

ON TIME-SEQUENTIAL POINT ESTIMATION OF THE MEAN OF AN
EXPONENTIAL DISTRIBUTION

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ABSTRACT

In the context of life testing, an asymptotically risk-efficient time-sequential procedure for estimating the mean of an exponential distribution is considered and its various properties are studied.

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1. INTRODUCTION

Let $\{X_i, i=1\}$ be a sequence of independent and identically distributed (i.i.d.) non-negative random variables (r.v.) with the distribution function (d.f.) $F_\theta(x) = 1 - e^{-x/\theta}$, $x \in [0, \infty)$, where $\theta(>0)$ is an unknown parameter. For $n(>1)$, items under *life testing*, the failures $X_{n,1}, \dots, X_{n,n}$ are the *order statistics* corresponding to X_1, \dots, X_n and, from cost and time considerations, one may curtail experimentation at the k^{th} failure $X_{n,k}$ and estimate θ by

$$(1.1) \quad \hat{\theta}_{nk} = k^{-1}V_{nk}, \quad \text{where } V_{nk} = \sum_{i=1}^k X_{n,i} + (n-k)X_{n,k} \quad \text{for } 1 < k < n.$$

Note that V_{nk} is the *total life under test upto the k^{th} failure*, $EV_{nk} = k\theta$ and $E(V_{nk} - k\theta)^2 = k\theta^2$, for $k=1, \dots, n$. Thus, if $a_1(>0)$ and $a_2(>0)$ be respectively the cost of recruitment (per individual) and of follow-up (per unit of test-life), then one may conceive of the *loss* incurred in estimating θ by $\hat{\theta}_{nk}$ as

$$(1.2) \quad L_{nk} = a_0(\hat{\theta}_{nk} - \theta)^2 + a_1n + a_2V_{nk} \quad (1 \leq k \leq n),$$

where the weights $a_0(>0)$, a_1 and a_2 are all known. Thus, the *risk* in estimating θ by $\hat{\theta}_{nk}$ is

$$(1.3) \quad R_{nk}(\underline{a}, \theta) = EL_{nk} = k^{-1}a_0\theta^2 + a_1n + a_2k\theta \quad (\underline{a} = (a_0, a_1, a_2)'),$$

and, naturally, one would seek to minimize (1.3) by a proper choice of k . However, as θ is unknown, no single value of k minimizes $R_{nk}(\underline{a}, \theta)$ for all $\theta(>0)$, and hence, a *time-sequential* procedure for choosing such a value of k is desirable.

Motivated by the works of Robbins (1959), Starr and Woodroffe (1972) and Ghosh and Mukhopadhyay (1979) [all dealing with the classical

sequential point estimation case], in Section 2, we formulate a time-sequential procedure for our problem and under asymptotic setup (similar to their cases) study its various properties. The derivation of the main results are postponed to concluding section.

2. TIME-SEQUENTIAL POINT ESTIMATION OF θ

Note that by (1.3),

$$(2.1) \quad R_{nk}(\underline{a}, \theta) - R_{nk+1}(\underline{a}, \theta) \begin{cases} \geq 0 \\ \leq 0 \end{cases} \text{ according as } k(k+1) \begin{cases} \geq \\ \leq \end{cases} \theta a_0/a_2.$$

Thus, if $n(n-1) < \theta a_0/a_2$, then $R_{nk}(\underline{a}, \theta)$ is \downarrow in $k(1 \leq k \leq n)$, and hence, $k=n$ is an optimal choice. On the other hand, if

$n(n-1) \geq \theta a_0/a_2$, then there exists an optimal $k_n (= k_n(\underline{a}, \theta))$ for which $k_n < n$ and $R_{nk}(\underline{a}, \theta)$ is minimized for $k=k_n$. Since $\hat{\theta}_{nk} = k^{-1}V_{nk}$ is an unbiased estimator of θ , motivated by the above, we consider the following *stopping number*

$$(2.2) \quad N_n = N_n(\underline{a}) = \begin{cases} \text{smallest } k(1 \leq k \leq n-1) \text{ for which } V_{nk} \leq k^2(k+1)a_2/a_0, \\ n \text{ if } V_{nk} > k^2(k+1)a_2/a_0, \text{ for every } 1 \leq k \leq n-1. \end{cases}$$

