INVARIANCE PRINCIPLES FOR SOME STOCHASTIC PROCESSES RELATING TO M-ESTIMATORS AND THEIR ROLE IN SEQUENTIAL STATISTICAL INFERENCE

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ABSTRACT

Weak convergence of certain two-dimensional time-parameter stochastic processes related to M-estimators is studied here. These results are then incorporated in the study of the asymptotic properties of bounded length (sequential) confidence intervals as well as sequential tests (for regression) based on M-estimators.

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1. INTRODUCTION

Let $X_1, \ldots, X_n$ be independent random variables (rv) with continuous distribution functions (df) $F_1, \ldots, F_n$, respectively, all defined on the real line R. It is assumed that $F_i(x) = F(x - \Delta^o c_i)$, $x \in R$, $i \geq 1$, where F is unknown, $c_1, \ldots, c_n$ are given constants and Δ^o is an unknown parameter. An *M-estimator* $\tilde{\Delta}_n$ of Δ^o [c.f. Huber (1973)] is a solution (for Δ) of the equation

$$\sum_{i=1}^{n} c_i \psi(X_i - \Delta c_i) = 0 \ . \tag{1.1}$$

where ψ is some specified *score function*. Various asymptotic properties of $\tilde{\Delta}_n$ have been studied by various workers; we may refer to Huber (1973) and Jurečková (1977) where other references are also cited. In this context, one encounters a stochastic process $M_n = \{M_n(\Delta) = \sum_{i=1}^{n} c_i [\psi(X_i - \Delta d_i) - \psi(X_i)], \ \Delta \in R\}$ (where $d_1, \ldots, d_n$ are suitable constants) and the asymptotic linearity of $M_n(\Delta)$ in Δ plays a vital role in the asymptotic theory of $\tilde{\Delta}_n$ [see Jurečková (1977)]. In the context of sequential confidence intervals for Δ^o as well as sequential tests for Δ^o based on M-estimators, we need to strengthen the asymptotic linearity results to random sample sizes, and, in this context, some invariance principles for certain two-dimensional time-parameter stochastic processes related to $\{M_n\}$ are found to be very useful. With these in mind, in Section 2, we formulate these invariance principles for $\{M_n\}$ and present their proofs in Section 3. Section 4 deals with the problem of bounded-length (sequential) confidence intervals for Δ^o based on M-estimators and some asymptotic properties of these procedures are studied with the aid of the invariance principles in Section 2. The last section is devoted to the study of the asymptotic properties of some sequential tests for Δ^o based on M-estimators.

2. SOME INVARIANCE PRINCIPLES RELATED TO M-ESTIMATORS

For technical reasons, in this section, we replace the X_i by a triangular array $\{(X_{n1}, \ldots, X_{nn});\ n \geq 1\}$ of rv's and assume that the following conditions hold.

(A) For every $n(\geq 1)$, X_{ni}, $i \geq 0$, are independent and identically distributed (i.i.d.) rv with a continuous df F, defined on R.

(B) ψ: $R \to R$ is nonconstant and absolutely continuous on any bounded interval in R. Let $\psi^{(1)}$ be the derivative of ψ and assume that

(i) $$\gamma_1(\psi, F) = \int_{-\infty}^{\infty} \psi^{(1)}(x)dF(x) \neq 0, \tag{2.1}$$

(ii) $$\int_{-\infty}^{\infty} [\psi^{(1)}(x)]^2 dF(x) < \infty, \tag{2.2}$$

(iii) $$0 < \sigma_1^2 = \int_{-\infty}^{\infty} [\psi^{(1)}(x)]^2 dF(x) - \gamma_1^2(\psi, F) < \infty, \tag{2.3}$$

(iv) $$\lim_{t \to 0} \int_{-\infty}^{\infty} \{\psi^{(1)}(x+t) - \psi^{(1)}(x)\}^2 dF(x) = 0 . \tag{2.4}$$

(C) Let $\{c_{ni},\ t \geq 0;\ n \geq 0\}$ and $\{d_{ni},\ i \geq 0;\ n \geq 0\}$ be two triangular arrays of real constants, such that if $\{k_n\}$ be any sequence of positive integers for which $k_n \uparrow \infty$ as $n \to \infty$, then

$$\lim_{n\to\infty}\left\{\max_{i \leq k_n} c_{ni}^2 \Big/ \textstyle\sum_{j \leq k_n} c_{nj}^2\right\} = 0, \tag{2.5}$$

$$\lim_{n\to\infty}\left\{\max_{i \leq k_n} d_{ni}^2 \Big/ \textstyle\sum_{j \leq k_n} d_{nj}^2\right\} = 0, \tag{2.6}$$

$$\lim_{n\to\infty}\left\{\max_{i \leq k_n} c_{ni}^2 d_{ni}^2 \Big/ \textstyle\sum_{j \leq k_n} c_{nj}^2 d_{nj}^2\right\} = 0 . \tag{2.7}$$

$$D_n^2 = \textstyle\sum_{j \leq k_n} d_{nj}^2 \leq D^{*2} < \infty , \ \forall\ n \geq 1. \tag{2.8}$$

