

BIOMETRICAL TRAINING PROGRAM

INEQUALITIES FOR POWER ONE TESTS  
FOR SUMS OF DEPENDENT VARIABLES

by

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ABSTRACT

Let  $S_n$  be the sum of  $n$  bounded dependent variables,  $L_n$  the sum of squares of these variables, and  $\phi$  a positive increasing concave function. We determine explicit and tight upper bounds on the probability that  $S_n \geq \phi(L_n)$  under the hypothesis that  $S_n$  is a martingale. We further determine explicit upper bounds for the mean intrinsic time for  $S_n$  to pass  $\phi(L_n)$  under the hypothesis that  $S_n$  has positive drift. These inequalities can be used in power one tests for dependent variables in the manner proposed by Darling, Robbins, and Siegmund for independent variables

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(I) Introduction

Let  $\phi: [0, \infty) \rightarrow [0, \infty)$  satisfy

$$(1.1) \quad \sqrt{t} < \phi(t) < t \quad \text{all } t \geq t_* > 0$$

$$(1.2) \quad \frac{\phi(t)}{\sqrt{t}} \uparrow \infty \quad \text{and} \quad \frac{\phi(t)}{t} \downarrow 0 \quad \text{as } t \uparrow \infty$$

(1.3)  $\phi$  has a continuous derivative  $\phi'$  such that

$$p(t) \equiv \frac{t\phi'(t)}{\phi(t)} \downarrow p \in [\frac{1}{2}, 1) \quad \text{as } t \uparrow \infty$$

$$(1.4) \quad \phi'(t) < \frac{\phi(t)}{t}, \quad \text{all } t \geq t_* > 0$$

Let  $(\Omega, B, P)$  be a fixed probability space supporting the adapted sequence of random variables  $\{f_n, B_n\}$ . We assume throughout that

$$(1.5) \quad \left\{ \begin{array}{l} \text{(a) } |f_n| \leq 1, \quad \text{all } n \geq 1 \\ \sum_1^\infty \sigma^2\{f_n | B_{n-1}\} = \infty \end{array} \right.$$

where  $\sigma^2\{f_n | B_{n-1}\} = E\{f_n^2 | B_{n-1}\} - E^2\{f_n | B_{n-1}\}$ .

We fix some notation:

$$(1.6) \quad \left\{ \begin{array}{l} S_n = \sum_1^n f_k \\ T_n = \sum_1^n E\{f_k^2 | B_{k-1}\} \\ L_n = \sum_1^n f_k^2 \\ R_n = \sum_1^n E\{f_k | B_{k-1}\} \end{array} \right.$$

We fix two hypotheses:

$$(1.7) \quad \begin{cases} H_0: E\{f_n | B_{n-1}\} = 0, \text{ all } n \geq 1 \\ H_1: E\{f_n | B_{n-1}\} \geq \theta \sigma^2 \{f_n | B_{n-1}\}, \text{ all } n \geq 1, \text{ some } \theta > 0. \end{cases}$$

We define, for each  $\theta > 0$  and small enough, a unique positive value  $t_\theta$  where

$$(1.8) \quad \frac{\phi(t_\theta)+1}{t_\theta} = \frac{\theta}{1+\theta}$$

We will prove the following two theorems:

Theorem (1)

Under  $H_0$ :

(A) For all  $t_0 \geq t_*$ :

$$P\{S_n \geq \phi(T_n), \text{ some } T_n \geq t_0\} \leq 3 \int_{t_0}^{\infty} \frac{\phi(t)}{t^{3/2}} e^{-\left\{1 - \frac{\sqrt{t}}{\phi(t)} - \frac{\phi(t)}{t}\right\} \frac{\phi^2(t)}{2t}} dt$$

(B) For all  $t_0 \geq t_{**}$  where  $\frac{\phi(t_{**})}{t_{**}} < \frac{1}{\sqrt{2}}$ :

$P\{S_n \geq \phi(L_n), \text{ some } L_n \geq t_0\}$

$$\leq 3 \left\{ \frac{1+2 \frac{\sqrt{t_0}}{\phi(t_0)}}{1-\sqrt{2} \frac{\phi(t_0)}{t_0}} \right\} \int_{t_0}^{\infty} \frac{\phi(t)}{t^{3/2}} e^{-\left\{1 - \frac{\sqrt{t}}{\phi(t)} - \frac{\phi(t)}{t}\right\} \frac{\phi^2(t)}{2t}} dt$$

Theorem (2)

Assume  $H_1$

Define

$$\tau_\theta = \inf n \ni S_n \geq \phi(T_n)$$

$$\tau_{\theta'} = \inf n \ni S_n \geq \phi(L_n)$$

Then  $\tau_\theta$  and  $\tau_{\theta'}$  are proper and

$$(A) \quad \frac{1}{t_\theta} \int_{T_{\tau_\theta}} \leq 1 + \frac{1}{(1-p(t_\theta))^2} \left\{ \sqrt{3\pi} + \frac{12\sqrt{t_\theta}}{\phi(t_\theta)} \right\} \frac{\sqrt{t_\theta}}{\phi(t_\theta)}$$

$$(B) \quad \frac{1}{t_\theta} \int_{L_{\tau_{\theta'}}} \leq 1 + 16 \left( \frac{\theta}{1+\theta} \right)^2 + \frac{2}{(1-p(t_\theta))^2} \left\{ 3.2 + \frac{12\sqrt{t_\theta}}{\phi(t_\theta)} \right\} \frac{\sqrt{t_\theta}}{\phi(t_\theta)}$$

