THE COX REGRESSION MODEL, INVARIANCE PRINCIPLES FOR SOME INDUCED QUANTILE PROCESSES AND SOME REPEATED SIGNIFICANCE TESTS

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ABSTRACT

For the Cox regression model, the partial likelihood functions involve linear combinations of induced order statistics. Some invariance principles pertaining to such linear combinations of induced order statistics are studied and the theory is incorporated in the formulation of suitable repeated significance tests (for the hypothesis of no regression) based on these partial likelihoods. Asymptotic power properties of these tests are also studied.

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1. INTRODUCTION

In the Cox (1972) regression model for survival data, it is assumed that the i-th subject (having *survival time* Y_i and a set of *covariates* $\underset{\sim}{Z}_i = (Z_{i1}, \ldots, Z_{ip})'$ for some $p \geq 1$) has the *hazard rate* (given $\underset{\sim}{Z}_i = \underset{\sim}{z}_i$)

(1.1) $$h_i(t) = h_0(t)\exp(\underset{\sim}{\beta}'\underset{\sim}{z}_i), \quad i = 1, \ldots, n,$$

where $h_0(t)$, the hazard rate for $\underset{\sim}{z}_i = \underset{\sim}{0}$, is an unknown, arbitrary non-negative function and $\underset{\sim}{\beta}' = (\beta_1, \ldots, \beta_p)$ parameterizes the regression of survival time on the covariates. We assume that $h_0(t)$ is continuous in t almost everywhere (a.e.), so that ties among the Y_i may be neglected, with probability 1. Let $\underset{\sim}{Q}_n = (Q_1, \ldots, Q_n)'$, the vector of *anti-ranks*, be defined by

(1.2) $$Y_{Q_i} = Y_{ni}, \quad 1 \leq i \leq n,$$

where $Y_{n1} < \cdots < Y_{nn}$ are the order statistics corresponding to $Y_1, \ldots, Y_n$. Then, following Bhattacharya (1974), $\underset{\sim}{Z}_{Q_1}, \ldots, \underset{\sim}{Z}_{Q_n}$ are termed the *induced order statistics*. In the event of no loss in the follow-up (*censoring*), the *partial* (*log*-) *likelihood function* when all the failures have been observed [c.f. Cox (1972, 1975)] is given by

(1.3) $$\log L_n = \sum_{i=1}^{n}\left\{\underset{\sim}{\beta}'\underset{\sim}{Z}_{Q_i} - \log\left[\sum_{j=i}^{n}\exp\left\{\underset{\sim}{\beta}'\underset{\sim}{Z}_{Q_j}\right\}\right]\right\} .$$

To incorporate censoring, Cox (1972) considered a more general case where there are $m(\geq 1)$ ordered failures $t_1, \ldots, t_m$ (a subset of the Y_{ni}), such that at time $t_j - 0$, there is a *risk set* R_j of r_j subjects who have neither failed nor been censored by that time, for $i \leq j \leq m$ (so that $R_m \subset \cdots \subset R_1$). If $W_1, \ldots, W_n$ be the *censoring times* for the n subjects, then, we have

(1.4) $$t_j = Y_{Q_j^*} \text{ and } R_j = \{i: \; Y_i^* = Y_i \wedge W_i \geq t_j, \; 1 \leq i \leq n\}, \; 1 \leq j \leq m,$$

where $\underset{\sim}{Q}^* = (Q_1^*, \ldots, Q_m^*)'$ is a sub-vector of Q and $a \wedge b = \min(a, b)$. Then, the Cox (1972) partial (log-) likelihood function is

$$\log L_{nm}^* = \sum_{j=1}^{m} \left\{ \underset{\sim}{\beta}' \underset{\sim}{Z}_{Q_j^*} - \log\left(\sum_{i \in R_i} \exp\left\{ \underset{\sim}{\beta}' \underset{\sim}{Z}_i \right\} \right) \right\} . \tag{1.5}$$

[Note that if $m = n$ and $Y_i = Y_i^*$, $1 \le i \le n$, then $L_{nm}^* = L_n$. A *discrete-time* version of (1.5) has also been considered by Cox (1972) and we shall refer to that later on.]

For testing the hypothesis of no regression viz.,

$$H_0: \ \underset{\sim}{\beta} = \underset{\sim}{0} \quad \text{vs.} \quad H_1: \ \underset{\sim}{\beta} \neq \underset{\sim}{0} , \tag{1.6}$$

Cox (1972) considered the test statistic

$$\mathcal{L}_{nm}^* = \underset{\sim}{U}_{nm}^{*\prime} \underset{\sim}{J}_{nm}^{*-} \underset{\sim}{U}_{nm}^* , \tag{1.7}$$

where

$$U_{nm}^* = (\partial/\partial\underset{\sim}{\beta}) \log L_{nm}^* \big|_{\underset{\sim}{\beta}=\underset{\sim}{0}}, \quad \underset{\sim}{J}_{nm}^* = (\partial^2/\partial\underset{\sim}{\beta}\partial\underset{\sim}{\beta}') \log L_{nm}^* \big|_{\underset{\sim}{\beta}=\underset{\sim}{0}} , \tag{1.8}$$

and $\underset{\sim}{A}^-$ stands for a generalized inverse of $\underset{\sim}{A}$. Cox (1972; 1975, Section 5) argued heuristically that under H_0, $\mathcal{L}_{nm}^*$ has asymptotically chi-square distribution with p degrees of freedom (DF).

In a variety of situations, relating to clinical trials and life-testing experimentations, one may be interested in monitoring the study from the beginning with the objective of an early termination if H_0 in (1.6) is not tenable. Such a plan is known as a *progressively censored scheme* (PCS) [viz., Chatterjee and Sen (1973) and Sen (1976, 1979)]. Thus, in a PCS, instead of making a *terminal test* at the m-th failure t_m, one may like to review the process at each failure t_j $(j \ge 1)$ and stop experimentation as soon as $\mathcal{L}_{nj}^*$ (defined as in (1.7) - (1.8), but, based on $\underset{\sim}{U}_{nj}^*$ and J_{nj}^*) leads to the rejection of H_0 for some $j \le m$; if $\mathcal{L}_{n1}^*, \ldots, \mathcal{L}_{nm}^*$ are all insignificant, then H_0 is accepted along with the termination of the study at the preplanned m-th failure. Hence, a *repeated significance testing* (RST) procedure is involved in a PCS. Since the L_{nj} are neither independent nor have independent increments, more elaborate analysis is needed to find out the critical values having prescribed overall level of significance.

We may note that by (1.5) and (1.8),

$$U_{nk}^* = \sum_{j=1}^{k}\left\{\underset{\sim}{Z}_{Q_j^*} - r_j^{-1}\sum_{i\varepsilon R_j}\underset{\sim}{Z}_i\right\}, \text{ for } k = 1, \ldots, m , \tag{1.9}$$

and hence, these are all linear combinations of induced order statistics. For this reason, we first study some *invariance principles* relating to induced order statistics and then incorporate these in the study of the asymptotic properties of the proposed RST. These *induced quantile processes* are introduced in Section 2 and their weak convergence results are presented there. Parallel results for the discrete time case are treated in Section 3. Section 4 is devoted to the study of these invariance principles under (local) contiguous alternatives. The concluding section incorporates all these results in the study of the asymptotic properties of the proposed RST procedures relating to the Cox model.