Conventionally, we let $c_{n0} = d_{n0} = 0$, $\forall\ n \ge 0$ and let

$$A_{nk}^2 = \sum_{j \le k} c_{nj}^2 d_{nj}^2,\ k \ge 0,\ n \ge 0;\quad A_n^2 = A_{nn}^2,\quad n \ge 0\ , \tag{2.9}$$

$$M_{nk}(\Delta) = \sum_{i=0}^{k} c_{ni}\{\psi(X_{ni} - \Delta d_{ni}) - \psi(X_{ni})\},\ k \ge 0,\quad \Delta \in R. \tag{2.10}$$

Let $K^* = \{t, \Delta\}$: $0 \le t \le k_j^*$, $0 \le |\Delta| \le k_2^*\} (= [0, k_1^*] \times [-k_2^*, k_2^*])$ be any compact set in R^2 (where $0 < k_1^*, k_2^* < \infty$). We define $W_n = \{W_n(t, \Delta),\ (t, \Delta) \in K\}$ by letting

$$W_n(t, \Delta) = \{M_{nn(t)}(\Delta) - EM_{nn(t)}(\Delta)\}/(\sigma_1 A_n),\quad (t, \Delta) \in K^*\ ; \tag{2.11}$$

$$n(t) = \max\{k: A_{nk}^2 \le tA_n^2\}\ ,\quad t \in K_1^* = [0, k_1^*];\quad k_n = n(k_1^*)\ . \tag{2.12}$$

Then W_n belongs to $D[K^*]$ for every $n \ge 1$. Finally, let $W = \{W(t, \Delta),\ (t, \Delta) \in K^*\}$ be defined by $W(t, \Delta) = \Delta\xi(t)$, $(t, \Delta) \in K^*$, where $\xi = \{\xi(t),\ t \in K_1^*\}$ is a standard Wiener process. Then, we have the following

Theorem 2.1. Under (A), (B) *and* (C), $\{W_n\}$ *converges weakly to* W.

We are also interested in replacing in (2.11), $EM_{nn(t)}(s)$ by a more explicit function of (t, s). For this, we assume that the following holds.

(B') In addition to (2.1) and (2.2), $\psi^{(1)}$ is absolutely continuous (a.e.), denote its derivative by $\psi^{(2)}$ and assume that

$$\lim_{t \to 0} \int_{-\infty}^{\infty} |\psi^{(2)}(x + t) - \psi^{(2)}(x)|\, dF(x) = 0\ , \tag{2.13}$$

(D) F admits of an absolutely continuous probability density function (pdf) f with a finite Fisher information $I(f) = \int (f'/f)^2 dF$ where $f'(x) = (d/dx)f(x)$.

Let then

$$\gamma_2(\psi, F) = \int_{-\infty}^{\infty} \psi^{(2)}(x)\, dF(x) \left[= \int_{\infty} \psi^{(1)}(x)\{-f'(x)/f(x)\}\, dF(x) \right], \tag{2.14}$$

so that by (2.2), (D) and the Schwarz inequality,

$\gamma_2^2(\psi, F) \le I(f)\int[\psi^{(1)}]^2 dF < \infty$. Also, let

$$h_n(t) = \sum_{i=0}^{n(t)} c_{ni} d_{ni}^2 / A_n, \quad t \in K_1^* . \tag{2.15}$$

Note that by (2.8), (2.12) and the Schwarz inequality,

$$h_n^2(t) \le A_n^{-2}(\sum_{i=0}^{n(t)} c_{ni}^2 d_{ni}^2)(\sum_{i=0}^{n(t)} d_{ni}^2) \le tD^{*2} < \infty, \ \forall\ t \in k_1^* . \tag{2.16}$$

Let us now define $\tilde{W}_n^0 = \{\tilde{W}_n^0(t, \Delta) = \{M_{nn(t)}(\Delta) + \Delta\gamma(\psi, F)\sum_{i=0}^{n(t)} c_{ni} d_{ni}\}/(\sigma, A_n)$, $(t, \Delta) \in K^*\}$ and $\tilde{W}_n = \{\tilde{W}_n(t,\Delta) = \tilde{W}_n^0(t, \Delta) - \frac{1}{2}\Delta^2\gamma_2(\psi, F)h_n(t)/\sigma_1$, $(t, \Delta) \in K^*\}$. Then, we have the following.

Theorem 2.2. Under (A), (B'), (C) *and* (D), $\{\tilde{W}_n\}$ *weakly converges to* W *and if, in addition,* $\lim_{n\to\infty} h_n(t) = 0$, $\forall\ t \in k_1^*$, *then,* $\{\tilde{W}_n^0\}$ *converges weakly to* W.

