COMBINING EXPERIMENTS TO ESTIMATE VARIANCE COMPONENTS

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Institute of Statistics
Mimeograph Series No. 1182
Raleigh - July 1978

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ABSTRACT

Occasionally information from a series of balanced experiments is to be combined to provide estimates of variance components. Combining the data typically leads to an unbalanced set and consequently makes the variance component estimation problem difficult. It is shown that the Minimum Norm Quadratic Unbiased Estimation theory leads a system of equations that can be formed by taking linear functions of the sums of squares obtained in the analyses of the separate data sets with weights proportional to the reciprocals of the squares of the expected values of the sums of squares evaluated at the prior values for the components. Limited simulation studies indicate that the iterative scheme implied above converges rapidly and is insensitive to choice of starting values. An intermediate result is the proof that MINQUE and the analysis of variance yield identical estimates for balanced data sets.

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1. INTRODUCTION

The analysis of variance method has long been accepted as the appropriate method for estimating variance components in the case of balanced data sets and the classical linear model. Under modest distributional assumptions, this method is known to yield minimum variance quadratic unbiased estimates and if one assumes normality then minimum variance unbiased estimates (Searle 1971, Graybill 1961). For unbalanced data sets Rao (1971a) has proposed a method which yields minimum norm quadratic unbiased estimates (MINQUE) of the variance components. If good prior information about the relative magnitudes of the variance components is available and normality holds, then Rao's (1971b) minimum variance quadratic unbiased estimation (MIVQUE) technique is available. These estimates are minimum variance in the sense that if the prior values agree with the true (but unknown) values, then the estimates have minimum variance among all unbiased quadratic estimates. Unfortunately computing either MINQUE or MIVQUE for large data sets can be rather formidable.

Patterson and Thompson (1971) proposed the restricted maximum liklihood (REML) method for estimating variance components. Harville (1975) has shown that if the prior values used to obtain MINQUE agree with the final estimates obtained then MINQUE and REML coincide. Giesbrecht and Burrows (1976) have shown that the REML equation can be arranged to correspond exactly with the MINQUE equations. The iterative scheme suggested by MINQUE corresponds exactly with REML. Sriburi (1978) has shown that the REML estimates are consistent, asymptotically normal and efficient in the sense of attaining the Cramér-Rao lower bound for the covariance matrix.

The object of this paper is to obtain the MINQUE or REML equations when the data set is sufficiently balanced to permit an orthogonal partitioning of the total sum of squares. This is particularly relevant when a number of balanced but dissimilar data sets are to be combined in order to estimate variance components. The quadratic forms in these equations are shown to be weighted linear functions of the sums of squares in the separate analyses of variance. The weights are simple functions of the prior values for the variance components. If one does not iterate but uses prior values for the variance components as fixed constants then the estimates are linear functions of orthogonal quadratic forms. Variances and covariances of the estimates can be obtained directly. If an iterative scheme is used to obtain the REML estimates then asymptotic variances and covariances of the estimates are available. A small Monte Carlo study indicates not only that the method tends to converge rapidly but also that (a) the answers are not seriously biased, (b) the formula for the asymptotic variances and covariances is applicable and (c) in some cases the REML estimators are much more efficient than estimators obtained by other methods.

2. THE MINQUE EQUATIONS

The linear model treated by Rao (1971a, 1971b) can be written as

$$\underset{\sim}{Y} = \underset{\sim}{X}_0\underset{\sim}{\beta} + X_1\epsilon_1 + \cdots + \underset{\sim}{X}_m\underset{\sim}{\epsilon}_m \tag{2.1}$$

where $\underset{\sim}{Y}$ is the n-vector of observations, the $\{\underset{\sim}{X}_i\}$ are known n x c_i matrices, $\underset{\sim}{\beta}$ is a c_0-vector of unknown parameters and the $\{\underset{\sim}{\epsilon}_i\}$ are independent c_i-vectors of independent random variables with mean zero and variance σ_i^2 respectively. It follows that the variance-covariance matrix of $\underset{\sim}{Y}$ is given by

$$\underset{\sim}{V}^* = \sum_{i=1}^{m} \sigma_i^2 \underset{\sim}{V}_i$$

where $\underset{\sim}{V}_i = \underset{\sim}{X}_i\underset{\sim}{X}_i'$ for i = 1, ⋯, m.