2. WEAK CONVERGENCE OF SOME INDUCED QUANTILE PROCESSES

By analogy to (1.3), (1.5) and (1.8), we let for every k: $1 \le k \le n$,

$$\log L_{nk} = \sum_{j=1}^{k}\left\{\underset{\sim}{\beta}'\underset{\sim}{Z}_{Q_j} - \log\left\{\sum_{i=j}^{n}\exp\left\{\underset{\sim}{\beta}'\underset{\sim}{Z}_{Q_i}\right\}\right\}\right\}, \tag{2.1}$$

$$\begin{aligned} \underset{\sim}{U}_{nk} &= (\partial/\partial\underset{\sim}{\beta})\log L_{nk}\big|_{\underset{\sim}{\beta}=\underset{\sim}{0}} \\ &= \sum_{j=1}^{k}\left\{\underset{\sim}{Z}_{Q_j} - (n-j+1)^{-1}\sum_{i=j}^{n}\underset{\sim}{Z}_{Q_i}\right\} , \end{aligned} \tag{2.2}$$

$$\begin{aligned} \underset{\sim}{J}_{nk} &= -(\partial^2/\partial\underset{\sim}{\beta}\partial\underset{\sim}{\beta}')\log L_{nk}\big|_{\underset{\sim}{\beta}=0} \\ &= \sum_{j=1}^{k}(n-j+1)^{-1}(n-j)\underset{\sim}{S}_{nj}, \quad \text{say} , \end{aligned} \tag{2.3}$$

where $\underset{\sim}{S}_{nn} = 0$ and

(2.4) $$\underset{\sim}{S}_{nj} = (n-j)^{-1}\sum_{i=j}^{n}(\underset{\sim}{Z}_{Q_i} - \overline{\underset{\sim}{Z}}_j^*)(\underset{\sim}{Z}_{Q_i} - \overline{\underset{\sim}{Z}}_j^*)', \quad 1 \le j \le n-1;$$

(2.5) $$\overline{\underset{\sim}{Z}}_j^* = (n-j+1)^{-1}\sum_{i=j}^{n}\underset{\sim}{Z}_{Q_i}, \quad 1 \le j \le n.$$

Conventionally, we let

(2.6) $$\underset{\sim}{U}_{n0} = \underset{\sim}{0}, \quad \underset{\sim}{J}_{n0} = \underset{\sim}{0}, \quad \forall\ n \ge 1.$$

First, we study suitable invariance principles relating to the triangular arrays $\{\underset{\sim}{U}_{nk}:\ 0 \le k \le n;\ n \ge 1\}$ and $\{\underset{\sim}{J}_{nk}:\ 0 \le k \le n;\ n \ge 1\}$.

Throughout this paper, the covariates $\underset{\sim}{Z}_1, \ldots, \underset{\sim}{Z}_n$ are assumed to be stochastic vectors; there are some simplifications when these are nonstochastic and these will be briefly considered later on. Also, in the usual custom of an analysis of covariance model, we assume that $\underset{\sim}{Z}_1, \ldots, \underset{\sim}{Z}_n$ are independent and identically distributed random vectors (i.i.d.r.v.) with

(2.7) $$\underset{\sim}{\mu} = E\underset{\sim}{Z}, \quad \underset{\sim}{\Gamma} = E(\underset{\sim}{Z} - \underset{\sim}{\mu})(\underset{\sim}{Z} - \underset{\sim}{\mu})' \quad \text{(both exist)},$$

and

(2.8) $$\underset{\sim}{\Gamma} \text{ is positive definite (p.d.) with det. } \underset{\sim}{\Gamma} < \infty .$$

For clarification of ideas, we note that $\underset{\sim}{S}_{n1} = (n-1)^{-1}\sum_{i=1}(\underset{\sim}{Z}_i - \overline{\underset{\sim}{Z}}_n)(\underset{\sim}{Z}_i - \overline{\underset{\sim}{Z}}_n)'$ (where $\overline{\underset{\sim}{Z}}_n = n^{-1}\sum_{i=1}^{n}\underset{\sim}{Z}_i$), while for $2 \le j \le n-1$,

(2.9) $$\underset{\sim}{S}_{nj} = (n-j)^{-1}\Big\{\sum_{i=1}^{n}\underset{\sim}{Z}_i\underset{\sim}{Z}_i' - \sum_{i=1}^{j-1}\underset{\sim}{Z}_{Q_i}\underset{\sim}{Z}_{Q_i}' - (n-j+1)^{-1}\Big(\sum_{i=1}^{n}\underset{\sim}{Z}_i - \sum_{i=1}^{j-1}\underset{\sim}{Z}_{Q_i}\Big)\Big(\sum_{i=1}^{n}\underset{\sim}{Z}_i' - \sum_{i-1}^{j-1}\underset{\sim}{Z}_{Q_i}'\Big)\Big\},$$

and $\underset{\sim}{S}_{nn} = 0$. Thus, if $\underset{\sim}{Z}_1, \ldots, \underset{\sim}{Z}_n$ are all observable at the beginning of the experimentation, then $\underset{\sim}{S}_{nk}$ depends only on $Q_i, \ldots, Q_{k-1}$ and hence, is observable at the $(k-1)$th failure, for $2 \le k \le n$. Also, $\underset{\sim}{U}_{nk}$ depends on $\underset{\sim}{Z}_1, \ldots, \underset{\sim}{Z}_n$ and $Q_1, \ldots, Q_k$ (but not individually on $Q_{k+1}, \ldots, Q_n$). We are primarily interested in the asymptotic behavior of the two partial sequences $\{\underset{\sim}{U}_{nk},\ 0 \le k \le n\}$ and $\{\underset{\sim}{J}_{nk},\ 0 \le k \le n\}$.

Let $\mathcal{B}_{nk} = \mathcal{B}(\underset{\sim}{Z}_1, \ldots, \underset{\sim}{Z}_n,\ Q_1, \ldots, Q_k)$ be the σ-field generated by $Q_1, \ldots, Q_k$ and $\underset{\sim}{Z}_1, \ldots, \underset{\sim}{Z}_n$ for $1 \le k \le n$, while $\mathcal{B}_{n0} = \mathcal{B}(\underset{\sim}{Z}_1, \ldots, \underset{\sim}{Z}_n)$. Then $\mathcal{B}_{nk}$ is nondecreasing in $k (\le n)$. For every $n (\ge 1)$, we introduce a sequence $\{k_n(t),\ t \in E = [0, 1]\}$ of nondecreasing and right-continuous non-negative integers by letting

$$k_n(t) = \max\{k:\ \text{Trace}(\underset{\sim}{J}_{nn}^{-1} \underset{\sim}{J}_{nk}) \le pt\},\ t \in E. \tag{2.10}$$

Let then $\underset{\sim}{\xi}_n = \{\underset{\sim}{\xi}_n(t),\ t \in E\}$ be defined by

$$\underset{\sim}{\xi}_n(t) = \underset{\sim}{J}_{nn}^{-\frac{1}{2}} \underset{\sim}{U}_{nk_n(t)},\quad t \in E. \tag{2.11}$$

Note that in (2.10) and (2.11), we have assumed that $\underset{\sim}{J}_{nn}$ is p.d. and $\underset{\sim}{J}_{nn}^{-\frac{1}{2}} = \underset{\sim}{B}_n$ is defined by $\underset{\sim}{B}_n \underset{\sim}{J}_{nn} \underset{\sim}{B}_n' = \underset{\sim}{I}_p$, the identify matrix of order p. Later on, we shall see that by virture of (2.8), $\underset{\sim}{J}_{nn}$ is p.d., in probability, and it is also possible to replace $k_n(t)$ by $[nt]$ (where $[s]$ denotes the largest integer $\le s$) and $\underset{\sim}{\xi}_n(t)$ by $n^{-\frac{1}{2}} \underset{\sim}{\Gamma}^{-\frac{1}{2}} \underset{\sim}{U}_{[nt]}$, $t \in E$. Then, the main theorem of this section is the following

<u>*Theorem 2.1.*</u> *Under* (2.7), (2.8) *and* H_0: $\underset{\sim}{\beta} = \underset{\sim}{0}$, $\underset{\sim}{\xi}_n$ ***weakly converges*** *to* $\underset{\sim}{\xi} = \{\underset{\sim}{\xi}(t),\ t \in E\}$ *where* $\underset{\sim}{\xi}(t) = (\xi_{(1)}(t), \ldots, \xi_{(p)}(t))'$, $t \in E$ *and* $\xi_{(j)} = \{\xi_{(j)}(t),\ t \in E\}$, $j = 1, \ldots, p$ *are independent copies of a standard Wiener process on* E.

Before we present the proof of the theorem, we consider the following lemmas.

<u>*Lemma 2.2.*</u> *Under* (2.7) *and* H_0: $\underset{\sim}{\beta} = \underset{\sim}{0}$, $\{\underset{\sim}{U}_{nk},\ \mathcal{B}_{nk},\ 0 \le k \le n\}$ *is a martingale* (*for every* $n \ge 1$).

Proof: Note that by (2,2), for every $k \ge 1$,

$$U_{nk} - U_{nk-1} = Z_{Q_k} - (n-k+1)^{-1}\sum_{i=k}^{n} Z_{Q_i} . \tag{2.12}$$

Now, given $\mathcal{B}_{nk-1}$, under H_0, Q_k can take on any one value in the set $(1, \ldots, n)\setminus(Q_1, \ldots, Q_{k-1})$ with the equal conditional probability $(n-k+1)^{-1}$, so that $E(Z_{Q_k} | \mathcal{B}_{nk-1}) = (n-k+1)^{-1}\left\{\sum_{i=1}^{n} Z_i - \sum_{i=1}^{k-1} Z_{Q_i}\right\} = (n-k+1)^{-1}\sum_{j=k}^{n} Z_{Q_j}$. Thus, under H_0, $E(U_{nk} - U_{nk-1} | \mathcal{B}_{nk-1}) = 0$ a.e., $\forall\ 1 \le k \le n$. Q.E.D.