In some applications, we may encounter a process $W_n^* = \{W_n^*(t, \Delta)$, $(t, \Delta) \in K^*\}$, where, defining the $M_{nn}(\Delta)$ as in (2.10) and $\{n(t)\}$ as in (2.12), for every $(t, \Delta) \in K^*$,

$$W_n^*(t, \Delta) = (\sigma_1 A_n)^{-1}\{M_{n(t)n(t)}(\Delta) - EM_{n(t)n(t)}(\Delta)\}. \tag{2.17}$$

The weak convergence of $\{W_n^*\}$ to some appropriate Gaussian function depends on the interrelationship between the elements of different rows of the triangular arrays $\{c_{ni}\}$ and $\{d_{ni}\}$. Suppose that

$$\lim_{n\to\infty} A_n^{-2}\sum_{i=0}^{n(t\wedge s)} c_{n(t)i}d_{n(s)i}c_{n(s)i}d_{n(s)i} = g(s, t) \quad s, t \in K^* \tag{2.18}$$

exists for every (s, t) and is a continuous function of s, t, $0 \le s, t \le K_1^*$. Further, we assume that there exists a positive number M, such that

$$\left[\sum_{i=0}^{q}(c_{ki}d_{ki} - c_{qi}d_{qi}) + \sum_{i=q+1}^{k} c_{ki}^2 d_{ki}^2\right] \Big/ (A_{nk}^2 - A_{nq}^2) \le M, \tag{2.19}$$

uniformly in $0 \le q < k \le k_n$, $n \ge 1$. It is also possible to replace (2.17) - (2.18) by alternative conditions. Then we have the following.

Theorem 2.3. Under (A), (B), (C) *and* (2.18) - (2.19), $\{W_n^*\}$ *converges weakly to a Gaussian function* W^* *where* $W^* = W^*(t, \Delta)$, $t, \Delta) \in K^*$ *has mean* 0 *and*

$$EW^*(s, \Delta)W^*(t, \Delta') = \Delta\Delta' g(s, t), \forall (s, \Delta), (t, \Delta') \in K^* \quad . \tag{2.10}$$

Under (A), (B'), (C), (D) *and* (2.18) - (2.19), in (2.17), *we may also replace* $EM_{n(t)n(t)}(\Delta)$ *by* $\Delta\gamma_1(\psi, F)\sum_{i=0}^{n(t)} c_{n(t)i}d_{n(t)i}$ $- \frac{1}{2}\Delta^2\gamma_2(\psi, F)\sum_{i=0}^{n(t)} c_{n(t)i}d_{n(t)i}^2$, *for* $(t, \Delta) \in K^*$. *Finally, if* $g(s, t) = s \wedge t$, $\forall (s, t) \in K_1^*$, *then* $W^* \equiv W$.

In Sections 4 and 5, we shall encounter a single sequence $\{c_i, i \geq 0\}$ and will have $c_{ki} = c_i$, $d_{ki} = c_{ki}/C_k$ where $C_k^2 = \sum_{i=1}^k c_{ki}^2$, for $k \geq 1$. In such a case, (2.18) and (2.19) are not difficult to verify. In fact, here, $g(s, t) = s \wedge t$, $\forall s, t \in K_1^*$.

3. PROOFS OF THEOREMS 2.1, 2.2 AND 2.3

First, consider Theorem 2.1. Let us define $W_n^0 = \{W_n^0(t, \Delta), (t, \Delta) \in K^*\}$ by letting

$$W_n^0(t, \Delta) = \Delta(\sum_{i=0}^{n(t)} c_{ni}d_{ni}\{\psi^{(1)}(X_{ni}) - \gamma_1(\psi, F)\})/\sigma_1 A_n \ , \ (t, \Delta) \in K^*, \tag{3.1}$$

where $n(t)$ is defined by (2.12). We like to approximate W_n by W_n^0. Note that for every $(t, \Delta) \in K^*$,

$$E\{W_n^0(t, \Delta) - W_n(t, \Delta)\}^2 = [\sum_{i=0}^{n(t)} c_{ni}^2 \text{Var}\{\psi(X_{ni} - \Delta d_{ni}) - \psi(X_{ni}) - d_{ni}\psi^{(1)}(X_{ni})\}]/\sigma_1^2 A_n^2$$

$$\leq \sum_{i=0}^{n(t)} c_{ni}^2 E\{\psi(X_{ni} - \Delta d_{ni}) - \psi(X_{ni}) - \Delta d_{ni}\psi^{(1)}(X_{ni})\}^2/\sigma_1^2 A_n^2 \tag{3.2}$$

$$= [\textstyle\sum_{n(t)}^* c_{ni}^2 d_{ni}^2 \Delta^2 E\{[\psi(X_{ni} - \Delta d_{ni}) - \psi(X_{ni})]/\Delta d_{ni} - \psi^{(1)}(X_{ni})\}^2/\sigma_1^2 A_n^2] \ ,$$

where $\sum_{n(t)}^*$ extends over all $\{i; i \leq n(t)$ and $d_{ni} \neq 0\}$. Note that by (2.2) and (2.4), for every $h > 0$, $\int_{-\infty}^{\infty} h^{-2}\{\psi(x + h) - \psi(x)\}^2 dF(x) =$

$$\int_{-\infty}^{\infty}[h^{-1}\int_0^t \psi^{(1)}(x+t)dt]^2 dF(x) \le \int_{-\infty}^{\infty} h^{-1}\left(\int_0^h [\psi^{(1)}(x+t)]^2 dt\right) dF(x) \to$$

$\int_{-\infty}^{\infty}[\psi^{(1)}(x)]^2 dF(x) (< \infty)$ as $h \to 0$. A similar case holds for $h < 0$.