The MINQUE principle leads one to:

a) Compute $\underset{\sim}{V} = \Sigma\, \alpha_i \underset{\sim}{V}_i$

where the $\{\alpha_i\}$ are proportional to prior estimates of the unknown $\{\sigma_i^2\}$. Note that $\underset{\sim}{V}$ is a function of the $\{\alpha_i\}$ and that the notation should indicate as much. However, this is cumbersome and the risk of confusion resulting from the imprecise notation is minimal.

b) Compute $\underset{\sim}{Q}_V = \underset{\sim}{V}^{-1} - \underset{\sim}{V}^{-1}\underset{\sim}{X}_0(\underset{\sim}{X}_0'\underset{\sim}{V}^{-1}\underset{\sim}{X}_0)^{-}\underset{\sim}{X}_0'\underset{\sim}{V}^{-1}$

where $\underset{\sim}{A}^{-}$ is the Moore-Penrose generalized inverse of $\underset{\sim}{A}$.

c) Compute the quadratic forms

$\underset{\sim}{Y}'\underset{\sim}{Q}_V\underset{\sim}{V}_i\underset{\sim}{Q}_V\underset{\sim}{Y}$ for i = 1, ⋯, m.

d) Express the expected values of the quadratic forms in (c) as linear functions of the components to be estimated and solve the system of equations obtained by equating observed quadratic forms to their expected values.

For this paper a model with somewhat more structure than that implied by (2.1) is required. In particular, it is assumed that

$\underset{\sim}{X}_i = (\underset{\sim}{X}_{i1} \mid \cdots \mid \underset{\sim}{X}_{is_i})$ for i = 0, 1, ⋯, m.

The model now becomes

$$\underset{\sim}{Y} = \sum_j \underset{\sim}{X}_{0j}\underset{\sim}{\beta}_j + \sum_{i>0} \sum_j \underset{\sim}{X}_{ij}\underset{\sim}{e}_{ij} \qquad (2.2)$$

and

$$\underset{\sim}{V}^* = \sum_{i>0} \sigma_i^2 \left(\sum_j \underset{\sim}{X}_{ij}\underset{\sim}{X}_{ij}'\right).$$

Define $\underset{\sim}{V}_{ij} = \underset{\sim}{X}_{ij}\underset{\sim}{X}_{ij}'$ for all ij. It follows that

$$\underset{\sim}{V}^* = \sum_{i>0} \sigma_i^2 \left(\sum_j \underset{\sim}{V}_{ij}\right).$$

It is assumed that no fixed effect in (2.2) is nested in a random effect, and that the model is balanced in the sense that there exists an analysis of variance of the form

$$\underset{\sim}{Y}'\underset{\sim}{Y} = \sum_i \sum_j \underset{\sim}{Y}'\underset{\sim}{A}_{ij}\underset{\sim}{Y}$$

with orthogonal sums of squares. Note that it is quite possible that the expected values of several sums of squares may be identical or possibly differ by no more than a constant. The $\{\underset{\sim}{A}_{ij}\}$ have the usual properties as spelled out by Cochran's theorem. It is convenient for our purposes to let

$$\underset{\sim}{A}_{0j} = \underset{\sim}{X}_{0j}(\underset{\sim}{X}_{0j}'\underset{\sim}{X}_{0j})^{-}\underset{\sim}{X}_{0j} \quad \text{for } j = 1, \cdots, s_0 .$$

It follows that there exist constants, $\{\theta_{iji'j'}\}$ and $\{\varphi_{iji'j'}\}$ such that

$$\underset{\sim}{A}_{ij} = \sum_{i'j'} \theta_{iji'j'} \underset{\sim}{V}_{i'j'}$$

and

$$\underset{\sim}{V}_{ij} = \sum_{i'j'} \varphi_{iji'j'} \underset{\sim}{A}_{i'j'}$$

for all ij. Also there exist $\{d_{ij}\}$, functions of $\{\varphi_{iji'j'}\}$ and $\{\alpha_i\}$ such that

$$\begin{aligned}\underset{\sim}{V} &= \sum_{i>0} \sum_j \alpha_i \underset{\sim}{V}_{ij} \\ &= \sum_{i>0} \sum_j \alpha_i (\sum_{i'} \sum_{j'} \varphi_{iji'j'} \underset{\sim}{A}_{i'j'}) \\ &= \sum_{i'} \sum_{j'} (\sum_{i>0} \sum_j \alpha_i \varphi_{iji'j'}) \underset{\sim}{A}_{i'j'} \\ &= \sum_{i'} \sum_{j'} d_{i'j'} \underset{\sim}{A}_{i'j'}\end{aligned}$$

and

$$\underset{\sim}{V}^{-1} = \sum_i \sum_j d_{ij}^{-1} \underset{\sim}{A}_{ij} .$$

The known non-singularity of $\underset{\sim}{V}$ guarantees that $d_{ij} \neq 0$ for all ij.