(1.9) Remarks

These theorems are analogues of results in the literature for sequences of i.i.d. random variables. For instance, when  $\{f_n\}$  is i.i.d., with  $\int f_1 = 0$ ,  $\int f_1^2 = 1$  and  $\int e^{\lambda f_1} < \infty$ , some positive  $\lambda$ , then modulo some further regularity conditions, we have, assuming

$$\int_1^\infty \frac{\phi(t)}{t^{3/2}} e^{-\frac{\phi^2(t)}{2t}} dt < \infty :$$

Darling-Robbins [ 3 ]

$$P\{S_n \geq \phi(n), \text{ some } n \geq t_0\} \leq \frac{\sqrt{e}}{2} \int_{t_0}^\infty \frac{\phi(t)}{t^{3/2}} e^{-\frac{\phi^2(t)}{2t}} dt$$

Strassen [10] Theorem (1,4)

For  $p$  as in (1.3):

$$\lim_{t_0 \uparrow \infty} \frac{P\left\{S_n \geq \phi(n), \text{ some } n \geq t_0\right\}}{\int_{t_0}^{\infty} \frac{\phi(t)}{t^{3/2}} e^{-\frac{\phi^2(t)}{2t}} dt} = \frac{p}{\sqrt{2\pi}}$$

Strassen's result establishes that  $\int_{t_0}^{\infty} \frac{\phi(t)}{t^{3/2}} e^{-\frac{\phi^2(t)}{2t}} dt$  is the

rate at which the probabilities under consideration tend to zero.

Darling-Robbins bound all such probabilities by a fixed constant

times such rates. As our Theorem (1) reveals, we bound the probabilities

of the conditional versions of the crossings  $\{S_n \geq \phi(n), \text{ some } n \geq t_0\}$

by a small perturbation of such rates.

Our second theorem establishes that

$$\lim_{\theta \downarrow 0} \frac{1}{t_\theta} \int T_{t_\theta} \leq 1$$

$$\lim_{\theta \downarrow 0} \frac{1}{t_\theta} \int L_{t_\theta} \leq 1$$

and gives rates for these lim sups.

If  $\{f_n\}$  is i.i.d., with  $\int f_1 = 0$ ,  $\int f_1^2 = 1$  and with other regularity conditions, very general but too numerous to mention here, for  $t_\theta$  defined through  $\frac{\phi(t_\theta)}{t_\theta} = \theta$  and for  $T_\theta$  defined through  $T_\theta = \inf n \ni S_n + n\theta \geq \phi(n)$ , we have

Lai [7] Corollary (1)

$$(*) \quad \lim_{\theta \downarrow 0} \frac{1}{t_\theta} \int T_\theta = P\{S_n < \phi(n), \text{ all } n \geq t_0\} \leq 1$$

Our second theorem is the closest we've been able to come to a conditional analogue of Lai's result. We mention here that Gut [ 6 ] has produced limit theorems related to (\*) for dependent sums.

(1.10) Consider again the following hypotheses:

$$H_0: E\{f_n | B_{n-1}\} = 0, \text{ all } n$$

$$H_1: E\{f_n | B_{n-1}\} \geq \theta \sigma^2 \{f_n | B_{n-1}\}, \text{ all } n$$

One statistical test of these hypotheses consists in deciding for  $H_1$  if  $S_n \geq \phi(L_n)$  for some  $L_n \geq t_0$ . Using our theorems and assuming  $t_0$  large,  $\theta$  small, we have

$$P\{S_n \geq \phi(L_n), \text{ some } L_n \geq t_0 | H_0\} \leq \varepsilon$$

and

$$P\{S_n \geq \phi(L_n), \text{ some } L_n \geq t_0 | H_1\} = 1$$

and

$$E\{L_{\tau_{\theta}} | H_1\} \leq (1 + \varepsilon)t_{\theta}.$$

Darling-Robbins [ 2 ], [ 3 ] have considered such "tests of power one" for sums of i.i.d.'s. There has been subsequent work in this area, [ 8 ], [ 9 ], with the focus on i.i.d.'s also.

Our theorems are an attempt at extending such tests to dependent sums. This extension does not seem statistically unrealistic.

Consider the following contexts:

(A) A drug level,  $d_n$ , is administered to a subject whose condition,  $C_{n-1}$ , at time  $n$ , depends on past conditions and dosage levels received. The objective of the treatment is the stabilization of

the subject's response where this response,  $f_n$ , is a random function of  $C_{n-1}$  and  $d_n$ , and where stabilization is interpreted as  $E\{f_n | B_{n-1}\} = 0$ .

(B) We wish to test, through discrete sampling, if a diffusion  $Z_t$ , has positive drift. Under reasonable assumptions on the variance coefficient,  $\sigma^2(Z_t)$ , and with  $f_n = Z_n - Z_{n-1}$  suitably truncated,  $H_0$  vs  $H_1$  as defined above will constitute a test of power one.

(C) To decide, as quickly as possible, that a game is not fair, i.e., that the win or loss increment,  $f_n$ , obtained at time  $n$  by a gambling house, favors the house in the sense that  $E\{f_n | B_{n-1}\} \geq \theta \sigma^2\{f_n | B_{n-1}\}$ .

(1.11) Under the mild hypothesis  $\int f_n^2 \geq a > 0$ , we see that

$$\int \tau_{\theta'} \leq \frac{1}{a} \int L_{\tau_{\theta'}}$$

so that upper bounds become available on the expected time  $\tau_{\theta'}$ .