The corresponding *stopping time* is X_{n, N_n} and the point estimator of θ is $\hat{\theta}_{nN_n}$. Then, the *risk* corresponding to $\hat{\theta}_{nN_n}$ is

$$(2.3) \quad R_n^*(\underline{a}, \theta) = a_0 E(\hat{\theta}_{nN_n} - \theta)^2 + a_1 n + a_2 E V_{nN_n}.$$

We may recall that by definition,

$$(2.4) \quad k_n = k_n(\underline{a}, \theta) = \begin{cases} \text{smallest } k(1 \leq k \leq n-1) \text{ for which } k(k+1) \geq \theta a_0/a_2, \\ n \text{ if } n(n-1) < \theta a_0/a_2. \end{cases}$$

Let then

$$(2.5) \quad R_n^0(\underline{a}, \theta) = R_{nk_n}(\underline{a}, \theta).$$

Our primary interest centers around the behavior of (a) N_n/k_n and

(b) $R_n^*(\underline{a}, \theta) / R_n^0(\underline{a}, \theta)$ when we impose some asymptotic considerations on \underline{a} and n .

In the classical sequential point estimation theory [c.f. Robbins (1959) and others], $a_2 = 0$, $L_n = a_0(\hat{\theta}_{nn} - \theta)^2 + a_1 n$ and the problem is to choose n in such a way that the corresponding risk is minimized. In this context, one lets $a_1 \rightarrow 0$ and, in this asymptotic sense, one obtains some optimal results. In our case, however, for a given n , the stopping number N_n depends on a_0 and a_2 , but not on a_1 , and we let $a_2/a_0 \rightarrow 0$ or, simply, $a_2 \rightarrow 0$, keeping a_0 fixed. Note that our main interest lies in the case where k_n in (2.4) is $< n$ and in this case, $a_2 n^2 > a_2 n(n-1) \geq \theta a_0 > 0$. We assume that the sample size $n = n(a_2)$ depends on a_2 in such a way that

$$(2.6) \quad \lim_{a_2 \rightarrow 0} a_2 [n(a_2)]^2 = a^*: \quad 0 < a^* < \infty.$$

We may note that by (1.3), $R_{n'k}(\underline{a}, \theta) = R_{nk}(\underline{a}, \theta) + a_1(n' - n) \geq R_{nk}(\underline{a}, \theta)$, $\forall n' \geq n$, and hence, there is no point in increasing $n(a_2)$ indefinitely even when we allow $a_2 \rightarrow 0$, so that the restriction that a^* in (2.6) is $< \infty$ is of no loss of generality. Secondly, we note that for $\{n\}$ satisfying (2.6), by (2.4),

$$(2.7) \quad \lim_{n \rightarrow \infty} k_n/n = \gamma = (\theta a_0/a_2)^{1/2} \quad \text{and we assume that } 0 < \gamma < 1.$$

In terms of (2.6), (2.7) demands that $a^* > \theta a_0$. Finally, as in the classical sequential point estimation case, we assume that $a_1 \rightarrow 0$. More explicitly, we let

$$(2.8) \quad a_1 = \rho a_2, \quad \text{where } \rho > 0, \quad \text{and allow } a_2 \rightarrow 0.$$

Then, we have the following

Theorem 1. Under (2.6) and (2.7),

$$(2.9) \quad N_n/k_n \rightarrow 1 \text{ almost surely (a.s.) as } a_2 \rightarrow 0.$$

Moreover, for every real x ($-\infty < x < \infty$), under (2.6) and (2.7),

$$(2.10) \quad \lim_{n \rightarrow \infty} P\{2(N_n - k_n)/((n\gamma)^{1/2}\theta) \leq x\} = (2\pi)^{-1/2} \int_{-\infty}^x e^{-1/2 t^2} dt.$$

Theorem 2. Under (2.6), (2.7) and (2.8),

$$(2.11) \quad \lim_{a_2 \rightarrow 0} R_n^*(a, \theta)/R_n^0(a, \theta) = 1.$$

We may remark that by (2.2), $N_n = N_n(a)$ is \downarrow in a_2 and for any given n , there exists an $a_2(n) (> 0)$, such that $N_n = n, \forall 0 < a_2 \leq a_2(n)$. Also, $N_n \leq n$, with probability 1, so that (2.7) and (2.9) insure that $EN_n/k_n \rightarrow 1$ as $n \rightarrow \infty$. Further, (2.11) holds even when in (2.8), $\rho = 0$. If $a_1/a_2 \rightarrow \infty$ as $a_2 \rightarrow 0$, then R_n^* or R_n^0 are both dominated by $a_1 n$, and hence, (2.11) holds trivially.