The MINQUE principle requires $\underset{\sim}{Y}'\underset{\sim}{Q}_V\underset{\sim}{V}_i\underset{\sim}{Q}_V\underset{\sim}{Y}$ for i = 1, ⋯, m, where

$$\underset{\sim}{Q}_V = \underset{\sim}{V}^{-1} - \underset{\sim}{V}^{-1}\underset{\sim}{X}_0(\underset{\sim}{X}_0'\underset{\sim}{V}^{-1}\underset{\sim}{X}_0)^{-}\underset{\sim}{X}_0'\underset{\sim}{V}^{-1} .$$

The balance in the structure implies

$$\underset{\sim}{X}_0 \underset{\sim}{A}_{ij} = 0 \quad \text{for all } j \text{ and } i > 0\text{, and in particular}$$

$$\underset{\sim}{X}_{0j} \underset{\sim}{A}_{0j'} = 0 \quad \text{for } j \neq j' .$$

It follows that

$$\underset{\sim}{V}^{-1} \underset{\sim}{X}_0 = \underset{\sim}{V}^{-1} (\underset{\sim}{X}_{01} | \cdots | \underset{\sim}{X}_{0s_0})$$

$$= (d_{01}^{-1} \underset{\sim}{A}_{01} \underset{\sim}{X}_{01} | \cdots | d_{0s_0}^{-1} \underset{\sim}{A}_{0s_0} \underset{\sim}{X}_{0s_0})$$

and

$$\underset{\sim}{X}_0' \underset{\sim}{V}^{-1} \underset{\sim}{X}_0 = \begin{bmatrix} d_{01}^{-1} \underset{\sim}{X}_{01}' \underset{\sim}{A}_{01} \underset{\sim}{X}_{01} & & 0 \\ & \ddots & \\ 0 & & d_{0s_0}^{-1} \underset{\sim}{X}_{0s_0}' \underset{\sim}{A}_{0s_0} \underset{\sim}{X}_{0s_0} \end{bmatrix}$$

$$= \begin{bmatrix} d_{01}^{-1} \underset{\sim}{X}_{01}' \underset{\sim}{X}_{01} & & 0 \\ & \ddots & \\ 0 & & d_{0s_0}^{-1} \underset{\sim}{X}_{0s_0}' \underset{\sim}{X}_{0s_0} \end{bmatrix} .$$

It follows that $\underset{\sim}{V}^{-1} \underset{\sim}{X}_0 (\underset{\sim}{X}_0' \underset{\sim}{V}^{-1} \underset{\sim}{X}_0)^{-} \underset{\sim}{X}_0' \underset{\sim}{V}^{-1}$ is equal to

$$\sum_j d_{0j}^{-1} \underset{\sim}{A}_{0j} \underset{\sim}{X}_{0j} (\underset{\sim}{X}_{0j}' \underset{\sim}{X}_{0j})^{-} \underset{\sim}{X}_{0j}' \underset{\sim}{A}_{0j} = \sum_j d_{0j}^{-1} \underset{\sim}{A}_{0j} ,$$

and

$$\underset{\sim}{Q}_v = \sum_{i>0} \sum_j d_{ij}^{-1} \underset{\sim}{A}_{ij} .$$

The desired quadratic forms follow as

$$\sum_j \underset{\sim}{Y}' \underset{\sim}{Q}_v \underset{\sim}{V}_{ij} \underset{\sim}{Q}_v \underset{\sim}{Y} = \sum_{i'} \sum_{j'} d_{i'j'}^{-2} (\sum_j \varphi_{iji'j'}) \underset{\sim}{Y}' \underset{\sim}{A}_{i'j'} \underset{\sim}{Y} \quad \text{for } i = 1, \cdots, m. \qquad (2.3)$$

In order to obtain expressions for d_{ij}^{-2}

define

$$\underset{\sim}{V}^* = \sum_{ij} d^*_{ij} \underset{\sim}{A}_{ij}$$

and consider

$$E[\underset{\sim}{Y}'\underset{\sim}{A}_{ij}\underset{\sim}{Y}] = \underset{\sim}{\beta}'\underset{\sim}{X}_0'\underset{\sim}{A}_{ij}\underset{\sim}{X}_0\underset{\sim}{\beta} + \mathrm{tr}(\underset{\sim}{A}_{ij}\underset{\sim}{V}^*)$$

$$= \sum_{i'j'} d^*_{i'j'} \mathrm{tr}(\underset{\sim}{A}_{ij}\underset{\sim}{A}_{i'j'})$$

$$= d^*_{ij} \ \mathrm{tr}(\underset{\sim}{A}_{ij}) \ .$$

It follows that d^*_{ij} is exactly the expected value of the mean square obtained from the quadratic for $\underset{\sim}{Y}'\underset{\sim}{A}_{ij}\underset{\sim}{Y}$ in the analysis of variance. Consequently the $\{d_{ij}\}$ are the expected values of the sums of squares evaluated at prior values of the variance components.