(1.12) The proofs of theorems (1) and (2) are almost entirely dependent on the use of three supermartingales devised by Freedman [4], [5]. We introduce these here.

(a) Let  $\{g_n, B_n\}$  be adapted, with  $0 \leq g_n \leq 1$ .

Let

$$C_n = \sum_1^n g_k$$

$$M_n = \sum_1^n E\{f_k | B_{k-1}\}$$

$$G(\lambda) = e^\lambda - 1$$

$$F(\lambda) = 1 - e^{-\lambda}$$

Then

$$e^{\lambda C_n - G(\lambda)M_n} \quad \text{and} \quad e^{F(\lambda)M_n - \lambda C_n}$$

are supermartingales.

(b) Let  $K(\lambda) = e^\lambda - 1 - \lambda$ . Then, with (1.5a) and the notation (1.6),

$$e^{\lambda S_n - K(\lambda)T_n}$$

is a supermartingale.

Proposition (1.1)

For all  $\theta \geq 0$ , all  $t \geq 0$ , and provided  $C_\infty = \sum_{k=1}^{\infty} g_k = \infty$  a.e.:

$$(1) \quad P\{M_n \geq (1+\theta)C_n, \text{ some } C_n \geq t\} \leq e^{-\frac{\theta^2 t}{2(1+\theta)}}$$

$$(2) \quad P\{M_n \leq \frac{1}{1+\theta} C_n, \text{ some } C_n \geq t\} \leq e^{-\frac{\theta^2 t}{2(1+\theta)}}$$

$$(3) \quad P\{M_n \notin [\frac{1}{1+\theta} C_n, (1+\theta)C_n], \text{ some } C_n \geq t\} \leq 2e^{-\frac{\theta^2 t}{2(1+\theta)}}$$

Proof:

Let  $0 \leq t < S$  and let

$$\tau_1 = \inf k \ni C_k \geq t$$

$$\tau_2 = \inf k \ni C_k \geq S$$

$$\tau = \begin{cases} \text{first } k \in [\tau_1, \tau_2) \text{ with } M_k \geq (1+\theta)C_k \\ \tau_2 \text{ if no such } k \text{ occurs} \end{cases}$$

Then

$$\begin{aligned}
 1 &\geq \int e^{F(\lambda)M_\tau - \lambda C_\tau} \\
 &\quad \{M_\tau \geq (1+\theta)C_\tau\} \\
 &\geq \int e^{F(\lambda)((1+\theta))C_\tau - \lambda C_\tau} \\
 &\quad \{M_\tau \geq (1+\theta)C_\tau\} \\
 &\geq e^{(F(\lambda)(1+\theta) - \lambda)t} P\{M_\tau \geq (1+\theta)C_\tau\}
 \end{aligned}$$

provided  $\frac{F(\lambda)}{\lambda} \geq \frac{1}{1+\theta}$ .

Under these circumstances, letting  $S \uparrow \infty$ :

$$P\{M_n \geq (1+\theta)C_n, \text{ some } C_n \geq t\} \leq \inf_{\lambda} e^{(\lambda - F(\lambda)(1+\theta))t}$$

Now the exponent minimizes for  $\lambda = \ell_n(1+\theta)$ ; we can use this value since

$$\frac{F(\lambda)}{\lambda} = \frac{\theta}{(1+\theta)\ell_n(1+\theta)} \geq \frac{1}{1+\theta}, \quad \theta \geq 0$$

Using  $\ell_n(1+\theta) - \theta \leq -\frac{\theta^2}{2(1+\theta)}$ , we have our first result.

For the second part, let  $0 < t < S$  as before, let  $\tau_1, \tau_2$  be as before, and define

$$\tau = \begin{cases} \text{first } k \in [\tau_1, \tau_2) & M_k \leq \frac{1}{1+\theta} C_k \\ \tau_2 & \text{if no such } k \text{ occurs} \end{cases}$$

Then

$$\begin{aligned}
 1 &\geq \int e^{\lambda C_\tau - G(\lambda)M_\tau} \\
 &\quad \{M_\tau \leq \frac{1}{1+\theta} C_\tau\} \\
 &\geq \int e^{(\lambda - \frac{1}{1+\theta} G(\lambda))C_\tau} \\
 &\quad \{M_\tau \leq \frac{1}{1+\theta} C_\tau\} \\
 &\geq e^{(\lambda - \frac{1}{1+\theta} G(\lambda))t} P\{M_\tau \leq \frac{1}{1+\theta} C_\tau\}
 \end{aligned}$$

provided  $\lambda \geq \frac{1}{1+\theta} G(\lambda)$ .

Letting  $S \rightarrow \infty$  we have

$$P\{M_n \leq \frac{1}{1+\theta} C_n, \text{ some } C_n \geq t\} \leq \inf_\lambda e^{(\frac{1}{1+\theta} G(\lambda) - \lambda)t}$$

where  $\lambda$  is subject to  $\lambda \geq \frac{1}{1+\theta} G(\lambda)$ .

The best choice is  $\lambda = \ln(1+\theta)$  which satisfies this constraint.

Using  $\frac{\theta}{1+\theta} - \ln(1+\theta) \geq -\frac{\theta^2}{2(1+\theta)}$ , we have the second case. From these

two cases the third case follows straightforwardly.

Q.E.D.