3. PROOFS OF THEOREMS 1 AND 2

Let us denote by $V_{n0} = 0$ and

$$(3.1) \quad Z_{nk} = V_{nk} - V_{nk-1} = (n - k + 1)(X_{n,k} - X_{n,k-1}), \quad 1 \leq k \leq n$$

(where $X_{n,0} = 0$). Then Z_{n1}, \dots, Z_{nn} are i.i.d.r.v. each having the d.f. $F_\theta(x) = 1 - e^{-x/\theta}$. Also, note that for every $n (\geq 1)$,

$$(3.2) \quad V_{nk} \text{ is } \uparrow \text{ in } k: 0 \leq k \leq n.$$

Further, note that for every $\eta > 0$,

$$\begin{aligned} (3.3) \quad & P\{X_{m,1} < m^{-1-\eta}, \text{ for some } m \geq n\} \\ & \leq \sum_{k=0}^{\infty} P\{X_{n2^k+j,1} < (n2^k+j)^{-1-\eta} \text{ for some } 0 \leq j \leq n2^k\} \\ & \leq \sum_{k \geq 0} P\{X_{2^{k+1}n,1} < (2^k n)^{-1-\eta}\} = \sum_{k \geq 0} \{1 - e^{-2(2^k n)^{-\eta}}\} \\ & \leq \sum_{k \geq 0} \{2(n2^k)^{-\eta}\} = 2n^{-\eta}(1 - 2^{-\eta})^{-1} \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Thus, by (3.1) and (3.3), for every $\eta > 0$,

$$(3.4) \quad V_{n1} > n^{-\eta} \text{ a.s., as } n \rightarrow \infty.$$

Let us now choose a positive number λ such that

$$(3.5) \quad \frac{1}{2} < \lambda < \frac{2}{3} \text{ i.e., } \xi = \frac{2}{3} - \lambda > 0.$$

Then, under (2.6) and (2.7), by using (2.4) and (3.2),

$$(3.6) \quad P\{N_m \leq m^\lambda \text{ for some } m \geq n\} \\ = P\{ \bigcup_{m \geq n} [V_{mk} \leq k^2(k+1)a_2/a_0, \text{ for some } k \leq m^\lambda] \} \\ \leq P\{ \bigcup_{m \geq n} [V_{m1} \leq (a_2 m^2/a_0) [m^{2\lambda}(m^\lambda + 1)/m^2]] \},$$

where $a_2 m^2/a_0 \rightarrow a^*/a_0 (> 0)$ while by (3.5), $m^{2\lambda}(m^\lambda + 1)/m^2 \sim m^{-\xi}$, so that by (3.4), the right-hand side (rhs) of (3.6) converges to 0 as $n \rightarrow \infty$. Let us now denote by

$$(3.7) \quad k_{n\epsilon}^{(1)} = \max\{k: k(k+1) \leq (1-\epsilon)k_n(k_n+1)\}, \quad 0 < \epsilon < 1,$$

where k_n is defined by (2.4). Also, we choose n so large that $n^\lambda \leq k_{n\epsilon}^{(1)}$. Then

$$(3.8) \quad P\{N_m \leq k_{m\epsilon} \text{ for some } m \geq n\} \\ \leq P\{N_m \leq m^\lambda \text{ for some } m \geq n\} + \\ P\{ \bigcup_{m \geq n} [V_{mk} < k^2(k+1)a_2/a_0 \text{ for some } k: m^\lambda \leq k \leq k_{m\epsilon}] \}.$$

By (3.6) the first term on the rhs of (3.8) converges to 0 as $n \rightarrow \infty$, while by (3.7), the second term is bounded by

$$(3.9) \quad \sum_{m \geq n} P\{(V_{mk} - k\theta)/k\theta < -\eta, \text{ for some } k: m^\lambda \leq k \leq k_{m\epsilon}\},$$

where $\eta (> 0)$ depends on $\epsilon (> 0)$ in (3.7). By (3.1), for every $n (\geq 1)$,