If one is willing to assume normality, then the result in (2.3) can be obtained via REML. The assumptions in model (2.2) together with normality imply the existence of k independent sums of squares $\{SS_i\}$ based on $\{v_i\}$ degrees of freedom and expected values $\{\sum_j^m w_{ij}\sigma_j^2\}$ respectively. Setting the partial derivatives of the log-likelihood equal to zero yields

$$\sum_i \frac{SS_i w_{ij}}{(\sum_h w_{ih} \hat{\sigma}_h^2)^2} = \sum_i \frac{v_i w_{ij}}{(\sum_h w_{ih} \hat{\sigma}_h^2)} \tag{2.4}$$

for $j = 1, \cdots, m$.

An approach to solving (2.4) is to use prior values for $\{\hat{\sigma}_h^2\}$ in the denominator and then equating the observed quadratic forms to their expected values. This, however, is exactly equivalent to (2.3) if one identifies $\{\underset{\sim}{Y}'\underset{\sim}{A}_{ij}\underset{\sim}{Y}\}$ with $\{SS_i\}$, $\{d_{ij}\}$ with $\{\sum_h w_{ih}\hat{\sigma}_h^2\}$ and $\{\sum_{j'} \varphi_{i'j'ij}\}$ with $\{w_{ij}\}$.

3. EXAMPLES

3.1 Combining a Series of r x c Tables with Subsampling.

We consider now the special case where data is available from a series of m experiments where r_k rows, c_k columns and n_k observations are available in each cell for $k = 1, \cdots, m$. The row, column, interaction and error variance components are assumed to be constant across all experiments. If RSS_k, CSS_k, $RCSS_k$ and ESS_k represent the row, column, interaction and error sums of squares respectively for the k'th experiment and α_R, α_C, α_{RC} and α_E prior values for the variance components then (2.3) implies that the following 4 quadratic forms be set equal to their expected values:

a) $$\sum_k \frac{c_k n_k \, RSS_k}{(c_k n_k \alpha_R + n_k \alpha_{RC} + \alpha_E)^2} \; .$$

b) $$\sum_k \frac{r_k n_k \, CSS_k}{(r_k n_k \alpha_C + n_k \alpha_{RC} + \alpha_E)^2} \; .$$

c) $$\sum_k \left\{ \frac{n_k \, RSS_k}{(c_k n_k \alpha_R + n_k \alpha_{RC} + \alpha_E)^2} + \frac{n_k \, CSS_k}{(r_k n_k \alpha_C + n_k \alpha_{RC} + \alpha_E)^2} + \frac{n_k \, RCSS_k}{(n_k \alpha_{RC} + \alpha_E)^2} \right\} \; .$$

d) $$\sum_k \left\{ \frac{RSS_k}{(c_k n_k \alpha_R + n_k \alpha_{RC} + \alpha_E)^2} + \frac{CSS_k}{(r_k n_k \alpha_C + n_k \alpha_{RC} + \alpha_E)^2} + \frac{RCSS_k}{(n_k \alpha_{RC} + \alpha_E)^2} + \frac{ESS_k}{\alpha_E^2} \right\} \; .$$

These quadratic forms are set equal to their expected values and the resulting system solved. The iterative process is obtained by replacing the prior values with estimates (negative values must be replaced) after the first cycle. If $n_k = 1$ then the value for ESS_k in d) is replaced by zero. Note also that allowing a different error variance component for some or all of the experiments introduces no new difficulties.

3.2 An Experiment with Common Row and Column Variance Component.

Consider the case where it is assumed that $\sigma_R^2 = \sigma_C^2 = \sigma_A^2$ and data is available from an r x C table with one observation per cell. Let RSS, CSS and ESS be the row, column and error sums of squares respectively. If α_A and α_E represent prior values then the appropriate quadratic forms are

$$\frac{cRSS}{(c\,\alpha_R + \alpha_E)^2} + \frac{rCSS}{(r\,\alpha_C + \alpha_E)^2} \quad \text{and ESS} \quad .$$

3.3 Numerical Example.

Corn yield data from a series of three experiments conducted by Dr. R. H. Moll in the Genetics Department at NCSU is used to illustrate the methodology developed. In experiment I, 64 males, grouped into blocks of 4 were each used on 4 different females. The identical matings were tested at two locations. Experiment II is similar, except that only 56 males were used. Experiment III again involved 64 males. However in this experiment, the blocks were replicated at each location.