By the same arguments, it follows that under H_0,

$$E\{(U_{nk} - U_{nk-1})(U_{nk} - U_{nk-1})' | \mathcal{B}_{nk-1}\} = \frac{n-k}{n-k+1} S_{nk}, \ \forall\ 1 \le k \le n, \tag{2.13}$$

where the S_{nk} are defined by (2.4). Note that if we let

$$\phi(a, b) = \frac{1}{2}(a - b)(a - b)', \quad \forall\ a, b \in R^p, \tag{2.14}$$

then we have

$$S_{nk} = \binom{n-k+1}{2} \sum_{k \le i \le j \le n} \phi(Z_{Q_i}, Z_{Q_j}), \quad \forall\ 1 \le k \le n-1 . \tag{2.15}$$

As such, by the same arguments as in the proof of Lemma 2.2, we have under H_0: $\beta = 0$,

$$E(S_{nk} | \mathcal{B}_{nq}) = S_{nq}, \quad \forall\ 1 \le q \le k \le n-1 . \tag{2.16}$$

Also, note that S_{n1} is a U-statistic (based on $Z_1, \ldots, Z_n$, which are i.i.d.r.v.), so that $ES_{n1} = \Gamma$. This leads us to the following

Lemma 2.3. *Under* (2.7) *and* H_0: $\beta = 0$, *for every* $n(\ge 2)$, $\{S_{nk} - \Gamma, \mathcal{B}_{nk}; 1 \le k \le n-1\}$ *is a martingale.*

For a $p \times p$ matrix $A = ((a_{ij}))$, we let $||A|| = \max\{|a_{ij}|: 1 \le i, j \le p\}$. Then, by Lemma 2.3, we have for every $n(\ge 2)$, under H_0: $\beta = 0$,

$$\{||S_{nk} - \Gamma||, \mathcal{B}_{nk}; 1 \le k \le n-1\} \text{ a non-negative submartingale.} \tag{2.17}$$

Let $\{k_n\}$ be any sequence of positive numbers $(k_n \le n)$, such that

$$k_n \to \infty \text{ but } n^{-1}k_n^2 \to 0 \text{ as } n \to \infty . \tag{2.18}$$

Then, by (2.17), (2.18) and the generalized Kolmogorov-inequality for submartingales, under H_0: $\beta = 0$ and (2.7), (2.19) $$P\{\max_{n-k_n+1\le k\le n-1} ||\underset{\sim}{S}_{nk} - \underset{\sim}{\Gamma}|| > k_n\} \le k_n^{-1} E||\underset{\sim}{S}_{nn-1} - \underset{\sim}{\Gamma}||,$$

(2.20) $$P\{\max_{1\le k\le n-k_n} ||\underset{\sim}{S}_{nk} - \underset{\sim}{\Gamma}|| > \varepsilon\} \le \varepsilon^{-1} E||\underset{\sim}{S}_{nn-k_n} - \underset{\sim}{\Gamma}||, \ \forall \ \varepsilon > 0 .$$

Note that under H_0: $\underset{\sim}{\beta} = \underset{\sim}{0}$, $(Q_1, \ldots, Q_n)$ takes on each permutation of $(1, \ldots, n)$ with the common probability $(n!)^{-1}$. Also, by (2.15), for every k: $1 \le k \le n-1$, $\underset{\sim}{S}_{nn-k} = \underset{\sim}{U}(\underset{\sim}{Z}_{Q_{n-k}}, \ldots, Z_{Q_n})$ is a U-statistic [Hoeffding (1948)], so that

(2.21) $$\begin{aligned} E||\underset{\sim}{S}_{nn-k} - \underset{\sim}{\Gamma}|| &= E||\underset{\sim}{U}(\underset{\sim}{Z}_{Q_{n-k}}, \ldots, \underset{\sim}{Z}_{Q_n}) - \underset{\sim}{\Gamma}|| \\ &= E\{E[||\underset{\sim}{U}(\underset{\sim}{Z}_{Q_{n-k}}, \ldots, \underset{\sim}{Z}_{Q_n}) - \underset{\sim}{\Gamma}|| \,|\mathcal{B}_{n0}]\} \\ &= E\left\{\binom{k+1}{2}^{-1} \sum_{1\le i_1<\cdots<i_{k+1}\le n} ||\underset{\sim}{U}(\underset{\sim}{Z}_{i_1}, \ldots, \underset{\sim}{Z}_{i_{k+1}}) - \underset{\sim}{\Gamma}||\right\} \\ &= E||\underset{\sim}{U}(\underset{\sim}{Z}_1, \ldots, \underset{\sim}{Z}_{k+1}) - \underset{\sim}{\Gamma}||. \end{aligned}$$

Thus, $E||\underset{\sim}{S}_{nn-1} - \underset{\sim}{\Gamma}|| = E||\underset{\sim}{U}(\underset{\sim}{Z}_1, \underset{\sim}{Z}_2) - \underset{\sim}{\Gamma}|| < \infty$ by (2.7) and

(2.22) $$E||\underset{\sim}{S}_{nn-k_n} - \underset{\sim}{\Gamma}|| = E||\underset{\sim}{U}(\underset{\sim}{Z}_1, \ldots, \underset{\sim}{Z}_{k_n+1}) - \underset{\sim}{\Gamma}||, \ \forall \ n.$$

On the other hand, $\{\underset{\sim}{U}(\underset{\sim}{Z}_1, \ldots, \underset{\sim}{Z}_m), \ m \ge 2\}$ is a reverse martingale sequence, so that by the reverse sub-martingale convergence theorem, $||\underset{\sim}{U}(\underset{\sim}{Z}_1, \ldots, \underset{\sim}{Z}_m) - \underset{\sim}{\Gamma}||$ converges in the first mean to 0 as $m \to \infty$. Hence, (2.22) converges to 0 as n (or k_n) $\to \infty$. Consequently, the righthand side of (2.19) is $0(k_n^{-1})$ and of (2.20) converges to 0 as $n \to \infty$. Note that, for every $\varepsilon > 0$,

(2.23) $$\begin{aligned} &P\{\max_{1\le k\le n} ||\tfrac{1}{n} \underset{\sim}{J}_{nk} - \tfrac{k}{n}\Gamma|| > \varepsilon\} \\ &= P\{\max_{1\le k\le n} ||\tfrac{1}{n} \textstyle\sum_{i=1}^{k}\{(n-i)(n-i+1)^{-1}(\underset{\sim}{S}_{ni} - \underset{\sim}{\Gamma}) + (n-i+1)^{-1}\underset{\sim}{\Gamma}\}|| > \varepsilon\} \\ &\le P\{\max_{1\le k\le n-1} ||n^{-1}\textstyle\sum_{i=1}^{k}(\underset{\sim}{S}_{nk} - \underset{\sim}{\Gamma})|| > \varepsilon/2\} + P\{||\underset{\sim}{\Gamma}|| > \tfrac{1}{2}\varepsilon n\}, \end{aligned}$$

where the second term on the righthand side of (2.23) converges to 0 as $n \to \infty$. On the other hand, using the inequality that

$$\max_{1 \le k \le n-1} \| n^{-1} \sum_{i=1}^{k} (\underset{\sim}{S}_{ni} - \underset{\sim}{\Gamma}) \| \tag{2.24}$$

$$\le \left\{ \max_{1 \le k \le n-k_n} \| \underset{\sim}{S}_{nk} - \underset{\sim}{\Gamma} \| \right\} + \frac{k_n}{n} \left\{ \max_{n-k_n < k \le n-1} \| \underset{\sim}{S}_{nk} - \underset{\sim}{\Gamma} \| \right\}$$

we obtain on using (2.18), (2.19), and the convergence results following (2.22) that the first term on the righthand side of (2.23) converges to 0 as $n \to \infty$. This leads us to the following.