$\{(V_{nk} - k\theta)/\theta, 0 \leq k \leq n\}$ is a martingale, so that $\{(V_{nk} - k\theta)^4/\theta^4, 0 \leq k \leq n\}$

is a sub-martingale, and hence, by the Chow (1961) extension of the Hájek-Rényi inequality,

$$\begin{aligned}
 (3.10) \quad & P\{(V_{mk} - k\theta)/k\theta < -\eta, \text{ for some } m^\lambda \leq k \leq k_{m\epsilon}\} \\
 & \leq P\{(V_{mk} - k\theta)^4/k^4\theta^4 > \eta^4 \text{ for some } k: m^\lambda \leq k \leq k_{m\epsilon}\} \\
 & \leq \sum_{k=[m^\lambda]}^{k_{m\epsilon}} \left\{ \eta^{-4} E(V_{mk} - k\theta)^4/\theta^4 \right\} \left\{ k^{-4} - (k+1)^{-4} \right\} \\
 & \leq \eta^{-4} \sum_{k \geq [m^\lambda]} [O(k^{-3})] = \eta^{-4} \cdot O(m^{-2\lambda}),
 \end{aligned}$$

so that by (3.5) and (3.10), the second term on the rhs of (3.8) converges to 0 as $n \rightarrow \infty$. Thus, for every $\epsilon > 0$,

$$(3.11) \quad N_n/k_n > 1 - \epsilon \text{ a.s., as } n \rightarrow \infty.$$

In a similar way, it follows that for every $\epsilon > 0$,

$$(3.12) \quad N_n/k_n < 1 + \epsilon \text{ a.s., as } n \rightarrow \infty,$$

and (2.9) follows from (3.11) and (3.12).

To prove (2.10), we note that for every (fixed) $u \in (-\infty, \infty)$,

$$(3.13) \quad P\{N_n \geq k_n + u\sqrt{n}\} = P\{V_{nk} > k^2(k+1)a_2/a_0, \forall k \leq k_n + u\sqrt{n}\},$$

and we choose n so large that $k_n + u\sqrt{n} > k_{n\epsilon}$, where $k_{n\epsilon}$ is defined by (3.7) and k_n by (2.4). Then, by using (3.11), the rhs of (3.13) can be written as

$$\begin{aligned}
 (3.14) \quad & P\{V_{nk} > k^2(k+1)a_2/a_0, \forall k_{n\epsilon} \leq k \leq k_n + u\sqrt{n}\} + o(1) \\
 & = P\left\{ \frac{V_{nk} - k\theta}{\theta\sqrt{n}} > \frac{k}{\sqrt{n}} \left[\frac{k(k+1)}{k_n(k_n+1)} - 1 \right], \forall k_{n\epsilon} \leq k \leq k_n + u\sqrt{n} \right\} + o(1).
 \end{aligned}$$

Let us now consider a sequence $\{W_n = \{W_n(t), t \in [0, 1]\}, n \geq 1\}$ of stochastic processes, where we let $W_n(t) = W_n(\frac{k}{n})$, for $\frac{k}{n} \leq t < \frac{k+1}{n}$, $0 \leq k \leq n-1$ and $W_n(k/n) = (V_{nk} - k\theta)/\theta\sqrt{n}$, $k = 0, 1, \dots, n$. Then by virtue of (3.1), the classical Donsker Theorem applies and we have

$$(3.15) \quad W_n \xrightarrow{D} W = \{W(t), t \in [0, 1]\},$$

where W is a standard Wiener process on $[0, 1]$. As a corollary to (3.15), we have that for every $\epsilon' > 0$ and $\eta' > 0$ there exist a $\delta: 0 < \delta < 1$ and an n_0 such that

$$(3.16) \quad P\{\sup\{|W_n(t) - W_n(s)| : 0 \leq s < t \leq s + \delta \leq 1\} > \epsilon'\} < \eta', \quad \forall n \geq n_0.$$