Table I shows the analysis of variance for each experiment. Note that a common model involving an effect due to replications within locations is used to obtain the expected mean squares for all three experiments, even though only one replicate was used in experiments I and II. We note that the analysis of variance for experiment III has ten lines while each of the other two has only seven. Ten variance components are to be estimated. It is assumed that all effects in each experiment are random.

The ten quadratic forms required in the first iteration are given in Table II. Rather than use a prior value for the unknown $\{\alpha_i\}$, all were

Table I. Analyses of Variance of Three Experiments

Experiment I.

Source	M.S.	d.f.	E.M.S.
Loc	4.501875	1	$\sigma_e^2 + \sigma_{LF(MB)}^2 + 4\sigma_{RM(LB)}^2 + 4\sigma_{LM(B)}^2 + 16\sigma_{R(LB)}^2 + 16\sigma_{LB}^2 + 256\sigma_L^2$
Block	.110619	15	$\sigma_e^2 + \sigma_{LF(MB)}^2 + 2\sigma_{F(MB)}^2 + 4\sigma_{RM(LB)}^2 + 4\sigma_{LM(B)}^2 + 8\sigma_{M(B)}^2 + 16\sigma_{R(LB)}^2 + 16\sigma_{LB}^2 + 32\sigma_B^2$
LxB	.121897	15	$\sigma_e^2 + \sigma_{LF(MB)}^2 + 4\sigma_{RM(LB)}^2 + 4\sigma_{LM(B)}^2 + 16\sigma_{R(LB)}^2 + 16\sigma_{LB}^2$
Male (B)	.016996	48	$\sigma_e^2 + \sigma_{LF(MB)}^2 + 2\sigma_{F(MB)}^2 + 4\sigma_{RM(LB)}^2 + 4\sigma_{LM(B)}^2 + 8\sigma_{M(B)}^2$
LxM (B)	.011032	48	$\sigma_e^2 + \sigma_{LF(MB)}^2 + 4\sigma_{RM(LB)}^2 + 4\sigma_{LM(B)}^2$
Female (M,B)	.012128	192	$\sigma_e^2 + \sigma_{LF(MB)}^2 + 2\sigma_{F(MB)}^2$
LxF (M,B)	.010947	192	$\sigma_e^2 + \sigma_{LF(MB)}^2$

Experiment II.

Source	M.S.	d.f.	E.M.S.
Loc	5.733175	1	$\sigma_e^2 + \sigma_{LF(MB)}^2 + 4\sigma_{RM(LB)}^2 + 4\sigma_{LM(B)}^2 + 16\sigma_{R(LB)}^2 + 16\sigma_{LB}^2 + 224\sigma_L^2$
Block	.024852	13	$\sigma_e^2 + \sigma_{LF(MB)}^2 + 2\sigma_{F(MB)}^2 + 4\sigma_{RM(LB)}^2 + 4\sigma_{LM(B)}^2 + 8\sigma_{M(B)}^2 + 16\sigma_{R(LB)}^2 + 16\sigma_{LB}^2 + 32\sigma_B^2$
LxB	.047801	13	$\sigma_e^2 + \sigma_{LF(MB)}^2 + 4\sigma_{RM(LB)}^2 + 4\sigma_{LM(B)}^2 + 16\sigma_{R(LB)}^2 + 16\sigma_{LB}^2$
Male (B)	.022411	42	$\sigma_e^2 + \sigma_{LF(MB)}^2 + 2\sigma_{F(MB)}^2 + 4\sigma_{RM(LB)}^2 + 4\sigma_{LM(B)}^2 + 8\sigma_{M(B)}^2$
LxM (B)	.010157	42	$\sigma_e^2 + \sigma_{LF(MB)}^2 + 4\sigma_{RM(LB)}^2 + 4\sigma_{LM(B)}^2$
Female (M,B)	.008474	168	$\sigma_e^2 + \sigma_{LF(MB)}^2 + 2\sigma_{F(MB)}^2$
LxF (M,B)	.005724	168	$\sigma_e^2 + \sigma_{LF(MB)}^2$

Experiment III.