<u>*Lemma 2.4.*</u> *Under* (2.7) *and* H_0: $\underset{\sim}{\beta} = \underset{\sim}{0}$,

$$\max_{1 \le k \le n} \| n^{-1} (\underset{\sim}{J}_{nk} - k\underset{\sim}{\Gamma}) \| \xrightarrow{p} 0, \text{ as } n \to \infty. \tag{2.25}$$

By (2.8) and (2,25), we conclude that under H_0: $\underset{\sim}{\beta} = \underset{\sim}{0}$,

$$n^{-1} \underset{\sim}{J}_{nn} \xrightarrow{p} \underset{\sim}{\Gamma} \text{ and is p.d., in probability.} \tag{2.26}$$

Note that by (2.2), with probability 1,

$$\max_{1 \le k \le n} \| \underset{\sim}{U}_{nk} - \underset{\sim}{U}_{nk-1} \| = \max_{1 \le k \le n} \| \underset{\sim}{Z}_{Q_k} - (n-k+1)^{-1} \sum_{i=1}^{k} \underset{\sim}{Z}_{Q_i} \| \tag{2.27}$$

$$\le 2 \left\{ \max_{1 \le i \le n} \| \underset{\sim}{Z}_i \| \right\} = 2Z_n^*, \quad \text{say}$$

where, by (2.7), for every $\varepsilon > 0$,

$$P\{Z_n^* > \varepsilon\sqrt{n}\} \to 0, \text{ as } n \to \infty . \tag{2.28}$$

[The proof of (2.28) follows from well known results on the maximum of n i.i.d.r.v.'s and hence is omitted.] Then, for every $\varepsilon > 0$,

$$\| n^{-1} \sum_{i=1}^{n} E\{ (\underset{\sim}{U}_{ni} - \underset{\sim}{U}_{ni-1})(\underset{\sim}{U}_{ni} - \underset{\sim}{U}_{ni-1})' I(\| \underset{\sim}{U}_{ni} - \underset{\sim}{U}_{ni-1} \| > \varepsilon\sqrt{n}) | B_{ni-1} \} \|$$

$$= \| n^{-1} \sum_{i=1}^{n} \frac{1}{n-i+1} \sum_{j=i}^{n} (\underset{\sim}{Z}_{Q_j} - \overline{\underset{\sim}{Z}}{}_i^*)(\underset{\sim}{Z}_{Q_j} - \overline{\underset{\sim}{Z}}{}_i^*)' I(\| \underset{\sim}{Z}_{Q_j} - \overline{\underset{\sim}{Z}}{}_i^* \| > \varepsilon\sqrt{n}) \|$$

$$\le \left\{ n^{-1} \sum_{i=1}^{n} \text{Trace } [(n-i)(n-i+1)^{-1} \underset{\sim}{S}_{ni}] \right\} I(Z_n^* > \tfrac{1}{2}\varepsilon\sqrt{n})$$

$$= (\text{Trace } [n^{-1} \underset{\sim}{J}_{nn}]) I(Z_n^* > \tfrac{1}{2}\varepsilon\sqrt{n})$$

$$\xrightarrow{p} 0, \quad \text{as } n \to \infty \quad [\text{by (2.6) and (2.28)}].$$

Let us now return to the proof of Theorem 2.1. By virtue of (2.10), (2.11), (2.13), (2.25), and Lemma 2.2, for every $0 \le s \le t \le 1$,

(2.30)
$$\begin{aligned} &E\{\underset{\sim}{\xi}_n(t)[\underset{\sim}{\xi}_n(s)]'|\mathcal{B}_{n0}\} \\ &= \underset{\sim}{J}_{nn}^{-\frac{1}{2}}\left\{\sum_{i=1}^{k_n(s)} \frac{n-i}{n-i+1} \underset{\sim}{S}_{ni}\right\}\underset{\sim}{J}_{nn}^{-\frac{1}{2}} \\ &= \underset{\sim}{J}_{nn}^{-\frac{1}{2}} \underset{\sim}{J}_{nk_n(s)} \underset{\sim}{J}_{nn}^{-\frac{1}{2}} \xrightarrow{p} sI_p = (s \wedge t)I_p \\ &= E\{\underset{\sim}{\xi}(t)[\underset{\sim}{\xi}(s)]'\} . \end{aligned}$$

This, intuitively suggests the process $\underset{\sim}{\xi}$ in Theorem 2.1. To prove the desired weak convergence result, we make use of the martingale property in Lemma 2.2 and appeal to functional central limit theorems for (a triangular array of) martingales [viz., Scott (1973)]; we only need to extend these to the multivariate case, but that poses no problem [the usual Cramér-Wold type arguments filter through without demanding any extra condition]. Our Lemma 2.4 and (2.29) insure that both the conditions of Scott (1973) hold in this multivariate case and the proof of the theorem is therefore complete. By (2.10), (2.11), and (2.25), it follows that we may also take $k_n(t) = [nt]$, $t \in E$ and $\underset{\sim}{\xi}_n(t) = n^{-\frac{1}{2}}\underset{\sim}{\Gamma}^{-\frac{1}{2}}\underset{\sim}{U}_{[nt]}$, $t \in E$.

Note that by virtue of the Courant theorem (on the ratio of two quadratic forms) and by (2.25),

(2.31)
$$\begin{aligned} &\max_{1\le k\le n} |\underset{\sim}{U}'_{nk}\underset{\sim}{J}_{nn}^{-1}\underset{\sim}{U}_{nk}/\underset{\sim}{U}'_{nk}(n\underset{\sim}{\Gamma})^{-1}\underset{\sim}{U}_{nk} - 1| \\ &\le \max\{|ch_1(n\underset{\sim}{\Gamma}\underset{\sim}{J}_{nn}^{-1}) - 1|,\ |ch_p(n\underset{\sim}{\Gamma}\underset{\sim}{J}_{nn}^{-1}) - 1|\} \\ &\xrightarrow{p} 0, \text{ as } n \to \infty, \end{aligned}$$

(where ch_1 and ch_p stand for the largest and smallest characteristic roots). On the other hand, by Lemma 2.2, under H_0: $\underset{\sim}{\beta} = \underset{\sim}{0}$, (2,32) $\{n^{-1}\underset{\sim}{U}'_{nk}\underset{\sim}{\Gamma}^{-1}\underset{\sim}{U}_{nk},\ B_{nk};\quad 0 \le k \le n\}$ is a submartingale,

Also, note that

$$E\{n^{-1}\underset{\sim}{U}'_{nk}\underset{\sim}{\Gamma}^{-1}\underset{\sim}{U}_{nk}\} = E\{n^{-1}\text{ Trace }[\underset{\sim}{\Gamma}^{-1}\underset{\sim}{U}_{nk}\underset{\sim}{U}'_{nk}]\}$$

$$= E\{n^{-1}\text{ Trace }[\underset{\sim}{\Gamma}^{-1}\underset{\sim}{J}_{nk}]\} = n^{-1}\sum_{i=1}^{k}\frac{n-i}{n-i+1}E\,(\text{Trace }[\underset{\sim}{\Gamma}^{-1}\underset{\sim}{S}_{ni}])$$

$$= pn^{-1}\sum_{i=1}^{k}(n-i)/(n-i+1) \quad [\text{as } E\underset{\sim}{S}_{ni} = \underset{\sim}{\Gamma},\ \forall\ 1 \le i \le n-1]$$

$$\le pk/n,\quad \forall\ 1 \le k \le n.$$

Therefore, by (2.32), (2.33) and Theorem 2.1 of Birnbaum and Marshall (1961) for any $\{a_{n1} \ge a_{n2} \ge \cdots a_{nn}\}, 0 < \delta < 1,\ \ \varepsilon > 0$.

(2.34) $$P\{\max_{1\le k\le[n\delta]} a_{nk}|n^{-1}\underset{\sim}{U}'_{nk}\underset{\sim}{\Gamma}^{-1}\underset{\sim}{U}_{nk}| > \varepsilon\}$$

$$\le \sum_{k=1}^{[n\delta]}(a_{nk} - a_{nk+1})pk/n\varepsilon$$

$$\le pn^{-1}(a_{n1} + \cdots + a_{n[n\delta]})/\varepsilon\ .$$

Thus, if $q = \{q(t),\ t \in E\}$ be a non-negative, and non-decreasing and continuous function of t such that

(2.35) $$\int_0^1 [q(t)]^{-2}dt < \infty,$$

then $n^{-1}(q^2(1/n) + \cdots + q^2([n\delta]/n)) \le \int_0^\delta [q(t)]^{-2}dt,\ \forall\ 0 < \delta \le 1$. From (2.35), it follows that by choosing $\delta(>0)$ sufficiently small, the righthand side of (2.34) can be made small when $a_{ni} = q^{-2}(i/n),\ \ i \ge 1$. On the other hand, for $t > \delta(>0)$, $q^{-2}(t) \le q^{-2}(\delta) < \infty$, so that for $\delta \le t \le 1$, the weak convergence of $\underset{\sim}{\xi}_n$ in the uniform (or Skorokhod metric) insures the same under the "sup norm" metric

(2,36) $$\rho_q(x, y) = \sup\{|x(t) - y(t)|/q(t),\ \ t \in E\}.$$

Thus, from Theorem 2,1, (2,31), (2,34), and (2,35), we arrive at the following, *Theorem 2.6. The weak convergence in Theorem 2.1 holds in the sup norm metric* ρ_q *when* (2.35) *holds.*

Both of these theorems are useful for the proposed RST procedures.