Proposition (1.2)

Let  $\{f_n, B_n\}$  satisfy

$$\sum_1^\infty f_n^2 = \infty \quad \text{a.e.}$$

Then

$$P\{T_n \notin \left[\frac{1}{1+\theta} L_n, (1+\theta)L_n\right], \text{ some } L_n \geq t\} \leq 2e^{-\frac{\theta^2 t}{2(1+\theta)^2}}.$$

Proof:

Let  $f_n^2 = g_n$  and use the previous proposition. We've replaced

$$\frac{\theta^2}{(1+\theta)} \text{ by } \frac{\theta^2}{(1+\theta)^2} \text{ for later technical reasons. Q.E.D.}$$

(II) Proof of Theorem (1)

Fix  $0 < t_0 < t_1$ . Define

$$\tau_1 = \text{first } n \ni T_n \geq t_0$$

$$\tau_2 = \text{last } n \ni T_n \leq t_1$$

$$\tau = \begin{cases} \text{first } n \in [\tau_1, \tau_2] & S_n \geq \phi(T_n) \\ \tau_2 & \text{if no such } n \text{ occurs} \end{cases}$$

Then, from Optimal Stopping:

$$\begin{aligned} 1 &\geq \int e^{\lambda S_\tau - K(\lambda)T_\tau} \{S_\tau \geq \phi(T_\tau)\} \\ &\geq e^{\lambda\phi(t_0) - K(\lambda)t_1} P\{S_\tau \geq \phi(T_\tau)\} \end{aligned}$$

Letting

$$P_0 = P\{S_n \geq \phi(T_n), \text{ some } T_n \in [t_0, t_1]\},$$

we have

$$P_0 \leq \inf_{\lambda \geq 0} e^{K(\lambda)t_1 - \lambda\phi(t_0)}$$

The infimum occurs for  $\lambda = \ln\left\{1 + \frac{\phi(t_0)}{t_1}\right\}$ .

Using  $(1+x)\ln(1+x) \geq x + \frac{x^2}{2} - \frac{x^3}{6}$ , valid for  $0 \leq x \leq 1$ , we have

$$P_0 \leq e^{\frac{\phi^3(t_0)}{6t_0^2} - \frac{\phi^2(t_0)}{2t_1}}$$

Define

$$t_0 = t_1 - \frac{1}{3} \frac{t_1^{3/2}}{\phi(t_1)}$$

Then

$$\frac{\phi^2(t_0)}{t_1} = \frac{\phi^2(t_1)}{t_1} - \left( \frac{\phi^2(t_1) - \phi^2(t_0)}{t_1} \right)$$

From the Mean Value Theorem and  $\phi'(t) \leq \frac{\phi(t)}{t}$ , we have

$$\begin{aligned} \frac{\phi^2(t_1) - \phi^2(t_0)}{t_1} &\leq 2\left(\frac{t_1}{t_0} - 1\right) \frac{\phi^2(t_1)}{t_1} \\ &\leq \frac{2\sqrt{t_1}}{3\phi(t_1) - \sqrt{t_1}} \frac{\phi^2(t_1)}{t_1} \end{aligned}$$

Thus

$$\frac{\phi^2(t_0)}{2t_1} \geq \left(1 - \frac{\sqrt{t_1}}{\phi(t_1)}\right) \frac{\phi^2(t_1)}{2t_1}$$

Next

$$\frac{\phi^3(t_0)}{t_0^2} \leq \frac{\phi^3(t_1)}{t_1^2} \left(\frac{t_1}{t_0}\right)^2 \leq \frac{9}{4} \frac{\phi^3(t_1)}{t_1^2}$$

Consequently

$$P_0 \leq e^{-\left(1 - \frac{\sqrt{t_1}}{\phi(t_1)} - \frac{\phi(t_1)}{t_1}\right) \frac{\phi^2(t_1)}{2t_1}}$$

Suppose we now repeat this argument over intervals  $[t_{n-1}, t_n]$

where

$$t_{n-1} = t_n - \frac{1}{3} \frac{t_n^{3/2}}{\phi(t_n)} .$$

We'd have

$$P_{n-1} \leq e^{-\left(1 - \frac{\sqrt{t_n}}{\phi(t_n)} - \frac{\phi(t_n)}{t_n}\right) \frac{\phi^2(t_n)}{2t_n}}$$

Then

$$P\{S_n \geq \phi(T_n), \text{ some } T_n \geq t_0\} \leq \sum_{n=1}^{\infty} e^{-\left(1 - \frac{\sqrt{t_n}}{\phi(t_n)} - \frac{\phi(t_n)}{t_n}\right) \frac{\phi^2(t_n)}{2t_n}}$$

Note next that  $t_n \uparrow \infty$  and

$$\frac{1}{t_n - t_{n-1}} = \frac{3\phi(t_n)}{t_n^{3/2}}$$

Thus, dominating the sum above by an integral:

$$P\{S_n \geq \phi(T_n), \text{ some } T_n \geq t_0\} \leq 3 \int_{t_0}^{\infty} \frac{\phi(t)}{t^{3/2}} e^{-\left(1 - \frac{\sqrt{t}}{\phi(t)} - \frac{\phi(t)}{t}\right) \frac{\phi^2(t)}{2t}} dt$$