Hence, by Fatou's lemma,

$$\lim_{h\to 0}\int_{-\infty}^{\infty} h^{-2}[\psi(x+h) - \psi(x)]^2 dF(x) = \int_{-\infty}^{\infty}[\psi^{(1)}(x)]^2 dF(x) . \quad (3.3)$$

Similarily,

$$\lim_{h\to 0}\int_{-\infty}^{\infty} h^{-1}[\psi(x+h) - \psi(x)]\psi^{(1)}(x) dF(x) = \int_{-\infty}^{\infty}[\psi^{(1)}(x)]^2 dF(x) . \quad (3.4)$$

From (2.6), (2.8), (3.2), (3.3) and (3.4), we obtain that

$$\lim_{h\to\infty}[E\{W_n(t, \Delta) - W_n^0(t, \Delta)\}^2/t\Delta^2] = 0, \ \forall\ (t, \Delta) \in K^*. \quad (3.5)$$

By (3.5) and the Chebychev in equality, for every (fixed) $m (\ge 1)$ and $(t_j, \Delta_j) \in K^*$, $1 \le j \le m$, as $n \to \infty$

$$\max_{1\le j\le m} |W_n(t_j, \Delta_j) - W_n^0(t_j, \Delta_j)| \xrightarrow{P} 0 . \quad (3.6)$$

Also, by (3.1), for any $\underset{\sim}{b} = (\underset{\sim}{b}_1, \ldots, \underset{\sim}{b}_m) \ne \underset{\sim}{0}$; $\sum_{j=1}^m b_j W_n^0(t_j, \Delta_j) = \sum_{i=0}^{k_n} e_{ij}\{\psi^{(1)}(X_{ni}) - \gamma_1(\psi, F)\}/\sigma_1$, where the e_{ni} depend on $\underset{\sim}{b}$, $t_1, \ldots, t_m$, $\Delta_1, \ldots, \Delta_m$, $\{c_{ni}\}$ and $\{d_{ni}\}$ and by (2.5) - (2.8),

$$\max_{0\le i\le k_n} e_{ni}^2/\sum_{j=0}^{k_n} e_{nj}^2 \to 0 \quad \text{as} \quad n \to \infty , \quad (3.7)$$

while the $\{\psi^{(1)}(X_{ni}) - \gamma_1(\psi, F)\}/\sigma_1$ are i.i.d.rv with 0 mean and unit variance. Hence, by a central limit theorem in Hájek and Šidak (1967, p. 153), we conclude that $\sum_{j=1}^m b_j W_n^0(t_j, \Delta_j)$ is asymptotically normal. Further, by (2.7) and (3.1),

$$EW_n^0(t_j, \Delta_j)W_n^0(t_\ell, \Delta_\ell) \to \Delta_j\Delta_\ell(t_j \wedge t_\ell) = EW(t_j, \Delta_j)W(t_\ell, \Delta_\ell), \quad (3.8)$$

for every $j, \ell = 1, \ldots, m$. Hence, for every $m (\ge 1)$, $(t_j, \Delta_j) \in K^*$,

$1 \le j \le m$, as $n \to \infty$,

$$[W_n^0(t_1, \Delta_1), \ldots, W_n^0(t_m, \Delta_m)] \xrightarrow{\mathcal{D}} [W(t_1, \Delta_1), \ldots, W(t_m, \Delta_m)] \ . \quad (3.9)$$

From (3.6) and (3.9), we conclude that the finite dimensional distributions (f.d.d.) of $\{W_n\}$ converge to those of W. Hence, to prove Theorem 2.1, it remains to show that $\{W_n\}$ is *tight*. For this, for a process x, we define the increment over a block $B = \{\underset{\sim}{t}: \underset{\sim}{t}_0 \le \underset{\sim}{t} \le \underset{\sim}{t}_1\}$ by

$$x(B) = x(\underset{\sim}{t}_1) - x(t_{11}, t_{02}) - x(t_{01}, t_{12}) + x(\underset{\sim}{t}_0), \quad (3.10)$$

where $\underset{\sim}{t}_j = (t_{j1}, t_{j2})$, $j = 0, 1$. Also, let $B_\delta(\underset{\sim}{t}_0)$ be the block $\{\underset{\sim}{t}: \underset{\sim}{t}_0 \le \underset{\sim}{t} \le \underset{\sim}{t}_0 + \delta\underset{\sim}{1}\}$ $(\delta > 0)$. Then, proceeding as in (3.2) - (3.5), we obtain that

$$E\{W_n(B_\delta(t, \Delta)) - W_n^0(B_\delta(t, \Delta))\}^2 \le \ell_n \delta^3 = \ell_n[\lambda(B_\delta(t, \Delta))]^{3/2} \quad (3.11)$$

where $\lambda(B)$ stands for the Lebesgue measure of the block B and

$$\lim_{n\to\infty} \ell_n = 0, \text{ uniformly in } \delta(>0) \text{ and } (t, \Delta) \in K^*. \quad (3.12)$$