3. WEAK CONVERGENCE IN THE DISCRETE TIME MODEL

As in Section 1, we conceive of m risk sets $\mathcal{R}_1 \supset \mathcal{R}_2 \supset \cdots \supset \mathcal{R}_m$ where $\mathcal{R}_k$ has r_k subjects whose survival times are $\geq t_k$, $k \geq 1$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = \infty$ and we denote by

(3.1) $$D_j = [t_j, t_{j+1}), \quad j = 0, 1, \ldots, m.$$

Let there be s_j failures in the interval D_j $(j \geq 0)$. Also, let

(3.2) $$\Omega_{ij} = I(Y_i \in D_j), \quad \Omega^*_{ij} = \sum_{k \geq j} \Omega_{ik}, \text{ for } 0 \leq j \leq m, \quad i = 1, \ldots, n.$$

Then, proceeding as in Section 6 of Cox (1972), we obtain the derivatives of the partial log-likelihood functions (evaluated at $\underset{\sim}{\beta} = \underset{\sim}{0}$) as

(3.3) $$\underset{\sim}{U}^*_{nk} = \sum_{j=1}^{k} \left\{ \sum_{i=1}^{n} (\Omega_{ij} - \Omega^*_{ij} s_j / r_j) \underset{\sim}{Z}_i \right\}, \quad k = 1, \ldots, m,$$

(3.4) $$\underset{\sim}{J}^*_{nk} = \sum_{j=1}^{k} \frac{s_j(r_j - s_j)}{r_j(r_j - 1)} \sum_{i=1}^{n} (\underset{\sim}{Z}_i \Omega^*_{ij} - \overline{\underset{\sim}{Z}}{}^*_j)(\underset{\sim}{Z}_i \Omega^*_{ij} - \overline{\underset{\sim}{Z}}{}^*_j)'$$

$$= \sum_{j=1}^{k} (s_j(r_j - s_j)/r_j) \underset{\sim}{S}^*_{nj}, \quad \text{say,}$$

where

(3.5) $$\overline{\underset{\sim}{Z}}{}^*_j = \left(\sum_{i=1}^{n} \underset{\sim}{Z}_i \Omega^*_{ij}\right) / r_j, \quad j = 1, \ldots, m.$$

[The close relationship between $\underset{\sim}{S}_{nk}$ in (2.4) and $\underset{\sim}{S}^*_{nk}$ in (3.4) need not be overemphasized.] In this case, we let $\mathcal{B}^*_{nk} = \mathcal{B}(\underset{\sim}{Z}_1, \ldots, \underset{\sim}{Z}_n, s_1, \ldots, s_m, r_1, \ldots, r_m, \Omega_{ij}, j \leq k, 1 \leq i \leq n)$ for $k = 1, \ldots, m$, while $\mathcal{B}^*_{n0} = \mathcal{B}(\underset{\sim}{Z}_1, \ldots, \underset{\sim}{Z}_n, s_1, \ldots, s_m, r_1, \ldots, r_m)$. Then, by arguments very similar to those in Lemma 2.2, we arrive at the following

Lemma 3.1. Under (2.7) *and* H_0: $\underset{\sim}{\beta} = \underset{\sim}{0}$, *for every* n, $\{\underset{\sim}{U}^*_{nk}, \mathcal{B}^*_{nk}, k \leq m\}$ *is a martingale.*

We may also note that by arguments very similar to those in

Section 2, under (2.7) and H_0: $\underset{\sim}{\beta} = \underset{\sim}{0}$,

$$\max_{1 \le k \le m} ||n^{-1}(\underset{\sim}{J}^*_{nk} - \textstyle\sum_{j=1}^k s_j(r_j - s_j)r_j^{-1}\underset{\sim}{\Gamma})|| \xrightarrow{p} 0, \text{ as } n \to \infty, \tag{3.6}$$

Now, to study the desired weak convergence results, we consider two different cases:

(I) m is fixed, so that as $n \to \infty$, s_j and r_j both increase for every $j(= 1, \ldots, m)$. This situation arises when we have a given number of ordered categories, while n is large.

(II) $m = m(n) \to \infty$ but $\max_{1 \le j \le m(n)} s_j/n \to 0$, as $n \to \infty$. This situation arises when the width of $D_j (j \ge 1)$ is small, so that there are a large number of cells and the possibility of ties is no longer negligible.

In case (I), by virtue of Lemma 3.1, $\underset{\sim}{U}^*_{nk} - \underset{\sim}{U}^*_{nk-1}$, $k \ge 1$ are all uncorrelated. Moreover, given $\mathcal{B}^*_{nk-1}$, the conditional distribution of

$$n^{-\frac{1}{2}}(\underset{\sim}{U}^*_{nk} - \underset{\sim}{U}^*_{nk-1}) = n^{-\frac{1}{2}}\textstyle\sum_{i=1}^n \underset{\sim}{Z}_i(\Omega_{ik} - s_k r_k^{-1}\Omega^*_{ik}) , \tag{3.7}$$

is generated by the $r_k!$ equally likely realizations of the Ω_{ik} (over the set $R_k = \{i: \Omega^*_{ik} = 1\}$) and by an appeal to the classical permutational central limit theorem [c.f. Hájek (1961)], we conclude that this conditional distribution is asymptotically (in probability) multinormal with null mean vector and dispersion matrix

$$s_k(r_k - s_k)r_k^{-1}\underset{\sim}{S}_{nk} = \underset{\sim}{\Sigma}_{(k)}, \text{ say, } (k = 1, \ldots, m), \tag{3.8}$$

Thus, using a chain of conditioning $(k = m - 1, \ldots, 1)$, if follows by some routine steps that given $\mathcal{B}^*_{n0}$, the joint conditional distribution of $\{n^{-\frac{1}{2}}(\underset{\sim}{U}^*_{nk} - \underset{\sim}{U}_{nk-1}), \ 1 \le k \le m\}$ is asymptotically (in probability) multinormal with null mean vector and dispersion matrix which is block-diagonal with the matrices $\underset{\sim}{\Sigma}_{(k)}$, $1 \le k \le m$ in (3.8). This, in turn, insures the asymptotic multinormality of $\{n^{-\frac{1}{2}}\underset{\sim}{U}_{nk},\ 1 \le k \le m\}$ when (2.7) and H_0: $\underset{\sim}{\beta} = \underset{\sim}{0}$ hold.

In case II, $m = m(n) \to \infty$ as $n \to \infty$. Also, it follows as in (2.27)-(2.29) that for every $\varepsilon > 0$

$$(3.9)\quad ||n^{-1} \sum_{i=1}^{m(n)} E\{(\underset{\sim}{U}^*_{nk} - \underset{\sim}{U}^*_{nk-1})(\underset{\sim}{U}^*_{nk} - \underset{\sim}{U}^*_{nk-1})' I(||\underset{\sim}{U}_{nk} - \underset{\sim}{U}_{nk-1}|| > \varepsilon\sqrt{n}) | \mathcal{B}_{nk-1}\} \xrightarrow{P} 0,$$

so that by Lemma 3.1, (3.6) and (3.9), the invariance principle for martingales, developed in Scott (1973), holds for the multivariate case treated here. The result is parallel to Theorem 2.1, where we replace $\underset{\sim}{U}_{nk}$, $\underset{\sim}{S}_{nk}$ and $\underset{\sim}{J}_{nk}$ by $\underset{\sim}{U}^*_{nk}$, $\underset{\sim}{S}^*_{nk}$ and $\underset{\sim}{J}^*_{nk}$, respectively.