Next, for  $\theta$  momentarily unspecified, and using Proposition (1.2):

$$\begin{aligned}
 P\{S_n \geq \phi(L_n), \text{ some } L_n \geq t_0\} &\leq P\{T_n \notin \left[\frac{1}{1+\theta} L_n, (1+\theta)L_n\right], \text{ some } L_n \geq t_0\} \\
 &\quad + P\left\{S_n \geq \phi\left(\frac{T_n}{1+\theta}\right), \text{ some } T_n \geq \frac{t_0}{1+\theta}\right\} \leq \\
 &2e^{-\frac{\theta^2 t_0}{2(1+\theta)^2}} + P\left\{S_n \geq \frac{1}{1+\theta} \phi(T_n), \text{ some } T_n \geq \frac{t_0}{1+\theta}\right\} \\
 &\leq 2e^{-\frac{\theta^2 t_0}{2(1+\theta)^2}} + 3 \int_{\frac{t_0}{1+\theta}}^{\infty} \frac{\phi_0(t)}{t^{3/2}} e^{-\left(1 - \frac{\sqrt{t}}{\phi_0(t)} - \frac{\phi_0(t)}{t}\right) \frac{\phi_0^2(t)}{2t}} dt
 \end{aligned}$$

where  $\phi_0(t) = \frac{1}{1+\theta} \phi(t)$ .

Substituting  $(1+\theta)t$  for  $t$  in the integral and letting  $\int_{t_0}^{\infty}$  denote this integral, and noting that the integral decreases in  $t$ ,

$$P\{S_n \geq \phi(L_n), \text{ some } L_n \geq t_0\} \leq 2e^{-\frac{\theta^2 t_0}{2(1+\theta)^2}} + 3(1+\theta) \int_{t_0}^{\infty}$$

Our choice for  $\theta$  will be made so that

$$\lim_{t_0 \uparrow \infty} e^{-\frac{\theta^2 t_0}{2(1+\theta)^2}} / \int_{t_0}^{\infty} = 0,$$

so that the right side in the inequality above is dominated by  $\int_{t_0}^{\infty}$  multiplied by some constant.

We reason as follows;

Letting  $\frac{\phi(t_0)}{\sqrt{t}} = x$ , we have

$$\int_{t_0}^{\infty} \frac{\phi(t_0)}{\sqrt{t_0}} e^{-\frac{x^2}{2}} dx;$$

this inequality follows from  $\phi'(t) \leq \frac{\phi(t)}{t}$ .

By a classical inequality:

$$\int_x^{\infty} e^{-\frac{z^2}{2}} dz \geq \left(\frac{1}{x} - \frac{1}{x^3}\right) e^{-\frac{x^2}{2}},$$

so that, subject to  $\frac{\phi(t_0)}{t_0} \leq \frac{1}{\sqrt{2}}$ , we have

$$\int_{t_0}^{\infty} \geq \frac{\sqrt{t_0}}{\phi(t_0)} \left\{1 - \frac{\sqrt{2}}{\phi(t_0)}\right\} e^{-\frac{\phi^2(t_0)}{2t_0}}$$

But then, letting  $\frac{\theta}{1+\theta} = \sqrt{2} \frac{\phi(t_0)}{t_0}$ , we have

$$e^{-\frac{\theta^2 t_0}{2(1+\theta)^2}} \leq e^{-\frac{\frac{1}{2} \frac{\phi^2(t_0)}{t_0}}{1 - \frac{\sqrt{2}}{\phi(t_0)}}} \frac{\phi(t_0)}{\sqrt{t_0}} \rightarrow 0 \text{ as } t_0 \rightarrow \infty.$$

Staying with this value of  $\theta$ , we have

$$P\{S_n \geq \phi(L_n), \text{ some } L_n \geq t_0\} \leq 2e^{-\frac{\phi^2(t_0)}{t_0}} + \frac{3}{1-\sqrt{2}} \frac{\phi(t_0)}{t_0} \int_{t_0}^{\infty}$$

from which Theorem (1B) follows.

Q.E.D.

(III) Proof of Theorem (2)

Lemma

$$P\{S_n \geq \phi(T_n) \text{ eventually}\} = 1$$

$$P\{S_n \geq \phi(L_n) \text{ eventually}\} = 1$$

Proof of Lemma

A special case of Brown [1], Theorem (1), shows that if

(a)  $E\{f_n | B_{n-1}\} \leq C E\{f_n | B_{n-1}\}$ , some  $C > 0$ , and

$$\sum_1^{\infty} E\{f_n | B_{n-1}\} = \infty \text{ a.e.,}$$

then

$$\frac{S_N}{R_N} = \frac{\sum_1^N f_n}{\sum_1^N E\{f_n | B_{n-1}\}} \rightarrow 1 \text{ a.e.}$$

This result clearly holds in the present context. Given this result and the hypotheses on the function  $\phi$ , we have for all  $k > 0$  and eventually on a set of measure one:

$$S_n \geq \frac{1}{2} R_n \geq \frac{\theta}{2(1+\theta)} T_n \geq \frac{k\theta}{2(1+\theta)} \phi(T_n) ; \text{ now let}$$

$$k = \frac{2(1+\theta)}{\theta} .$$

Next, since, from Freedman [5]

$$\frac{L_n}{T_n} \rightarrow 1 \text{ a.e.,}$$

and since  $\frac{\phi(s)}{\phi(t)} \rightarrow 1$  as  $t \rightarrow \infty$ ,  $\frac{s}{t} \rightarrow 1$ , we have

$$S_n \geq \phi(L_n) \text{ eventually.} \quad \text{Q.E.D.}$$

Now define

$$g_n = \frac{E\{f_n | B_{n-1}\} - f_n}{2}$$

$$\tilde{S}_n = \sum_1^n g_k = \frac{1}{2} (R_n - S_n)$$

$$\tilde{T}_n = \sum_1^n E(g_k^2 | B_{k-1}) = \frac{1}{4} \sum_1^n \sigma^2(f_k | B_{k-1})$$

Then

$$e^{\lambda \tilde{S}_n - K(\lambda) \tilde{T}_n} \text{ is a}$$

positive supermartingale.