Source	M.S.	d.f.	E.M.S.									
Loc	1.476984	1	σ_e^2	$+ 2\sigma_{LF(MB)}^2$		$+ 4\sigma_{RM(LB)}^2$	$+ 8\sigma_{LM(B)}^2$		$+ 16\sigma_{R(LB)}^2$	$+ 32\sigma_{LB}^2$		$+ 512\sigma_L^2$
Block	.078833	15	σ_e^2	$+ 2\sigma_{LF(MB)}^2$	$+ 4\sigma_{F(MB)}^2$	$+ 4\sigma_{RM(LB)}^2$	$+ 8\sigma_{LM(B)}^2$	$+ 16\sigma_{M(B)}^2$	$+ 16\sigma_{R(LB)}^2$	$+ 32\sigma_{LB}^2$	$+ 64\sigma_B^2$	
LxB	.045938	15	σ_e^2	$+ 2\sigma_{LF(MB)}^2$		$+ 4\sigma_{RM(LB)}^2$	$+ 8\sigma_{LM(B)}^2$		$+ 16\sigma_{R(LB)}^2$	$+ 32\sigma_{LB}^2$		
Rep (L,B)	.011495	32	σ_e^2			$+ 4\sigma_{RM(LB)}^2$			$+ 16\sigma_{R(LB)}^2$			
Males (B)	.051411	48	σ_e^2	$+ 2\sigma_{LF(MB)}^2$	$+ 4\sigma_{F(MB)}^2$	$+ 4\sigma_{RM(LB)}^2$	$+ 8\sigma_{LM(B)}^2$	$+ 16\sigma_{M(B)}^2$				
LxM (B)	.006574	48	σ_e^2	$+ 2\sigma_{LF(MB)}^2$		$+ 4\sigma_{RM(LB)}^2$	$+ 8\sigma_{LM(B)}^2$					
RxM (B)	.004850	96	σ_e^2			$+ 4\sigma_{RM(LB)}^2$						
Female (M,B)	.017559	192	σ_e^2	$+ 2\sigma_{LF(MB)}^2$	$+ 4\sigma_{F(MB)}^2$							
LxF (M,B)	.004980	192	σ_e^2	$+ 2\sigma_{LF(MB)}^2$								
RxF (M,L,B)	.005076	384	σ_e^2									

TABLE II Quadratic Forms for First Iteration

Symbolic Form	Quadratic Form (first iteration)
$\Sigma(\underset{\sim}{Y}'\underset{\sim}{Q}\underset{\sim}{V}_L\underset{\sim}{Q}\underset{\sim}{Y})_\ell$	.033425
$\Sigma(\underset{\sim}{Y}'\underset{\sim}{Q}\underset{\sim}{V}_B\underset{\sim}{Q}\underset{\sim}{Y})_\ell$	.01249
$\Sigma(\underset{\sim}{Y}'\underset{\sim}{Q}\underset{\sim}{V}_{LB}\underset{\sim}{Q}\underset{\sim}{Y})_\ell$	.03627
$\Sigma(\underset{\sim}{Y}'\underset{\sim}{Q}\underset{\sim}{V}_R\underset{\sim}{Q}\underset{\sim}{Y})_\ell$	.04589
$\Sigma(\underset{\sim}{Y}'\underset{\sim}{Q}\underset{\sim}{V}_M\underset{\sim}{Q}\underset{\sim}{Y})_\ell$	.07050
$\Sigma(\underset{\sim}{Y}'\underset{\sim}{Q}\underset{\sim}{V}_{LM}\underset{\sim}{Q}\underset{\sim}{Y})_\ell$	.09222
$\Sigma(\underset{\sim}{Y}'\underset{\sim}{Q}\underset{\sim}{V}_{RM}\underset{\sim}{Q}\underset{\sim}{Y})_\ell$	.15545
$\Sigma(\underset{\sim}{Y}'\underset{\sim}{Q}\underset{\sim}{V}_F\underset{\sim}{Q}\underset{\sim}{Y})_\ell$	.76186
$\Sigma(\underset{\sim}{Y}'\underset{\sim}{Q}\underset{\sim}{V}_{LF}\underset{\sim}{Q}\underset{\sim}{Y})_\ell$	1.37352
$\Sigma(\underset{\sim}{Y}'\underset{\sim}{Q}\underset{\sim}{V}_{RF}\underset{\sim}{Q}\underset{\sim}{Y})_\ell$	3.16347

set equal to one for the first cycle. In particular, we note that the first quadratic form is computed from only the location lines in the three analysis of variance tables. In particular, it is given by

$$\frac{256 \times 4.501875}{298 \times 298} + \frac{224 \times 5.733175}{266 \times 266}$$

$$+ \frac{512 \times 1.476984}{575 \times 575} .$$

The third quadratic form is based on the first three lines in the analyses of variance. In particular, it is given by

$$+ 16 \times \left\{ \frac{15 \times .121897}{42 \times 42} + \frac{15 \times .110619}{84 \times 84} + \frac{4.501875}{298 \times 298} \right\}$$

$$+ 16 \times \left\{ \frac{13 \times .047801}{42 \times 42} + \frac{13 \times .02452}{84 \times 84} + \frac{5.733175}{266 \times 266} \right\}$$

$$+ 32 \times \left\{ \frac{15 \times .045938}{63 \times 63} + \frac{13 \times .078833}{147 \times 147} + \frac{1.476984}{575 \times 575} \right\} .$$

We notice that the first quadratic form involves only the sums of squares whose expected value contain σ_L^2 . Similarly the third quadratic form involves all sums of squares whose expected value contain σ_{LB}^2 . The tenth quadratic form is based on all lines in all three analyses.