4. INVARIANCE PRINCIPLES UNDER LOCAL ALTERNATIVES

With a view to study the asymptotic power properties of the RST (to be considered in Section 5), we proceed now to extend the results of Sections 2 and 3, when H_0: $\underset{\sim}{\beta} = \underset{\sim}{0}$ may not hold. We conceive of a sequence $\{K_n\}$ of local (Pitman-type) alternative hypotheses, where

$$(4.1)\qquad K_n:\ \underset{\sim}{\beta} = \beta_{(n)} = n^{-\frac{1}{2}}\underset{\sim}{\lambda},\ \text{ for some } \underset{\sim}{\lambda} \in R^p,$$

and desire to study the weak convergence results under $\{K_n\}$. This task is greatly simplified by employing the concept of contiguity of probability measures [as in Chapter 6 of Hájek and Šidak (1967)]. Let P_n be the joint distribution of $(Y_i, \underset{\sim}{Z}_i)$, $i = 1, \ldots, n$ under H_0: $\underset{\sim}{\beta} = \underset{\sim}{0}$ and P^*_n be the same under k_n. As a first step, we like to show that under (1.1), (2.7), and (4.1),

$$(4.2)\qquad \{P^*_n\} \text{ is contiguous to } \{P_n\}.$$

Let $H_0(x)$ be a non-decreasing and non-negative function for which $(d/dx)H_0(x) = h_0(x)$, defined in (1.1), and let $g(\underset{\sim}{z})$ be the joint density function of $\underset{\sim}{Z}$. Then, by (1.1), the joint density of $(Y, \underset{\sim}{Z})$ is given by

$$(4.3)\qquad f(y,\ \underset{\sim}{z};\ \underset{\sim}{\beta}) = g(\underset{\sim}{z})h_0(y)\exp\{\underset{\sim}{\beta}'\underset{\sim}{z} - e^{\underset{\sim}{\beta}'\underset{\sim}{z}}H_0(y)\}$$
$$= f(y,\ \underset{\sim}{z};\ \underset{\sim}{0})\cdot\exp\{\underset{\sim}{\beta}'\underset{\sim}{z} + H_0(y)(1 - e^{\underset{\sim}{\beta}'\underset{\sim}{z}})\},$$

Thus, for testing the simple null hypothesis H_0: $\underset{\sim}{\beta} = \underset{\sim}{0}$ vs. K_n: $\underset{\sim}{\beta} = n^{-\frac{1}{2}}\underset{\sim}{\lambda}$, the (log-) likelihood ratio statistic is

$$(4.4)\qquad \log L_n = \sum_{i=1}^n \log\{f(Y_i\underset{\sim}{Z}_i;\ n^{-\frac{1}{2}}\underset{\sim}{\lambda})/f(Y_i,\ \underset{\sim}{Z}_i;\ 0)$$
$$= n^{-\frac{1}{2}}\sum_{i=1}^n \{\underset{\sim}{\lambda}'\underset{\sim}{Z}_i + H_0(Y_i)n^{\frac{1}{2}}(1 - e^{n^{-\frac{1}{2}}\underset{\sim}{\lambda}\underset{\sim}{Z}_i})\}$$

where we note that

$$(4.5)\qquad n^{\frac{1}{2}}(1 - e^{n^{-\frac{1}{2}}\underset{\sim}{\lambda}'\underset{\sim}{Z}_i}) = -\underset{\sim}{\lambda}'\underset{\sim}{Z}_i - \frac{1}{2}n^{-\frac{1}{2}}(\underset{\sim}{\lambda}'\underset{\sim}{Z}_i)^2$$
$$+ \frac{1}{2}n^{-\frac{1}{2}}(\underset{\sim}{\lambda}'\underset{\sim}{Z}_i)^2[1 - \exp(\theta n^{-\frac{1}{2}}\underset{\sim}{\lambda}'\underset{\sim}{Z}_i)] \quad (0 < \theta < 1)$$

and by (2.28), the last term on the rhs of (4.5) is $o_p(n^{-\frac{1}{2}})(\underset{\sim}{\lambda}'\underset{\sim}{Z}_i)^2$, uniformly in i: $1 \le i \le n$. Thus, by (4.5) and (4.6),

$$(4.7)\qquad \log L_n = n^{-\frac{1}{2}}\sum_{i=1}^n(\underset{\sim}{\lambda}'\underset{\sim}{Z}_i)[1 - H_0(Y_i)] - \frac{1}{2n}\sum_{i=1}^n(\underset{\sim}{\lambda}'\underset{\sim}{Z}_i)^2 + o_p(1).$$

Now, by the Kintchine law of large numbers, under (2.7),

$$(4.8)\qquad n^{-1}\sum_{i=1}^n(\underset{\sim}{\lambda}'\underset{\sim}{Z}_i)^2 \xrightarrow{p} \underset{\sim}{\lambda}'\underset{\sim}{\Gamma}\underset{\sim}{\lambda}\ .$$

Also, under H_0: $\underset{\sim}{\beta} = \underset{\sim}{0}$, Y and $\underset{\sim}{Z}$ are independent with

$$(4.9)\qquad E[H_0(Y)]^r = \int[H_0(y)]^r\exp(-H_0(y))dH_0(y) = r!,\quad \forall\ r = 1, 2, \ldots\ .$$

Thus, $E(\underset{\sim}{\lambda}'\underset{\sim}{Z}_i)[1 - H_0(Y_i)] = (\underset{\sim}{\lambda}'\underset{\sim}{\mu})\ 0 = 0$ and $E[(\underset{\sim}{\lambda}'\underset{\sim}{Z}_i)(1 - H_0(Y_i))^2] = \underset{\sim}{\lambda}'\underset{\sim}{\Gamma}\underset{\sim}{\lambda}$. Hence, from (4.7), (4.8), (4.9), and the above, we conclude that under H_0: $\underset{\sim}{\beta} = \underset{\sim}{0}$, $\log L_n$ is asymptotically normal with mean $-\frac{1}{2}\underset{\sim}{\lambda}'\underset{\sim}{\Gamma}\underset{\sim}{\lambda}$ and variance $\underset{\sim}{\lambda}'\underset{\sim}{\Gamma}\underset{\sim}{\lambda}$. By virtue of LeCam's third lemma [c.f. Hájek and Šidak (1967, p. 208)], this ensures that $\{P_n^*\}$ is contiguous to $\{P_n\}$.

By using (4.3), we obtain by some simple steps that the conditional density of $\underset{\sim}{Z}$ given $Y = y$, under K_n, is given by

(4.10) $$g(\underset{\sim}{z}|y) = f(y, \underset{\sim}{z};\ \underset{\sim}{\beta} = n^{-\frac{1}{2}}\underset{\sim}{\lambda}) / \int f(y, \underset{\sim}{z};\ \underset{\sim}{\beta} = n^{-\frac{1}{2}}\underset{\sim}{\lambda})d\underset{\sim}{z}$$
$$= g(\underset{\sim}{z})[1 + n^{-\frac{1}{2}}(1 - H_0(y))(\underset{\sim}{z} - \underset{\sim}{\mu})'\underset{\sim}{\lambda} + o(n^{-\frac{1}{2}})] ,$$

so that under K_n,

(4.11) $$E(\underset{\sim}{Z}_i | Y_i = y) = \underset{\sim}{\mu} + n^{-\frac{1}{2}}(1 - H_0(y))\underset{\sim}{\Gamma}\underset{\sim}{\lambda} + o(n^{-\frac{1}{2}}).$$

Also, if $F_0(x)$ be the marginal density of Y (same in both the null and non-null cases), then $1 - F_0(y) = \exp(-H_0(y))$, so that $H_0(y)$ has the simple exponential distribution. Hence, for any y,

(4.12) $$[1 - H_0(y)] - \{1 - F_0(y)\}^{-1}\int_y^\infty [1 - H_0(\underset{\sim}{z})dF_0(\underset{\sim}{z})$$
$$= \{1 - F_0(y)\}^{-1}\int_y^\infty [1 - F_0(\underset{\sim}{z})dH_0(\underset{\sim}{z}) = \{1 - F_0(y)\}^{-1}\int_y^\infty e^{-H_0(y)} dH_0(\underset{\sim}{z}) = 1.$$

From (2.2), (4.11), and (4.12), we obtain by some routine steps that under $\{K_n\}$ in (4.1),

(4.13) $$n^{-1}k \to \alpha (0 < \alpha < 1) \Rightarrow$$
$$E\{n^{-\frac{1}{2}}\underset{\sim}{U}_{nk} | K_n\} \to \alpha\underset{\sim}{\Gamma}\underset{\sim}{\lambda}.$$

Thus, if we define $\xi_n(t)$ as in (2.11), then

(4.14) $$\lim_{n\to\infty} E\{\xi_n(t) | K_n\} \to t\underset{\sim}{\Gamma}^{\frac{1}{2}}\underset{\sim}{\lambda} ,\ \forall\ t \in E.$$