Thus, from (3.11) and the results of Bickel and Wichura (1971), we conclude that $\{W_n - W_n^0\}$ is tight. Hence, it suffices to show that $\{W_n^0\}$ is tight. Toward this, we note that $W_n^0(B_\delta(t, \Delta)) = \delta[W_n(t + \delta) - W_n(t)]$, where

$$W_n(t) = \left(\sum_{i=0}^{n(t)} c_{ni} d_{ni} \{\psi^{(1)}(X_{ni}) - \gamma_1(\psi, F)\}\right) / \sigma_1 A_n, \ t \in K_1^*. \quad (3.13)$$

Hence, $E[W_n(t) - W_n(s)]^2 \le (t - s)$, so that $E[W_n^0(B_\delta(t, \Delta))]^2 \le \delta^3 = [\lambda(B_\delta(t, \Delta)]^{3/2}$, for every $\delta > 0$ and $(t, \Delta) \in K^*$, and hence, the tightness of $\{W_n^0\}$ follows from the results of Bickel and Wichura (1971). Q.E.D.

To prove Theorem 2.2, we note that for every $(t, \Delta) \varepsilon K^*$,

$$A_n^{-1}|EM_{nn(t)}(\Delta) + \Delta\gamma_1(\psi, F)\sum_{i=0}^{n(t)} c_{ni}d_{ni} - \tfrac{1}{2}\Delta^2\gamma_2(\psi, F)\sum_{i=0}^{n(t)} c_{ni}d_{ni}^2|$$

$$= A_n^{-1}|\sum_{i=0}^{n(t)} c_{ni}E\{\psi(X_{ni} - \Delta d_{ni}) - \psi(X_{ni}) + \Delta d_{ni}\psi^{(1)}(X_{ni}) - \tfrac{1}{2}\Delta^2 d_{ni}^2\psi^{(2)}(X_{ni})\}|$$

$$= \tfrac{1}{2}\Delta^2|\sum_{i=0}^{n(t)} c_{ni}d_{ni}^2 E\{\psi^{(2)}(X_{ni} - \theta\Delta d_{ni}) - \psi^{(2)}(X_{ni})\}|/A_n \qquad (0 < \theta < 1)$$

$$\le \tfrac{1}{2}\Delta^2\left\{\sum_{i=0}^{n(t)} |c_{ni}|d_{ni}^2\left\{\max_{1\le i\le n(t)} E|\psi^{(2)}(X_{ni} - \theta\Delta d_{ni}) - \psi^{(2)}(X_{ni})|\right\}\right\}/A_n$$

$$\le \tfrac{1}{2}\Delta^2 D^* t \sup_{h: |h|\le\Delta d_n^*} \int_{-\infty}^{\infty} |\psi^{(2)}(x+h) - \psi^{(2)}(x)|dF(x) \qquad (3.14)$$

where $d_n^* = \max_{1\le i\le k_n} |d_{ni}| \to 0$ as $n\to\infty$. Hence, by (2.6), (2.13) and (3.14), $\sup\{|W_n(t, \Delta) - W_n(t, \Delta)|: (t,\Delta) \varepsilon K^*\} \xrightarrow{P} 0$ as $n\to\infty$, so that Theorem 2.2 follows from Theorem 2.1.

For Theorem 2.3, in (3.1), we take for every $(t, \Delta) \varepsilon K^*$

$$W_n^{*0}(t, \Delta) = \Delta\left(\sum_{i=0}^{n(t)} c_{n(t)i}d_{n(t)i}\{\psi^{(1)}(X_{ni}) - \gamma_1(\psi, F)\}\right)/\sigma_1 A_n. \qquad (3.15)$$

Then, as in (3.2) - (3.5), $\lim_{n\to\infty} E\{[W_n^*(t, \Delta) - W_n^{*0}(t, \Delta)]^2\} = 0$, $\forall (t, \Delta) \varepsilon K^*$, and hence, as in (3.6), $\max_{1\le j\le m} |W_n^*(t_j, \Delta_j) - W_n^{*0}(t_j, \Delta_j) \xrightarrow{P} 0$ as $n\to\infty$.

The convergence of f.d.d.'s of $\{W_n^{*0}\}$ to those of W^* follows [under (2.18)] as in (3.7) - (3.9). Further, in (3.11), we may replace W_n and W_n^0 by W_n^* and W_n^{*0}, respectively; (2.19) insures the same inequality. Finally, by (3.10) and (3.15), for every $(t, \Delta) \varepsilon K^*$,

$$W_n^*(B_\delta(t, \Delta)) = \delta[\hat{W}_n^*(t+\delta) - \hat{W}_n^*(t)] \;; \qquad (3.16)$$

$$\hat{W}_n^*(t) = \left(\sum_{i=0}^{n(t)} c_{n(t)i}d_{n(t)i}\{\psi^{(1)}(X_{ni}) - \gamma_1(\psi, F)\}\right)/\sigma_1 A_n. \qquad (3.17)$$