To make use of (3.15) and (3.16) in (3.14), we note that for $k = k_n + [u\sqrt{n}]$ $n^{-\frac{1}{2}}k[k(k+1)/k_n(k_n+1) - 1] \rightarrow 2u$. Thus, the rhs of (3.14) can be expressed as

$$(3.17) \quad P\left\{\frac{V_{nk} - k\theta}{\sqrt{n}\theta} > \frac{k}{\sqrt{n}\theta} \left[\frac{k(k+1)}{k_n(k_n+1)} - 1 \right], \quad \forall k_{n\epsilon} \leq k \leq k_n + u\sqrt{n}, \quad W_n(\gamma) > \frac{2u - \epsilon}{\theta}\right\} +$$

$$P\left\{\frac{V_{nk} - k\theta}{\theta\sqrt{n}} > \frac{k}{\theta\sqrt{n}} \left[\frac{k(k+1)}{k_n(k_n+1)} - 1 \right], \quad \forall k_{n\epsilon} \leq k \leq k_n + u\sqrt{n}, \quad W_n(\gamma) \leq \frac{2u - \epsilon}{\theta}\right\} + o(1),$$

where $\epsilon > 0$. The second term is bounded by $P\{W_n(n^{-1}k_n + un^{-\frac{1}{2}}) - W_n(\gamma) > \epsilon/\theta\}$ and, by (3.16), it converges to 0 as $n_2 \rightarrow \infty$ (or $a_2 \rightarrow 0$). Similarly, the first term is convergent-equivalent to

$$(3.18) \quad P\{W_n(\gamma) > (2u - \epsilon)/\theta\} \rightarrow P\{W(\gamma) > (2u - \epsilon)/\theta\}$$

$$= P\{W(1) > (2u - \epsilon)/\theta\sqrt{\gamma}\} = (2\pi)^{-\frac{1}{2}} \int_{(2u-\epsilon)/\theta\sqrt{\gamma}}^{\infty} \exp(-\frac{1}{2}t^2) dt.$$

Thus, (2.10) follows from (3.13), (3.14), (3.17) and (3.18) by letting $u = \theta\sqrt{\gamma}x/2$ and $\epsilon \rightarrow 0$. This completes the proof of Theorem 1.

To prove Theorem 2, we first note that under (2.6), (2.7) and (2.8),

$$(3.19) \quad (a^*/a_2)^{\frac{1}{2}} R_n^0(a, \theta) \rightarrow a_0 \theta^2 / \gamma + \rho a^* + a^* \gamma \theta \quad \text{as } a_2 \rightarrow 0.$$

Also, recalling that $n^{-1}N_n \leq 1$, with probability 1, we have by (2.9),

$$(3.20) \quad \lim_{n \rightarrow \infty} E(n^{-1}N_n)^m = \gamma^m (< 1), \quad \forall m = 1, 2, \dots$$

Further, by (2.2), (3.1), and the fact that Z_{nk} is ≥ 0 , $\forall k \geq 1$, we

$$(3.21) \quad N_n(N_n - 1)^2 a_2 / a_0 < V_{nN_n - 1} < V_{nN_n} \leq N_n^2 (N_n + 1) a_0^{-1} a_2 I_{(N_n < n)} + V_{nn} I_{(N_n = n)}.$$

Note that by (2.6) and (2.7),

$$(3.22) \quad \begin{aligned} P\{N_n = n\} &= P\{V_{nk} > k^2(k+1)a_2/a_0, \quad \forall 1 \leq k \leq n-1\} \\ &\leq P\{V_{nn-1} > (n-1)^2 na_2/a_0\} = P\{V_{nn-1} - (n-1)\theta > (n-1)[n(n-1)a_2/a_0 - \theta]\} \\ &\leq \theta^2 / (n-1)[n(n-1)a_2/a_0 - \theta]^2 \sim \theta^2 / [(n-1)\{a^*/a_0 - \theta\}^2] = o(n^{-1}). \end{aligned}$$

Also, $E(V_{nn}^2) = n(n+1)\theta^2$, so that by the Schwarz inequality and (3.22),

$$(3.23) \quad |E\{V_{nn} I_{(N_n=n)}\}| \leq \theta^2 \sqrt{n(n+1)} \{o(n^{-1/2})\} = o(n^{1/2}) = o(a_2^{-1/4}), \quad \text{by (2.6)}.$$

Further, by (3.21)

$$(3.24) \quad a_2 |V_{nN_n} - a_0^{-1} a_2 N_n^3| \leq \frac{a_2^2}{a_0} 2N_n^2 I_{(N_n < n)} + a_2 |V_{nn} - a_0^{-1} a_2 n^3| I_{(N_n = n)},$$

where by (2.6), $a_0^{-1} a_2 n^3 \rightarrow a_0^{-1} \sqrt{a_2} (a^*)^{3/2}$, while $a_2^2 N_n^2 I_{(N_n < n)} \leq a_2^2 n^2 \rightarrow a_2 a^*$.