Now, under the null hypothesis H_0: $\underset{\sim}{\beta} = \underset{\sim}{0}$ and (1.1), $\underset{\sim}{Z}$ is independent of Y, so that by using Lemma 1 of Bhattacharya (1974), it follows [as the conditional df of $\underset{\sim}{Z}$ given y does not depend on y] that under H_0, $\underset{\sim}{Z}_{Q_1}, \ldots, \underset{\sim}{Z}_{Q_n}$ are i.i.d.r.v. with the density $q(\underset{\sim}{z})$. Hence, by an appeal to Theorem 2.1, the contiguity of $\{P_n^*\}$ to $\{P_n\}$ and a theorem of Neuhaus and Behnen (1975), we conclude that for any k: $k/n \to \alpha$: $0 < \alpha \le 1$, $n^{-\frac{1}{2}}\underset{\sim}{U}_{nk}$ is asymptotically normal (under $\{K_n\}$) with mean $\alpha\underset{\sim}{\Gamma}\underset{\sim}{\lambda}$ and dispersion matrix $\alpha\underset{\sim}{\Gamma}$. The same technique can be employed to show that for finitely many (m) points, $n^{-\frac{1}{2}}(\underset{\sim}{U}_{nk_1}, \ldots, \underset{\sim}{U}_{nk_m})$

(where $k_j/n \to \alpha_j$: $0 \le \alpha_j \le 1$) has asymptotically (under $\{K_n\}$) a pm variate normal distribution with mean vector $(\alpha_1, \ldots, \alpha_m) \otimes \underset{\sim}{\Gamma}\underset{\sim}{\lambda}$ and dispersion matrix $((\alpha_{j'} \wedge \alpha_j)) \otimes \underset{\sim}{\Gamma}$. This establishes the convergence of finite-dimensional distributions of $\{\xi_n\}$ under $\{K_n\}$ in (4.1). As in the proof of Theorem 2 of Sen (1976), we conclude that the contiguity of $\{P_n^*\}$ to $\{P_n\}$ and the tightness of $\{\xi_n\}$ under H_0 (insured by Theorem 2.1) imply the tightness of $\{\xi_n\}$ under $\{K_n\}$ as well. This leads us to the following

<u>*Theorem 4.1.*</u> *Under* (2.7) *and* $\{K_n\}$ *in* (4.1) [*relating to* (1.1)], $\underset{\sim}{\xi}_n$ *defined by* (2.11) *converges weakly to* $\underset{\sim}{\xi} + \underset{\sim}{\zeta}$, *where* $\underset{\sim}{\xi}$ *is defined in Theorem* 2.1 *and* $\underset{\sim}{\zeta} = \{\underset{\sim}{\zeta}(t) = t\underset{\sim}{\Gamma}^{\frac{1}{2}}\underset{\sim}{\lambda},\ t \in E\}$.

We conclude this section with the remark that similar weak convergence results hold for the discrete time case treated in Section 3.

5. RST PROCEDURES RELATED TO THE COX REGRESSION MODEL

We start with the remark that if one works with $\underset{\sim}{U}_{nn}$ in (2.2) and constructs

$$L_{nn} = \underset{\sim}{U}'_{nn}\underset{\sim}{J}^{-}_{nn}\underset{\sim}{U}_{nn}, \tag{5.1}$$

where $\underset{\sim}{J}_{nn}$ is defined by (2.3), then, under H_0: $\underset{\sim}{\beta} = \underset{\sim}{0}$ and (2.7)-(2.8), L_{nn} has asymptotically chi-square distribution with p degrees of freedom (DF) (by Theorem 2.1) and by Theorem 4.1, under $\{K_n\}$ in (4.1), L_{nn} has asymptotically a non-central chi-square distribution with p DF and the non-centrality parameter

$$\Delta_L = \underset{\sim}{\lambda}'\underset{\sim}{\Gamma}\underset{\sim}{\lambda}\ . \tag{5.2}$$

Now, suppose we work with L^*_{nm}, defined by (1.7). Since $\underset{\sim}{Q}^*$ is a sub-vector of $\underset{\sim}{Q}$, the martingale property in Lemmas 2.2 and 2.3 remain true for these subsequences $\{\underset{\sim}{U}^*_{nk}\}$ and $\{\underset{\sim}{J}^*_{nk}\}$ and also

Lemma 2.4 extends to $\{\underset{\sim}{J}^*_{nk}\}$. Thus, the weak convergence of $\{L^*_{nm}\}$ follows on the same line as in the proof of Theorem 2.1, provided $m(= m(n)) \to \infty$ as $n \to \infty$. Further, the contiguity being assured by (4.2), Theorem 4.2 also holds for this subsequence. As a result, we claim that under (2.7)-(2.8) and H_0: $\underset{\sim}{\beta} = \underset{\sim}{0}$, L^*_{nm} has asymptotically chi-square distribution with p DF , provided $m(n) \to \infty$ as $n \to \infty$. Also, if, in addition,

(5.3) $$\lim_{n\to\infty} n^{-1} m(n) = \tau: \quad 0 \le \tau \le 1 ,$$

then under $\{K_n\}$ in (4.1); L^*_{nm} has asymptotically a non-central chi-square distribution with p DF and non-centrality parameter

(5.4) $$\Delta_{L^*} = \tau^2(\underset{\sim}{\lambda}'\underset{\sim}{\Gamma}\underset{\sim}{\lambda}) = \tau^2 \cdot \Delta_L .$$

Thus, the asymptotic relative efficiency (A.R.E.) of L^*_{nm} with respect to L_{nn} is equal to

(5.5) $$e(L^*, L) = \Delta_{L^*}/\Delta_L = \tau^2 .$$

In particular, if τ is close to 1, then (5.5) is also so and in this sense, if $m(n)/n \to 1$ as $n \to \infty$, then the Cox (1972) test for no regression based on L^*_{nm} in (1.7) is asymptotically fully efficient — this result has been obtained for the discrete time model by Efron (1977) from a somewhat different consideration. However, we may note that for $\tau < 1$, the A.R.E. in (5.5) is less than 1 indicating that too much of censoring takes away some information.

A common feature of the two tests based on L_{nn} and L^*_{nm} is that they are based on the partial likelihood L_{nn} and L^*_{nm}, respectively, and, thereby, demand that the experimentation be continued until all or m of the failures occur. In the proposed RST procedure, we like to update the picture as successive failures occur and consider the following time-sequential procedures. These procedures are based on the partial sequence $\{L_{nk}, \; k \le n\}$ of partial likelihood ratio statistics. Note that by (2.10)-(2.11),

(5.6) $$L_{nk_n}(t) = [\underset{\sim}{\xi}_n(t)]'[\underset{\sim}{J}_{nn}^{\frac{1}{2}} \underset{\sim}{J}_{nk_n}^{-1}(t) \underset{\sim}{J}_{nn}^{\frac{1}{2}}][\underset{\sim}{\xi}(t)], \; \forall \; t \in E ,$$

where by Lemma 2.4, under (2.7), (2.8), and H_0: $\underset{\sim}{\beta} = \underset{\sim}{0}$,

(5.7) $$\underset{\sim}{J}_{nn}^{\frac{1}{2}} \underset{\sim}{J}_{nk_n}(t) \underset{\sim}{J}_{nn}^{\frac{1}{2}} \overset{p}{\sim} t^{-1} \underset{\sim}{I}_p, \quad \forall \; t \in E.$$

Thus, $L_{nk_n}(t)$ behaves like $t^{-1}[\underset{\sim}{\xi}(t)]'[\xi_n(t)]$ as $n \to \infty$. On the other hand, for $q^2(t) = t^{-1}$, (2.35) does not hold and $t^{-1}[\underset{\sim}{\xi}(t)]'[\underset{\sim}{\xi}(t)]$ does not behave very smoothly when $t \to 0$; in fact, by the law of iterated logarithm for Brownian notions,

(5.8) $$\limsup_{t \to 0} t^{-1}[\underset{\sim}{\xi}(t)]'[\underset{\sim}{\xi}(t)] = \infty, \quad \text{with probability 1.}$$

Hence, to make statistically informative use of the partial sequence $\{L_{nk}, \; k \le n\}$, either we use some appropriate weight function or we start monitoring of the experimentation only when a given number of failures have already taken place. This leads us to the following two types of RST procedures.