Note that $E[\hat{W}_n^*(t) - \hat{W}_n^*(s)]^2 = A_n^{-2}\left\{\sum_{i=0}^{n(t)} c_{n(t)i}^2 d_{n(t)i}^2 + \sum_{i=0}^{n(s)} c_{n(s)i}^2 d_{n(s)i}^2 - 2\sum_{i=0}^{n(t\wedge s)} c_{n(t)i}d_{n(t)i}c_{n(s)i}d_{n(s)i}\right\}$ which, by (2.19), is bounded from above by $M(t-s)$, $\forall\ t \geq s$. Thus, $E[W_n^*(B_\delta(t, \Delta))]^2 \leq M\delta^3 = M[\lambda(B_\delta(t, \Delta))]^{3/2}$ for every $(t, \Delta) \in K^*$, $\delta > 0$, and this insures the tightness of $\{W_n^{*0}\}$.

Q.E.D.

4. SEQUENTIAL CONFIDENCE INTERVALS FOR Δ^o BASED ON M-ESTIMATORS

Let us consider the simple regression model: $X_i = \Delta^o c_i + X_i^o$, $i \geq 1$, where the X_i^o are i.d.rv with a continuous and symmetric df F, defined on R^1, the c_i are known regression constants and we want to provide a bounded-length confidence interval for the unknown parameter Δ^o. Our procedure rests on the M-estimators [viz. (1.1)]. Since F is not specified, no fixed-sample size procedure exists and, therefore, we take recourse to sequential procedures.

Let us define $C_n^2 = \sum_{i=1}^n c_i^2$ and assume that

$$\lim_{n\to\infty} n^{-1} C_n^2 = C^2 \text{ exists } (0 < C < \infty) , \tag{4.1}$$

$$\lim_{n\to\infty}\left\{\max_{1\leq i\leq n} c_i^2/C_n^2\right\} = 0 . \tag{4.2}$$

Define an M-estimator of Δ^o as

$$\hat{\Delta}_n = \tfrac{1}{2}(\hat{\Delta}_n^{(1)} + \hat{\Delta}_n^{(2)});$$

$$\hat{\Delta}_n^{(1)} = \sup\left\{a:\ S_n(a) = \sum_{i=1}^n c_i\psi(x - ac_i) > 0\right\},\ \hat{\Delta}_n^{(2)} = \inf\{a:\ S_n(a) < 0\}. \tag{4.3}$$

We assume that ψ is non-constant, nondecreasing and skew-symmetric on R^1 ($\Rightarrow S_n(a)$ is $\searrow$ in $a \in R^1$, and hence, $\hat{\Delta}_n$ exists), and, further, we assume that

$$0 < \sigma_0^2 = \int_{-\infty}^{\infty} \psi^2(x)dF(x) < \infty . \tag{4.4}$$

Then [c.f. Jurečková (1977)], as $n \to \infty$,

$$C_n(\hat{\Delta}_n - \Delta) \sim N(0, \nu^2(\psi, F)) \tag{4.5}$$

where defining $\gamma_1(\psi, F)$ by (2.1),

$$\nu(\psi, F) = \sigma_0/\gamma_1(\psi, F). \tag{4.6}$$

Thus, if $\nu(\psi, F)$ is specified, then for a given $d(>0)$, one can define a desired sample size n_d by letting

$$n_d = \min\{n: \tau_{\alpha/2}\nu(\psi, F) \le dC_n\} \tag{4.7}$$

(where $\tau_{\alpha/2}$ is the upper $50\alpha\%$ point of the standard normal df) and take

$$I^*_{n_d} = [\hat{\Delta}_{n_d} - d,\ \hat{\Delta}_{n_d} + d]\ , \tag{4.8}$$

so that by (4.5), (4.7) and (4.8), we have

$$\lim_{d\downarrow 0} P\{\Delta^o \varepsilon I^*_{n_d}\} = 1 - \alpha \quad (0 < \alpha < 1)\ . \tag{4.9}$$

Thus, for small d, $I^*_{n_d}$ provides a bounded length confidence interval for Δ^o with *asymptotic coverage probability* $1 - \alpha$. In our case, neither F nor $\nu(\psi, F)$ is specified, and hence, the procedure in (4.7) - (4.9) is not usable. For this reason, we take recourse to the Chow-Robbins (1965) type sequential procedure. [See Gleser (1965) for the sequential least squares procedure and Ghosh and Sen (1972) for the sequential rank procedure.] Let us define

$$s_n^2 = n^{-1}\sum_{i=1}^n \psi^2(X_i - \hat{\Delta}_n c_i) - \left\{n^{-1}\sum_{i=1}^n \psi(X_i - \hat{\Delta}_n c_i)\right\}^2, \tag{4.10}$$

and

$$\left.\begin{aligned} \hat{\Delta}_{L,n} &= \sup\{a;\ S_n(a) > \tau_{\alpha/2} C_n s_n\} \\ \hat{\Delta}_{U,n} &= \inf\{a: S_n(a) < -\tau_{\alpha/2} C_n s_n\} \end{aligned}\right\} \tag{4.11}$$