Hence, by (3.22), (3.23) and (3.24), we have

$$(3.25) \quad (a^*/a_2)^{1/2} E\{a_2 |V_{nN_n} - a_0^{-1} a_2 N_n^3|\} = o(a_2^{1/4}) \rightarrow 0 \quad \text{as } a_2 \rightarrow 0.$$

On the other hand, by (3.20) and (2.4) - (2.7),

$$(3.26) \quad (a^*/a_2)^{1/2} E(a_2^2 N_n^3)/a_0 \rightarrow \gamma \theta a^* \quad \text{as } a_2 \rightarrow 0.$$

Also, by (2.8),

$$(3.27) \quad (a^*/a_2)^{1/2} a_1 n \rightarrow \rho a^*, \quad \text{as } a_2 \rightarrow 0.$$

Hence, by (2.3), (3.19), (3.26) and (3.27), for (2.11), it suffices to show that

$$(3.28) \quad \lim_{a_2 \rightarrow 0} (a^*/a_2)^{1/2} E(N_n^{-1} V_{nN_n} - \theta)^2 = \theta^2 / \gamma.$$

Note that

$$(3.29) \quad \begin{aligned} (a^*/a_2)^{1/2} E(N_n^{-1} V_{nN_n} - \theta)^2 &= (a^*/a_2)^{1/2} k_n^{-2} E(V_{nN_n} - \theta N_n)^2 + \\ &\quad (a^*/a_2)^{1/2} k_n^{-2} E\{(V_{nN_n} - \theta N_n)^2 [(k_n/N_n)^2 - 1]\}. \end{aligned}$$

Now, for every $n(\geq 1)$, $\{V_{nk} - k\theta = \sum_{j=1}^k (Z_{nj} - \theta), \quad 1 \leq k \leq n\}$ is a martingale,

$E(Z_{nj} - \theta)^2 = \theta^2$ and $EN_n < \infty$. Hence, by the Wald second lemma [viz. Theorem 2 of Chow, Robbins and Teicher (1965)], we have $E(V_{nN_n} - \theta N_n)^2 = \theta^2 EN_n$, so that by (2.6), (2.7) and (3.20), the first term on the rhs of (3.29) is equal to

$$(3.30) \quad (a^*/a_2)^{1/2} k_n^{-2} \theta^2 EN_n \rightarrow \theta^2/\gamma \quad \text{as } a_2 \rightarrow 0.$$

Thus, we need to show that the second term on the rhs of (3.29) converges to 0 as $a_2 \rightarrow 0$. Now, by the same technique as in (3.24) - (3.25), it follows that

$$(3.31) \quad (a^*/a_2)^{1/2} k_n^{-2} E\{|V_{nN_n} - a_0^{-1} a_1 N_n^3\|^2 |(k_n/N_n)^2 - 1|\} \\ = O(n^{-1}) = O(\sqrt{a_2}) \rightarrow 0 \quad \text{as } a_2 \rightarrow 0.$$

On the other hand, by (2.4), (2.6) and (2.7),

$$(3.32) \quad (a^*/a_2)^{1/2} k_n^{-2} E\{[a_0^{-1} a_2 N_n^3 - N_n \theta]^2 |(k_n/N_n)^2 - 1|\} \\ = \sqrt{a^*} a_2^{3/2} a_0^{-2} E\{(N_n^2 - \theta a_0/a_2)^2 |1 - (N_n/k_n)^2|\} \\ = \sqrt{a^*} a_2^{3/2} a_0^{-2} E\{(N_n^2 - k_n^2)^2 |1 - (N_n/k_n)^2|\} + O(\sqrt{a_2}) \\ = \sqrt{a^*} a_2^{3/2} a_0^{-2} k_n^{-2} E|N_n^2 - k_n^2|^3 + O(\sqrt{a_2}) \\ \leq \sqrt{a^*} a_2^{3/2} a_0^{-2} 8n^3 k_n^{-2} E|N_n - k_n|^3 + O(\sqrt{a_2}).$$