(i) *Type A RST procedures.* We choose some ε: $0 < \varepsilon < 1$ and let $k_n^0 = [n\varepsilon]$. Define then the *stopping number*

(5.9) $$N_\varepsilon = \begin{cases} \min\{k(k_n^0 \le k \le n): \; L_{nk} > c_n^*\}, \\ n, \; \text{ if } \; L_{nk} \le c_n^*, \; \forall \; k_n^0 \le k \le n. \end{cases}$$

where c_n^* is a suitable constant, to be defined later on. Then, operationally, starting at the k_n^0-th failure $Y_{nk_n^0}$, we monitor experimentation and at each failure $Y_{nk}(k \ge k_n^0)$, we compute L_{nk}: if L_{nk} is > c_n^*, we stop experimentation at that time along with the rejection of H_0: $\underset{\sim}{\beta} = \underset{\sim}{0}$ and if $L_{nk} \le c_n^*$, we continue experimentation until the next failure occurs and reinspect the process at that failure. If, we do not stop on or before the n-th failure, experimentation terminates along with the acceptance of H_0. In (5.9), we may modify N_ε by allowing k to range over $k_n^0 \le k \le m$ for some predetermined $m(\le n)$ and, in that case, $N_\varepsilon \le m$ with probability 1. In theory, it does not make any difference, but, in most practical problems, we may be inclined to choose an m well below n, depending on the time and cost of experimentation (at our disposal). We need to choose c_n^* in such a way that

$$(5.10) \qquad P\{L_{nk} > c_n^* \text{ for some } k:\ k_n^0 \le k \le n \mid H_0:\ \underset{\sim}{\beta} = \underset{\sim}{0}\} = \alpha,$$

where $\alpha (0 < \alpha < 1)$ is the desired level of significance of the RST.

(ii) *Type B RST procedure.* Here, we do not use a constant boundary and choose a sequence $\{c_{nk}^*,\ 1 \le k \le n\}$ of positive numbers and denote the *stopping number* by

$$(5.11) \qquad N^* = \begin{cases} \min k(1 \le k \le n):\ L_{nk} > c_{nk}^*, \\ n, \quad \text{if } L_{nk} \le c_{nk}^*,\ \forall\ k \le n\ . \end{cases}$$

Then, we proceed as in the Type A procedure, but using the variable boundary instead of the constant c_n^*. Here, we need to have

$$(5.12) \qquad P\{L_{nk} > c_{nk}^* \text{ for some } k:\ 1 \le k \le n \mid H_0:\ \underset{\sim}{\beta}\ \ \underset{\sim}{0}\} = \alpha.$$

[We do not need to wait until the occurrence of $Y_{nk_n^0}$ and the process may terminate even prior to the k_n^0-th failure.]

To determine c_n^* in (5.9)-(5.10), we now appeal to Theorem 2.1 and conclude that for every $\varepsilon > 0$ and $\lambda > 0$, under H_0: $\underset{\sim}{\beta} = \underset{\sim}{0}$,

$$(5.13) \qquad \begin{aligned} &P\{L_{nk} > \lambda, \text{ for some } k;\ k_n^0 \le k \le n \mid H_0\} \\ &\to P\{t^{-1}[\underset{\sim}{\xi}(t)]'[\underset{\sim}{\xi}(t)] > \lambda \text{ for some } \varepsilon \le t \le 1\} \\ &= P(\lambda;\ \varepsilon), \text{ say.} \end{aligned}$$

TABLE 1

Table for the simulated values of $\lambda_{\alpha\varepsilon}$ for typical (α, ε)

	p = 1			p = 2			p = 3			p = 4		
	α=.01	.05	.10	.01	.05	.10	.01	.05	.10	.01	.05	.10
.005	12.11	9.24	7.78	15.42	12.46	10.68	18.19	15.14	13.10	20.97	16.83	14.99
.010	11.97	9.06	7.62	15.24	12.23	10.52	18.10	14.52	12.85	20.44	16.76	14.86
.050	11.49	8.35	6.86	14.85	11.90	9.88	16.99	14.19	12.37	19.72	16.18	13.94
.100	11.09	7.79	6.35	14.69	11.48	9.39	16.92	13.98	11.87	19.58	15.72	13.46
.150	10.63	7.51	6.10	13.94	10.99	8.97	16.75	13.60	11.47	18.62	15.24	13.18
.200	10.48	7.34	5.90	13.92	10.55	8.87	16.50	13.02	11.21	18.10	14.96	12.96

Thus, if $\lambda_{\alpha\varepsilon}$ be the solution of $P(\lambda; \varepsilon) = \alpha$, then we may choose $\{c_n^*\}$ any sequence of positive numbers for which $c_n^* \to \lambda_{\alpha\varepsilon}$ as $n \to \infty$. Analytical solutions for $\lambda_{\alpha\varepsilon}$ are not yet available. However, certain simulation studies are made by Majumdar and Sen (1977) and Majundar (1978). We report above, in Table 1, some of these simulated values for $p \le 4$.

For (5.11)-(5.12), we take $c_{nk}^* = (n/k)c_n^*$, $1 \le k \le n$, so that by (5.6), (5.7), (5.12), and Theorem 2.1, we have for $c_n^* \to \lambda$,

(5.14) $$P\{L_{nk} > c_{nk}^*, \text{ for some } k \le n | H_0\}$$
$$\to P\{[\underset{\sim}{\xi}(t)]'[\underset{\sim}{\xi}(t)] > \lambda, \text{ for some } t \in E\} = P_\lambda^*, \text{ say }.$$

Thus, if λ_α^* be the solution for $P_\lambda^* = \alpha$, then we have choose $c_n^* = \lambda_\alpha^*$. [For this choice of c_{nk}^*, we really work with the partial sequence $\left\{\frac{k}{n} L_{nk}, 1 \le k \le n\right\}$.] Again, analytical solutions for λ_α^* are not known for $p > 1$ (for $p = 1$, this is, however, known), and, we quote the following simulated values obtained by Majumdar and Sen (1978).

By virtue of Theorem 4.1, we are also able to express the asymptotic power of these RST procedures interms of the boundary crossing probabilities of $\{\underset{\sim}{\xi}(t), t \in E\}$. From (5.9), (5.10), and Theorem 4.1,

TABLE 2

Table for the simulated values of λ_α^* for some typical (α, p)

$p \backslash \alpha$	λ_α^* .01	.05	.10
1[a]	7.90	5.02	3.84
2	10.37	7.29	5.52
3	13.76	9.30	7.73
4	15.13	10.96	9.24

a) Exact values.

we have that under $\{K_n\}$ in (4.1), the asymptotic power of the Type A RST procedure is equal to

(5.15) $$P\{t^{-1}(\underset{\sim}{\xi}(t) + t\underset{\sim}{\Gamma}^{\frac{1}{2}}\underset{\sim}{\lambda})'(\underset{\sim}{\xi}(t) + t\underset{\sim}{\Gamma}^{\frac{1}{2}}\underset{\sim}{\lambda}) > \lambda_{\alpha\varepsilon} \text{ for some } \varepsilon \le t \le 1\} .$$

Similarly, for the Type B RST procedure, the asymptotic power under $\{K_n\}$ in (4.1) (when we choose $c^*_{nk} = (n/k)c^*_n$, $k \le n$) is given by

(5.16) $$P\{(\underset{\sim}{\xi}(t) + t\underset{\sim}{\Gamma}^{\frac{1}{2}}\underset{\sim}{\lambda})'(\underset{\sim}{\xi}(t) + t\underset{\sim}{\Gamma}^{\frac{1}{2}}\underset{\sim}{\lambda}) > \lambda^*_\alpha \text{ for some } t \in E\}.$$

A comparison of (5.15) and (5.16) demands extensive numerical integration and simulation studies.

So far, we have considered RST procedures based on $\{L_{nk}, k \le n\}$. It is also possible to use $\{L^*_{nk}, k \le m\}$ when $m \to \infty$ with $n \to \infty$. The stopping variables are similar to those in (5.9) and (5.11) and when $m/n \to 1$, (5.13) and (5.14) also stand valid. However, if $m/n \to \tau$, for some $\tau < 1$, then in (5.13) or (5.14), we have to replace the range of $t(\varepsilon \le t \le 1$ or $t \in E)$ by $\varepsilon \le t \le \tau$ (or $t \in E[0, \tau]$) and similar changes are needed in (5.15)-(5.16). For the discrete time parameter case, treated in Section 3, when $m = m(n) \to \infty$ with $n \to \infty$ (i.e., case II), the same results apply. In case I (i.e., m fixed), instead of the process $\xi_n(t)$, $t \in E$, we have only finitely many $\xi_n(t_j)$, $t_j \in E$, $j = 1, \ldots, m$. As such, the use of $\lambda_{\alpha\varepsilon}$ or λ^*_α in (5.13) or (5.14) will result in a more conservative test i.e., the actual level of significance may be much below the specified α. However, as in Majumdar and Sen (1977), this will continue to be a nice test. Finally, throughout the paper, in the usual fashion of an analysis of covariance model, we have taken Z_i as stochastic. If, on the other hand, the Z_i are non-stochastic, the problem becomes a little more simple and a direct multivariate extension of Sen (1975) yields the desired results.

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