Also, we assume in addition that

$$\lim_{h\downarrow 0} E\{\sup_{t:|t|\le h} [\psi(X_i^o - t) - \psi(X_i^o)]^2\} = 0\ . \tag{4.12}$$

Then [c.f. Jurečková (1977)], it follows that

$$\hat{\nu}_n = C_n(\hat{\Delta}_{U,n} - \hat{\Delta}_{L,n})/2\tau_{\alpha/2} \xrightarrow{P} \nu(\psi, F), \text{ as } n \to \infty \quad (4.13)$$

[Actually, by (2.2) and the Kintchine law of large numbers, as $n \to \infty$,

$$n^{-1}\sum_{i=1}^{n} \psi^r(X_i^o) \to \int_{-\infty}^{\infty} \psi^r(x)dF(x) \text{ a.s., for } r = 1, 2 , \quad (4.14)$$

Let $\omega_n^* = \max\{|(\hat{\Delta}_n - \Delta^o)c_i|: 1 \le i \le n\}(\le C_n|\hat{\Delta}_n - \Delta_0| \max_{1\le i\le n} |c_{ni}|$ where $c_{ni} = c_i/C_n$, $1 \le i \le n)$. Then, by (2.5), (4.2) and (4.5), $\omega_n^* \xrightarrow{P} 0$, as $n \to \infty$. On the other hand, $[\omega_n^* < \varepsilon](\varepsilon > 0)$ insures that

$$|n^{-1}\sum_{i=1}^{n}[\psi^r(X_i - \hat{\Delta}_n c_i) - \psi^r(X_i - \Delta^o c_i)]| \le n^{-1}\sum_{i=1}^{n}\omega_{ni}^r(\varepsilon), \quad (4.15)$$

where

$$\omega_{ni}^r(\varepsilon) = \sup_{t:|t|\le\varepsilon} |\psi^r(X_i^o - t) - \psi^r(X_i^o)|, \ 1 \le i \le n, \ (r = 1, 2) \quad (4.16)$$

are i.i.d.r.v. By (4.12) and (4.16) along with Markov inequality, for $\varepsilon > 0$ arbitrarily small, the right hand side of (4.15) can be made arbitrarily small, in probability. Hence, by (4.14) and (4.15),

$$s_n^2 \xrightarrow{P} \sigma_0^2 , \text{ as } n \to \infty, \quad (4.17)$$

while the rest of the proof of (4.13) follows as in Jurečková (1977)].

With (4.7) - (4.9) and (4.13) in mind, we consider a sequential procedure where a *stopping variable* N_d is defined by

$$\begin{aligned} N_d &= \min\{n \ge n_0: \hat{\Delta}_{U,n} - \hat{\Delta}_{L,n} \le 2d\} \\ &= \min\{n \ge n_0: \tau_{\alpha/2}\hat{\nu}_n \le dC_n\} \end{aligned} \quad (4.18)$$

where $n_0(\ge 2)$ is an initial sample size and we propose

$$I_{N_d} = [\hat{\Delta}_{L,N_d} \le \Delta^o \le \hat{\Delta}_{u,N_d}] \quad (4.19)$$

as the desired confidence interval for Δ^o. Note that by (4.18), I_{N_d} has width $\le 2d$, while, it remains to show that $P\{\Delta^o \varepsilon I_{N_d}\} \to 1 - \alpha$ as $d \downarrow 0$. Our interest centers around the asymptotic properties of

I_{N_d} and $\hat{\nu}_{N_d}$ as $d \downarrow 0$.

Theorem 4.1. Under (4.1), (4.2), (4.4) *and* (4.12), *as* $d \downarrow 0$,

$$N_d/n_d \to 1, \quad \textit{in probability}, \tag{4.20}$$

$$d^2 N_d \xrightarrow{p} \theta^2 = [\tau_{\alpha/2}\nu(\psi, F)/C]^2 = \lim_{d \downarrow 0} (d^2 n_d)(<\infty), \tag{4.21}$$

$$P\{\Delta^o \in I_{N_d}\} \to 1-\alpha. \tag{4.22}$$

Remark; In the Chow-Robbins (1965) procedure, (4.20)-(4.21) have been established up to an a.s. convergence as well as convergence in the first mean; parallel results for rank estimates are due to Ghosh and Sen (1972). But, the later authors needed considerably stringent regularity conditions on the score functions for such stronger results. Here also, under stronger regularity conditions on the c_i and ψ, such a.s. results can be established. Hcwever, we do not intend to pursue these results.

