18	3.60
100	2.38	2.79	3.14	3.53
250	2.38	2.79	3.12	3.49
500	2.38	2.78	3.11	3.48
∞	2.38	2.78	3.11	3.46
s.e.	0.004	0.005	0.006	0.009

Table 7. Empirical Distribution of Φ_1

Sample size n	Probability of a smaller value							
	0.01	0.025	0.05	0.10	0.90	0.95	0.975	0.99
25	0.29	0.38	0.49	0.65	4.12	5.18	6.30	7.88
50	0.29	0.39	0.50	0.66	3.94	4.86	5.80	7.06
100	0.29	0.39	0.50	0.67	3.86	4.71	5.57	6.70
250	0.30	0.39	0.51	0.67	3.81	4.63	5.45	6.52
500	0.30	0.39	0.51	0.67	3.79	4.61	5.41	6.47
∞	0.30	0.40	0.51	0.67	3.78	4.59	5.38	6.43
s.e.	0.002	0.002	0.002	0.002	0.01	0.02	0.03	0.05

Table 8. Empirical Distribution of Φ_2

Sample size n	Probability of a smaller value							
	0.01	0.025	0.05	0.10	0.90	0.95	0.975	0.99
25	0.61	0.75	0.89	1.10	4.67	5.68	6.75	8.21
50	0.62	0.77	0.91	1.12	4.31	5.13	5.94	7.02
100	0.63	0.77	0.92	1.12	4.16	4.88	5.59	6.50
250	0.63	0.77	0.92	1.13	4.07	4.75	5.40	6.22
500	0.63	0.77	0.92	1.13	4.05	4.71	5.35	6.15
∞	0.63	0.77	0.92	1.13	4.03	4.68	5.31	6.09
s.e.	0.003	0.003	0.003	0.003	0.01	0.02	0.03	0.05

Table 9. Empirical Distribution of Φ_3

Sample size n	Probability of a smaller value							
	0.01	0.025	0.05	0.10	0.90	0.95	0.975	0.99
25	0.74	0.90	1.08	1.33	5.91	7.24	8.65	10.61
50	0.76	0.93	1.11	1.37	5.61	6.73	7.81	9.31
100	0.76	0.94	1.12	1.38	5.47	6.49	7.44	8.73
250	0.76	0.94	1.13	1.39	5.39	6.34	7.25	8.43
500	0.76	0.94	1.13	1.39	5.36	6.30	7.20	8.34
00	0.77	0.94	1.13	1.39	5.34	6.25	7.16	8.27
s.e.	.004	.004	.003	.004	.015	.020	.032	.058