The estimates are obtained by equating the 10 quadratic forms to their expected values and solving the resulting system. These estimates are then used as prior values for the next cycle. Table III shows the results of 10 cycles of iteration. Table IV shows the estimated asymptotic variance-covariance matrix of the estimates.

TABLE III Results of 10 Iterations of the Estimation Procedure

Variance Component	Iteration No. 1	2	3	7	8,9 & 10
σ^2_E	.005258	.005538	.005050	.005714	.005715
$\sigma^2_{LF(MB)}$	.001602	.001074	.000904	.000810	.000809
$\sigma^2_{F(MB)}$	.001984	.002177	.002223	.002248	.002249
$\sigma^2_{RM(LB)}$	-.000059*	-.000082*	-.000096*	-.000104*	-.000104*
$\sigma^2_{LM(B)}$	.000413	.00461	.000477	.000486	.000486
$\sigma^2_{M(B)}$	.001356	.001324	.001323	.001323	.001323
$\sigma^2_{R(LB)}$	.000628	.000447	.000427	.000421	.000417
σ^2_{LB}	.002594	.003007	.003087	.003087	.003100
σ^2_B	-.000427*	-.000550*	-.000572*	-.000572*	-.000576*
σ^2_L	.014684	.015072	.015071	.015070	.015070

* These values were replaced by zeros when used for the next iteration.

Table IV Estimated Asymptotic Variance - Covariance Matrix of Estimated Variance Components.

	σ^2_E*	σ^2_{LF}	σ^2_F	σ^2_{RM}	σ^2_{LM}	σ^2_M	σ^2_R	σ^2_{LB}	σ^2_B	σ^2_L
σ^2_E*	$.783x10^{-4}$	$.116x10^{-7}$	$.231x10^{-7}$	$.466x10^{-10}$	0	0	0	0	0	0
σ^2_{LF}	$.116x10^{-7}$	$.229x10^{-6}$	$-.177x10^{-6}$	$-.876x10^{-10}$	-.156x10-7	$.268x10^{-8}$	$.175x10^{-9}$	0	0	0
σ^2_F	$-.231x10^{-7}$	$-.177x10^{-6}$	$.367x10^{-6}$	$-.158x10^{-7}$	$.268x10^{-8}$	$-.825x10^{-8}$	$.363x10^{-8}$	0	0	0
σ^2_{RM}	$-.466x10^{-10}$	$-.875x10^{-10}$	$-.158x10^{-7}$	$.199x10^{-7}$	$.175x10^{-9}$	$.363x10^{-8}$	$-.500x10^{-8}$	0	0	0
σ^2_{LM}	0	$-.156x10^{-7}$	$-.268x10^{-8}$	$.175x10^{-9}$	$.659x10^{-7}$	$.120x10^{-7}$	$-.856x10^{-9}$	$-.135x10^{-7}$	$.135x10^{-8}$	$.628x10^{-9}$
σ^2_M	0	$.268x10^{-8}$	$-.825x10^{-8}$	$.363x10^{-8}$	$-.120x10^{-7}$	$.379x10^{-7}$	$-.177x10^{-7}$	$.506x10^{-8}$	$-.197x10^{-7}$	$.127x10^{-7}$
σ^2_R	0	$.175x10^{-9}$	$.363x10^{-8}$	$-.500x10^{-8}$	$-.856x10^{-9}$	$-.177x10^{-7}$	$.246x10^{-9}$	$.628x10^{-9}$	$.127x10^{-7}$	$-.186x10^{-7}$
σ^2_{LB}	0	0	0	0	$-.135x10^{-7}$	$.507x10^{-8}$	$.628x10^{-9}$	$.539x10^{-7}$	$-.203x10^{-7}$	$-.251x10^{-8}$
σ^2_B	0	0	0	0	$.507x10^{-8}$	$-.197x10^{-7}$	$.127x10^{-7}$	$-.203x10^{-7}$	$.790x10^{-7}$	$-.510x10^{-7}$
σ^2_L	0	0	0	0	$.628x10^{-9}$	$.127x10^{-7}$	$-.186x10^{-7}$	$-.251x10^{-8}$	$-.510x10^{-7}$	$.743x10^{-7}$

*This component could also be labeled σ^2_{RF}.