Define

$$\tau = \inf n \geq 0 \quad S_n \geq \phi(T_n)$$

Our lemma tells us that  $\tau$  is proper, so that

$$\begin{aligned} 1 &\geq \int_{\{T_\tau \geq t\}} e^{\lambda \tilde{S}_\tau - K(\lambda) \tilde{T}_\tau} \\ &\geq \int_{\{T_\tau \geq t\}} e^{-\frac{\lambda}{2}(S_\tau - R_\tau) - \frac{K(\lambda)}{4} T_\tau} \\ &\geq \int_{\{T_\tau \geq t\}} e^{-\frac{\lambda}{2}((\phi(T_\tau)+1) - \frac{\theta}{1+\theta} T_\tau) - \frac{K(\lambda)}{4} T_\tau} = \int_{\{T_\tau \geq t\}} e^{-\frac{\lambda}{2} T_\tau \left\{ \frac{\theta}{1+\theta} - \frac{K(\lambda)}{2\lambda} - \frac{(\phi(T_\tau)+1)}{T_\tau} \right\}} \end{aligned}$$

Suppose

$$(*) \quad \frac{K(\lambda)}{2\lambda} \leq \frac{\theta}{1+\theta} - \frac{(\phi(t)+1)}{t}$$

Then the integrand increases in  $T_\tau$ , so that

$$(**) \quad P\{T_\tau \geq t\} \leq e^{-\left[ \frac{K(\phi)t}{4} + \frac{(\phi(t)+1)\lambda}{2} - \frac{\theta t \lambda}{2(1+\theta)} \right]}$$

It is easily seen that the right side minimizes for

$$\lambda_0 = \ln\left\{1 + \frac{2\theta}{1+\theta} - 2 \frac{(\phi(t)+1)}{t}\right\}$$

We have this  $\lambda$  available to us if it satisfies (\*). This follows immediately from

$$\lambda(e^\lambda - 1) \geq k(\lambda), \quad \text{valid for all } \lambda \geq 0.$$

Let's use  $\lambda_0$  in (\*\*) and also

$$(1+x)\ln(1+x) \geq x + \frac{x^2}{2(1+x)}, \quad \text{all } x \geq 0$$

to arrive at

$$P\{T_\tau \geq t\} \leq \exp\left\{-\frac{t}{1+2\left(\frac{\theta}{1+\theta} - \frac{(\phi(t)+1)}{t}\right)} \left(\frac{\theta}{1+\theta} - \frac{(\phi(t)+1)}{t}\right)^2\right\}$$

Defining  $t_\theta$  through

$$\frac{\phi(t_\theta)+1}{t_\theta} = \frac{\theta}{1+\theta}, \quad \text{we have}$$

$$P\{T_\tau \geq t\} \leq e^{-\frac{t}{6}\left(\frac{\phi(t_\theta)}{t_\theta} - \frac{\phi(t)}{t}\right)^2}, \quad \text{all } t \geq t_\theta$$

But then

$$\int_{T_{\tau} \leq t_{\theta}} + \int_{t_{\theta}}^{\infty} e^{-\frac{t}{6} \left( \frac{\phi(t_{\theta})}{t_{\theta}} - \frac{\phi(t)}{t} \right)^2} dt$$

Now

$$\begin{aligned} (***) \quad \int_{t_{\theta}}^{\infty} e^{-\frac{t}{6} \left( \frac{\phi(t_{\theta})}{t_{\theta}} - \frac{\phi(t)}{t} \right)^2} dt &\leq \int_{t_{\theta}}^{2t_{\theta}} e^{-\frac{t}{6} \left( \frac{\phi(t_{\theta})}{t_{\theta}} - \frac{\phi(t)}{t} \right)^2} dt \\ (**) \quad t_{\theta} &+ \int_{2t_{\theta}}^{\infty} e^{-\frac{t}{6} \left( \frac{\phi(t_{\theta})}{t_{\theta}} - \frac{\phi(2t_{\theta})}{2t_{\theta}} \right)^2} dt \end{aligned}$$

Use the Mean Value Theorem on the exponent in the first integral on the right. One has

$$\int_{t_{\theta}}^{2t_{\theta}} e^{-\frac{t}{6} \left( \frac{\phi(t_{\theta})}{t_{\theta}} - \frac{\phi(t)}{t} \right)^2} dt \leq \int_{t_{\theta}}^{2t_{\theta}} e^{-\frac{(1-p(t_{\theta}))^2}{48} \frac{\phi^2(t_{\theta})}{t_{\theta}^3} (t-t_{\theta})^2} dt$$

Letting  $t = (1+r)t_{\theta}$  in the integral on the right, and then letting

$$z = \frac{(1-p(t_{\theta}))}{\sqrt{24}} \frac{\phi(t_{\theta})}{\sqrt{t_{\theta}}} s, \quad \text{we have}$$

$$\begin{aligned} \int_{t_{\theta}}^{2t_{\theta}} e^{-\frac{t}{6} \left( \frac{\phi(t_{\theta})}{t_{\theta}} - \frac{\phi(t)}{t} \right)^2} dt &\leq \frac{\sqrt{24}}{(1-p(t_{\theta}))} \frac{t_{\theta}^{3/2}}{\phi(t_{\theta})} \int_0^1 \frac{(1-p(t_{\theta}))}{\sqrt{24}} \frac{\phi(t_{\theta})}{\sqrt{t_{\theta}}} e^{-\frac{z^2}{2}} dz \\ &\leq \frac{\sqrt{12\pi}}{(1-p(t_{\theta}))} \frac{t_{\theta}^{3/2}}{\phi(t_{\theta})} \end{aligned}$$