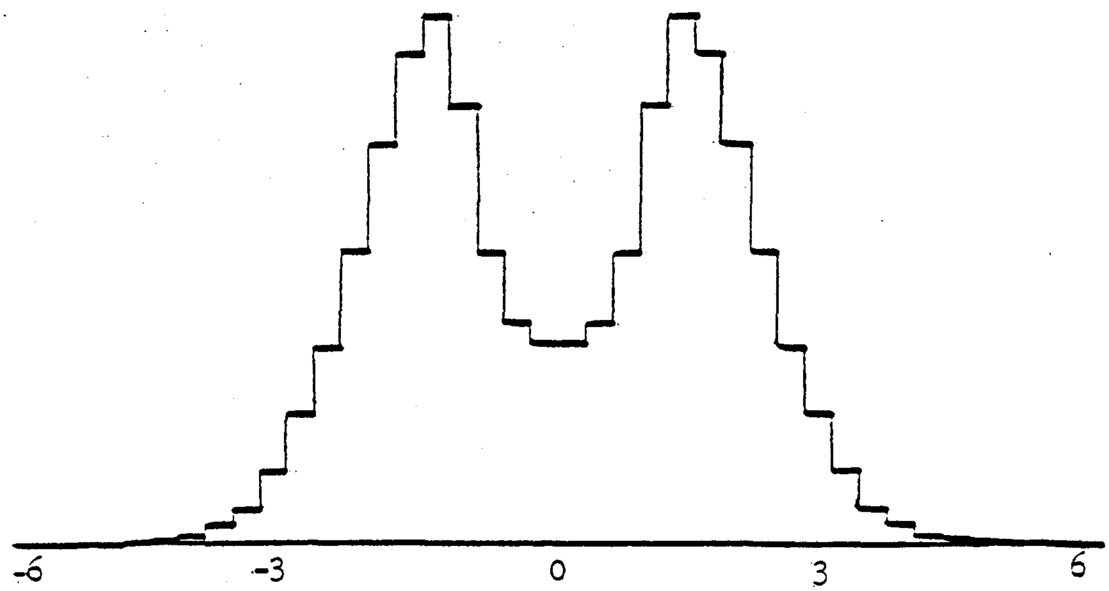

Figure 1. Histogram for 50,000 values of $\tau_{\alpha\mu}$ constructed with $n = 25$

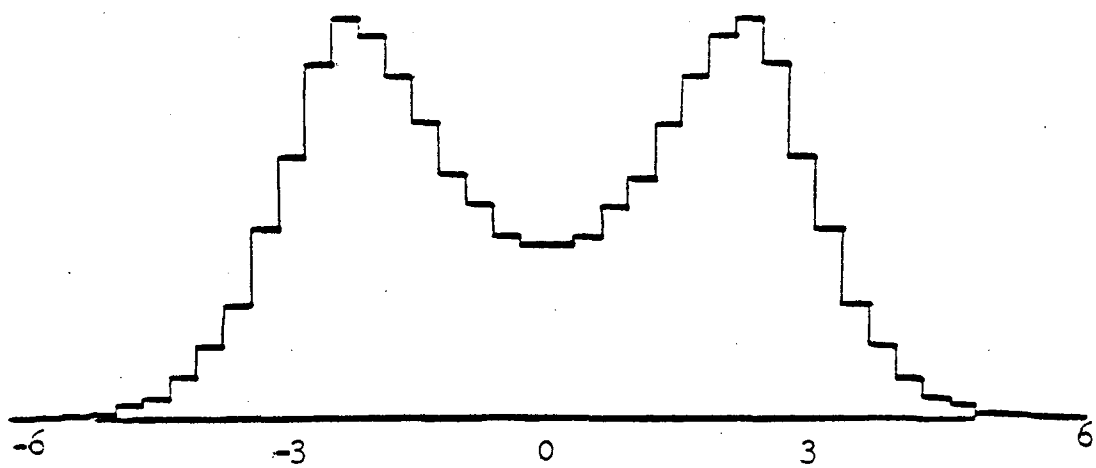

Figure 2. Histogram for 50,000 values of $\tau_{\alpha\tau}$ constructed with $n = 25$

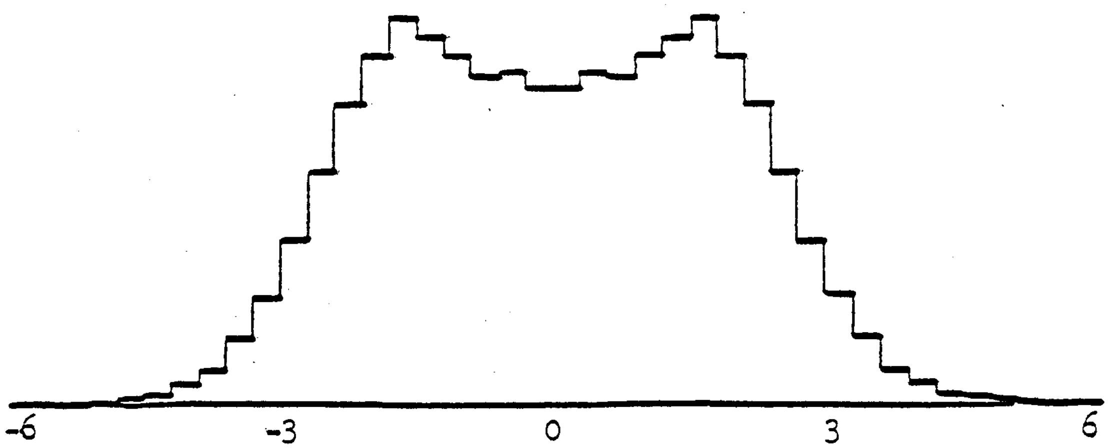

Figure 3. Histogram for 50,000 values of $\tau_{\beta T}$ constructed with $n = 25$