4. ASYMPTOTIC PROPERTIES

The complexity of the formulas implied by (2.3) makes it difficult to assess exact distributional properties of the estimators obtained. However, several points can be made. First of all, if one does not iterate then the resulting estimators are unbiased (Rao 1971a, 1971b), regardless of the prior values used. If the prior values are proportional to the true values and one assumes normality, then the estimators also have minimum variance among all unbiased quadratic estimators (Rao 1971b).

Sriburi (1978) has shown that if one iterates and obtains the REML solutions then the estimates are consistent, asymptotically normal and efficient in the sense of attaining the Cramér-Rao lower bound for the covariance matrix. The asymptotic covariance matrix is twice the inverse of matrix of coefficients obtained when the expected values of the quadratic forms (2.3) are expressed as linear functions of the variance components. We note that there is no guarantee that the solutions to the MINQUE equations will be non-negative. Consequently, any iterative procedure must employ some method to replace negative values if they appear.

The adequacy of the asymptotic theory can be judged from a Monte Carlo run involving 1000 sets of five experiments, each involving rows, columns interaction and subsampling with $\sigma_r^2 = .5$, $\sigma_c^2 = .2$, $\sigma_{rc}^2 = .9$ and $\sigma_e^2 = 1$. The sizes of the five experiments are given in Table V.

Table VI gives the exact variances and covariances of estimators for the four variance components, using three alternative methods of estimation. Note that the variances and covariances for the restricted M.L. estimators are for the case where actual values of the components are used in (2.4) and the procedure is not iterated. Table VII shows the results of the simulation experiment. Several points are rather striking. First of

Table V. Size of experiments used combined to generate variances and covariances of estimates given in Tables VI and VII.

No. of Rows	No. of Cols.	No. of Subsamples
10	10	2
10	4	2
4	4	2
10	4	10
4	4	10

Table VI. Exact Variance-covariance matrices for estimates of σ_r^2, σ_c^2, σ_{rc}^2 and σ_e^2 using three methods of estimation.

Method		Variance-Covariance Matrix σ_r^2	σ_c^2	σ_{rc}^2	σ_e^2
Averaging	σ_r^2	.0471	.0019	-.0416	0
Unbiased	σ_c^2	.0019	.0182	-.0076	0
AOV	σ_{rc}^2	-.0416	-.0076	.0351	-.0040
Estimates	σ_e^2	0	0	-.0040	.0086
Pooling	σ_r^2	.0460	.0007	-.0054	.0000
Sums of	σ_c^2	.0007	.0188	-.0033	.0000
Squares	σ_{rc}^2	-.0054	-.0033	.0252	-.0008
	σ_e^2	.0000	.0000	-.0008	.0030
Restricted	σ_r^2	.0342	.0005	-0040	.0000
M.L.	σ_c^2	.0005	.0127	-.0025	-.0000
or	σ_{rc}^2	-.0040	-.0025	.0211	-.0010
MIVQUE	σ_e^2	.0000	-.0000	-.0010	.0030

Table VII. Means and variances of 1000 simulated sets of estimates.

Method of Estimation	σ_r^2	σ_c^2	σ_{rc}^2	σ_e^2
	Means			
Averaging Unbiased AoV Estimates	.495	.199	.905	1.000
Averaging Estimates with Neg. Values Eliminated	.512	.231	.907	1.000
Pooling Sums of Squares	.492	.201	.902	1.001
First Iterate, Restricted ML	.504	.204	.897	1.001
Fourth Iterate, Restricted ML	.498	.198	.903	1.001
Restricted ML Using True Values	.498	.198	.903	1.001
	Variances			
Averaging Unbiased AoV Estimates	.0477	.0200	.0394	.0080
Averaging Estimates with Neg. values Eliminated	.0437	.0157	.0387	.0080
Pooling Sums of Squares	.0417	.0194	.0264	.0030
First Iterate, Restricted ML	.0330	.0125	.0210	.0030
Fourth Iterate, Restricted ML	.0331	.0125	.0217	.0029
Restricted ML Using True Values	.0331	.0125	.0217	.0029

all it appears that bias is not a problem, except for the obvious case of computing an average after replacing negative values by zeros. Secondly, the iterative scheme appears to converge very rapidly. In all cases, the initial values used were obtained by averaging the unbiased AoV estimator (with negative values replaced by zero). Finally, the variances observed for the estimator obtained via the iterative scheme agree very closely with the exact values given in Table VI.

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