As for the second integral on the right in (\*\*\*) , we have

$$\int_{2t_\theta}^{\infty} e^{-\frac{t}{6} \left( \frac{\phi(t_\theta)}{t_\theta} - \frac{\phi(2t_\theta)}{2t_\theta} \right)^2} dt \leq \int_{2t_\theta}^{\infty} e^{-\frac{t}{12} (1-p(t_\theta))^2 \frac{\phi^2(t_\theta)}{t_\theta^2}} dt$$

$$= \frac{12t_\theta^2}{(1-p(t_\theta))^2 \phi^2(t_\theta)} e^{-\left( \frac{1-p(t_\theta)}{6} \right)^2 \frac{\phi^2(t_\theta)}{t_\theta}}$$

$$\frac{1}{t_\theta} \int_{t_\theta}^{\infty} P\{T_\tau \geq t\} dt \leq \frac{1}{(1-p(t_\theta))^2} \left\{ \sqrt{3\pi} + 12 \frac{\sqrt{t_\theta}}{\phi(t_\theta)} \right\} \frac{\sqrt{t_\theta}}{\phi(t_\theta)}$$

Thus

$$\frac{1}{t_\theta} \int T_\tau \leq 1 + \frac{1}{(1-p(t_\theta))^2} \left\{ \sqrt{3\pi} + 12 \frac{\sqrt{t_\theta}}{\phi(t_\theta)} \right\} \frac{\sqrt{t_\theta}}{\phi(t_\theta)}$$

We turn next to the consideration of upper bounds for

$$\frac{1}{t_\theta} \int L_\tau$$

where

$$\tau = \inf n \rightarrow S_n \geq \phi(L_n).$$

Arguing as before;

$$1 \geq \int e^{-\frac{\lambda}{2} (S_\tau - R_\tau) - \frac{K(\lambda)}{4} T_\tau} \{L_\tau \geq t\} \{T_\tau \in [(1-\epsilon)L_\tau, (1+\epsilon)L_\tau]\}$$

For notational convenience, let the set in the integral above be denoted  $A(t, \epsilon)$ . We have

$$1 \geq \int_{A(t, \epsilon)} e^{-\frac{\lambda}{2} L_{\tau} \left\{ (1-\epsilon) \frac{\theta}{1+\theta} - \frac{K(\lambda)}{2\lambda} (1+\epsilon) - \frac{(\phi(L_{\tau})+1)}{L_{\tau}} \right\}}$$

Continuing to mimic the previous argument, we choose

$$\lambda = \ln \left\{ 1 + \frac{2}{1+\epsilon} \left( \frac{\theta}{1+\theta} (1-\epsilon) - \left( \frac{\phi(t)+1}{t} \right) \right) \right\}$$

and we have

$$P\{A(t, \epsilon)\} \leq \exp \left\{ \frac{t}{2(1+\epsilon)} \frac{\left( \frac{\theta(1-\epsilon)}{1+\theta} - \left( \frac{\phi(t)+1}{t} \right)^2 \right)^2}{1 + \frac{2}{1+\epsilon} \left( \frac{\theta(1-\epsilon)}{1+\theta} - \left( \frac{\phi(t)+1}{t} \right) \right)} \right\}$$

We define  $t_{\epsilon}$  through

$$\frac{\phi(t_{\epsilon})+1}{t_{\epsilon}} = \frac{\theta(1-\epsilon)}{1+\theta}, \text{ and we then}$$

have for  $t \geq t_{\epsilon}$ :

$$P\{A(t, \epsilon)\} \leq e^{-\frac{t}{6} \left\{ \frac{\phi(t_{\epsilon})}{t_{\epsilon}} - \frac{\phi(t)}{t} \right\}^2}$$

But then

$$\frac{1}{t_{\epsilon}} \int_{t_{\epsilon}}^{\infty} P\{A(t, \epsilon)\} dt \leq \frac{1}{(1-p(t_{\epsilon}))^2} \left\{ \sqrt{3\pi} + 12 \frac{\sqrt{t_{\epsilon}}}{\phi(t_{\epsilon})} \right\} \frac{\sqrt{t_{\epsilon}}}{\phi(t_{\epsilon})}$$

Next, for  $\varepsilon \leq \frac{1}{2}$ , and using Proposition (1.2):

$$\int_{t_\varepsilon}^{\infty} P\{T_\tau \notin [(1-\varepsilon)L_\tau, (1+\varepsilon)L_\tau], L_\tau \geq t\} dt \leq$$

$$2 \int_{t_\varepsilon}^{\infty} e^{-\frac{-2\varepsilon^2 t}{(1+2\varepsilon)^2}} dt \leq \frac{4}{\varepsilon^2} e^{-\frac{\varepsilon^2}{2} t_\varepsilon}$$

$$\leq \frac{4}{\varepsilon^2} e^{-\frac{\varepsilon^2}{2} t_\theta} \quad \text{since } t_\theta \leq t_\varepsilon .$$