Thus, it suffices to show (by virtue of (2.6) - (2.7)) that

$$(3.33) \quad \lim_{n \rightarrow \infty} k_n^{-2} E|N_n - k_n|^3 = 0 \quad [\text{as } a_2^{3/2} n^3 \rightarrow (a^*)^{3/2}].$$

Define λ as in (3.5). Then

$$(3.34) \quad k_n^{-2} E|N_n - k_n|^3 = k_n^{-2} E(|N_n - k_n|^3 I_{(|N_n - k_n| \leq n^\lambda)}) + k_n^{-2} E(|N_n - k_n|^3 I_{(|N_n - k_n| > n^\lambda)}),$$

where by (2.7), the first term on the rhs of (3.34) converges to 0 as $n \rightarrow \infty$. On the other hand, the second term is bounded by

$$(3.35) \quad n^3 k_n^{-2} P\{|N_n - k_n| > n^\lambda\}.$$

Let $b_1 = 1/3 - \epsilon$, $\epsilon > 0$. Then

$$\begin{aligned}
 (3.36) \quad P\{N_n \leq n^{b_1}\} &= P\{V_{nk} \leq k^2(k+1)a_2/a_0 \text{ for some } k \leq n^{b_1}\} \\
 &\leq P\{V_{n1} \leq \frac{a_2}{a_0} n^{2b_1}(n^{b_1} + 1)\} = P\{V_{n1} \leq 0(n^{-2+3b_1})\} \\
 &= 0(n^{-2+3b_1}) = 0(n^{-1-3\epsilon}).
 \end{aligned}$$

Also, let $k_{n\epsilon}$ be defined as in (3.7). Then proceeding as in (3.10)

but using the 8th order moment of $(V_{nk} - k\theta)$, we obtain that

$$(3.37) \quad P\{n^{b_1} \leq N_n < k_{n\epsilon}\} = 0([n^{b_1}]^{-4}) = 0(n^{-4/3-4\epsilon}).$$

Finally, let $k_n^* = k_n - n^\lambda$ and assume n so large that $k_n^* > k_{n\epsilon}$. Then

$$\begin{aligned}
 (3.38) \quad P\{k_{n\epsilon} < N_n \leq k_n^*\} &\leq \sum_{k=k_{n\epsilon}}^{k_n^*} P\left\{\frac{1}{k}(V_{nk} - k\theta) < \theta\left(\frac{k(k+1)}{k_n(k_n+1)} - 1\right)\right\} \\
 &\leq \sum_{k=k_{n\epsilon}}^{k_n^*} k^{-2r} E(V_{nk} - k\theta)^{2r} / \theta^{2r} \left[\frac{k(k+1)}{k_n(k_n+1)} - 1\right]^{2r},
 \end{aligned}$$

for any $r(> 0)$. Now,

$$(3.39) \quad E(V_{nk} - k\theta)^{2r} = 0(k^r), \text{ for every } r = 2, 3, 4, \dots$$

Also, for $k_{n\epsilon} \leq k \leq k_n^*$,

$$(3.40) \quad \left|k(k+1)/k_n(k_n+1) - 1\right| = 0\left(\frac{k_n - k}{k_n}\right),$$

so that the rhs of (3.38) is

$$\begin{aligned}
 (3.41) \quad &0\left(\sum_{k=k_{n\epsilon}}^{k_n^*} k^{-r} k_n^{2r} (k_n - k)^{-2r}\right) \\
 &= 0(n^r) \left(\sum_{k=k_{n\epsilon}}^{k_n^*} (k_n - k)^{-2r}\right) \\
 &= 0(n^r) \cdot 0(n^{-\lambda(2r-1)}) = 0(n^{-(2\lambda-1)r+\lambda}).
 \end{aligned}$$

Since (3.38) and (3.39) hold for every positive integer r and $\lambda > \frac{1}{2}$,

we may choose r so large that $(2\lambda - 1)r - \lambda > 1$, and this leads to the

rhs of (3.41) as $0(n^{-1-\eta})$, for some $\eta > 0$. A similar treatment holds

for the case of $N_n \geq k_n + n^\lambda$. Thus, $P\{|N_n - k_n| > n^\lambda\} = 0(n^{-1-\eta})$ for some

$\eta > 0$, and this proves that (3.35) converges to 0 as $n \rightarrow \infty$.

Q.E.D.

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