We then have, again using  $t_\theta \leq t_\varepsilon$  :

$$\frac{1}{t_\varepsilon} \int_{t_\varepsilon}^{\infty} P\{L_\tau \geq t\} dt \leq \frac{4}{\varepsilon^2 t_\theta} e^{-\frac{\varepsilon^2}{2} t_\theta}$$

$$+ \frac{1}{(1-p(t_\theta))^2} \left\{ \sqrt{3\pi} + 12 \frac{\sqrt{t_\theta}}{\phi(t_\theta)} \right\} \frac{\sqrt{t_\theta}}{\phi(t_\theta)}$$

Now

$$\frac{t_\varepsilon}{t_\theta} \leq \frac{1}{1-\varepsilon} \frac{\phi(t_\varepsilon)}{\phi(t_\theta)} \leq \frac{1}{1-\varepsilon} \left\{ 1 + (t_\varepsilon - t_\theta) \frac{\phi'(t_\theta)}{t_\theta} \right\}$$

$$= \frac{1}{1-\varepsilon} \left\{ 1 + p(t_\theta) \left( \frac{t_\varepsilon}{t_\theta} - 1 \right) \right\} ,$$

from which

$$\frac{t_\varepsilon}{t_\theta} \leq 1 + \frac{\varepsilon}{1-p(t_\theta)-\varepsilon}$$

Replace  $\varepsilon$  with  $(1 - p(t_\theta))\varepsilon$ , and we have

$$\frac{t_\varepsilon}{t_\theta} \leq 1 + \frac{\varepsilon}{1-\varepsilon}$$

But then

$$\begin{aligned} \frac{1}{t_\theta} \int_{t_\theta}^{\infty} P\{L_\tau \geq t\} dt &\leq \left(\frac{t_\varepsilon}{t_\theta} - 1\right) + \frac{t_\varepsilon}{t_\theta} \cdot \frac{1}{t_\varepsilon} \int_{t_\varepsilon}^{\infty} P\{L_\tau \geq t\} dt \\ &\leq \frac{\varepsilon}{1-\varepsilon} + \frac{1}{1-\varepsilon} \frac{1}{t_\varepsilon} \int_{t_\varepsilon}^{\infty} P\{L_\tau \geq t\} dt \end{aligned}$$

Setting  $\varepsilon = \frac{\sqrt{t_\theta}}{\phi(t_\theta)}$ , we have

$$\begin{aligned} \frac{1}{t_\theta} \int_{t_\theta}^{\infty} P\{L_\tau \geq t\} dt &\leq 16 \frac{\phi^2(t_\theta)}{t_\theta^2} + \frac{1}{4(1-p(t_\theta))^2} \left\{ 1 + 8 \left[ \sqrt{3\pi} + 12 \frac{\sqrt{t_\theta}}{\phi(t_\theta)} \right] \right\} \frac{\sqrt{t_\theta}}{\phi(t_\theta)} \\ &\leq \frac{4}{(1-p(t_\theta))^2} \frac{\phi^2(t_\theta)}{t_\theta^2} + \frac{1}{4(1-p(t_\theta))^2} \left\{ 1 + 8 \left[ \sqrt{3\pi} + 12 \frac{\sqrt{t_\theta}}{\phi(t_\theta)} \right] \right\} \frac{\sqrt{t_\theta}}{\phi(t_\theta)} \\ &\leq 16 \frac{\phi^2(t_\theta)}{t_\theta^2} + \frac{2}{(1-p(t_\theta))^2} \left\{ 3.2 + 12 \frac{\sqrt{t_\theta}}{\phi(t_\theta)} \right\} \frac{\sqrt{t_\theta}}{\phi(t_\theta)} \end{aligned}$$

Q.E.D.

REFERENCES

- [1] Brown, B.M. (1971). A Conditional Setting for Some Theorems Associated with the Strong Law. *Zeit. Wahr.*, Vol. 19,
- [2] Darling, D.A., Robbins, H. (1967). Iterated Logarithm Inequalities *Proc. N.S.A.*, Vol. 57, No. 5, 1188-1192.
- [3] Darling, D.A., Robbins, H. (1968). Some Further Remarks on Inequalities for Sample Sums. *Proc. N.A.S.*, Vol. 60, No. 4, 1175-1182.
- [4] Freedman, D.A. (1975). On Tail Probabilities for Martingales. *Annals of Prob.*, Vol. 3, No. 1, 100-118.
- [5] Freedman, D.A., (1973). Another Note on the Borel-Cantelli Lemma and the Strong Law, with the Poisson Approximation as a by-Product. *Annals of Prob.*, Vol. 1, No. 6, 910-925.
- [6] Gut, A. (1974). On the Moments of Some First Passage Times for Sums of Dependent Random Variables. *Stochastic Processes and Their Applications*, Vol. 2, No. 1, 115-126.
- [7] Lai, T.L. (1977). Power One Tests Based on Sample Sums. *Annals of Stat.*, Vol. 5, No. 5, 866-880.
- [8] Lai, T.L. (1976). Boundary Crossing Probabilities for Sample Sums and Confidence Sequences. *Annals of Prob.*, Vol. 4, No. 2, 299-312.
- [9] Robbins, H., Siegmund, D. (1968). Iterated Logarithm Inequalities and Related Statistical Procedures, *Mathematics of the Decision Sciences. A.M.S. Vol.12*, 267-282.
- [10] Strassen, V. (1965). Almost sure Behavior of Sums of Independent Random Variables and Martingales. *Fifth Berkeley Symposium on Mathematical Statistics and Probability*, Vol. 2, 315-344.