Proof of Theorem 4.1. For every $\varepsilon > 0$ and $d > 0$, let $n^o_{d\varepsilon} = [n_d(1+\varepsilon)]$. Then, by (4.18)

$$\begin{aligned} P\{N_d/n_d > 1+\varepsilon\} &= P\{N_d > n^o_{d\varepsilon}\} \\ &= P\{\hat{\Delta}_{U,n} - \hat{\Delta}_{L,n} > 2d, \ \forall\ n_0 \le n \le n^o_{d\varepsilon}\} \le P\{\hat{\Delta}_{U,n^o_{d\varepsilon}} - \hat{\Delta}_{L,n^o_{d\varepsilon}} > 2d\} \\ &= P\{C_{n^o_{d\varepsilon}}(\hat{\Delta}_{U,n^o_{d\varepsilon}} - \hat{\Delta}_{L,n^o_{d\varepsilon}}) > 2dC_{n^o_{d\varepsilon}}\}. \end{aligned} \tag{4.23}$$

By (4.1), (4.7) and the definition of $n^o_{d\varepsilon}$, $2dC_{n^o_{d\varepsilon}} \to (1+\varepsilon)^{\frac{1}{2}} 2\tau_{\alpha/2}\nu(\psi, F) > 2\tau_{\alpha/2}\nu(\psi, F)$, so that by (4.13) and (4.23), $P\{N_d/n_d > 1+\varepsilon\} \to 0$ as $d \downarrow 0$. Similarly, for every $\varepsilon > 0$, $P\{N_d/n_d < 1+\varepsilon\} \to 0$ ad $d \downarrow 0$, and hence, (4.20) holds. Then, (4.21) follows from (4.20), (4.1) and (4.7). To prove (4.22), we define $Z_n = \{Z_n(t) = S_{\tilde{n}(t)}(\Delta^o)/\sigma_0 C_n,\ t \in K^*_1$, where

$\tilde{n}(t) = \max\{k: C_k^2 \le tC_n^2\}$, $t \in K_1^*$. Since $X_i^o = X_i - \Delta^o c_i$, $i \ge 1$ are i.i.d.r.v., by the same technique as in the proof of Theorem 2.3, it follows by some routine steps that under (4.1), (4.2), (4.4) and (4.12), $\{Z_n\}$ converges weakly to a standard Wiener process $\xi = \{\xi(t),\ t \in K_1^*\}$. This insures that

$$\sup_{t\in K_1^*} |Z_n(t)| = 0_p(1) \text{ and } \lim_{\delta\downarrow 0} \sup_{0<s\le t\le s+\delta\le 1} |Z_n(t) - Z_n(s)| = 0, \text{ in prob.} \tag{4.24}$$

Also, we now appeal to Theorem 2.3, where we take $c_{ki} = c_i$ and $d_{ki} = c_{ki}/C_k$, for $1 \le i \le k$, $k \le k_n$. Then

$$A_n^2 = \left(\textstyle\sum_{i=0}^n c_{ni}^2 d_{ni}^2\right) \le \max_{1\le i\le n} d_{ni}^2 \left(\textstyle\sum_{i=1}^n c_{ni}^2\right) , \tag{4.25}$$

so that

$$A_n^2/C_n^2 \le \max_{1\le i\le n} d_{ni}^2 \to 0 \quad \text{as} \quad n \to \infty . \tag{4.26}$$

Finally, by (4.1), (2.18) reduces to $g(s, t) = s \wedge t$, $\forall\ s, t \in K_1^*$. Hence, from Theorem 2.3, (4.3) and (4.23), we conclude that for every $t_0 > 0$ $(0 < t_0 < k_1^*)$,

$$\sup_{t_0\le t\le k_1^*} C_n|\hat{\Delta}_{n(t)} - \Delta^o| = 0_p(1) , \tag{4.27}$$

$$\sup_{t_0\le t\le k_1^*} |C_n^{-1} C_{n(t)}^2 (\hat{\Delta}_{n(t)} - \Delta^o) + Z_n(t) \cdot \nu(\psi, F)| \xrightarrow{p} 0 , \tag{4.28}$$

$$\sup_{t_0\le s<t\le s+\delta\le k_1^*} |C_n^{-1}\{C_{n(t)}^2 (\hat{\Delta}_{n(t)} - \Delta^o) - C_{n(s)}^2 (\hat{\Delta}_{n(s)} - \Delta^o)\}| \xrightarrow{p} 0 \text{ as } \delta\downarrow 0. \tag{4.29}$$

Similarly, by Theorem 2.3, (4.11), (4.12), (4.24), (4.26), (4.27) and (4.28),

$$\sup_{t_0\le t\le k_1^*} |C_n^{-1} C_{n(t)} \{C_{n(t)} (\hat{\Delta}_{L,n(t)} - \hat{\Delta}_{n(t)}) + \tau_{\alpha/2}\nu(\psi, F)\}| \xrightarrow{p} 0 , \tag{4.30}$$

$$\sup_{t_0\le t\le k_1^*} |C_n^{-1} C_{n(t)} \{C_{n(t)} (\hat{\Delta}_{U,n(t)} - \hat{\Delta}_{n(t)}) - \tau_{\alpha/2}\nu(\psi, F)\}| \xrightarrow{p} 0, \tag{4.31}$$

$$\sup_{t_0\le s<t\le s+\delta\le k_1^*} |C_n^{-1}\{C_{n(t)}^2 (\hat{\Delta}_{L,n(t)} - \Delta^o) - C_{n(s)}^2 (\hat{\Delta}_{L,n(s)} - \Delta^o)\}| \xrightarrow{p} 0 \ (\text{as } \delta \downarrow 0) \tag{4.32}$$

$$\sup_{t_0 \le s < t \le s+\delta \le k_1^*} |C_n^{-1}\{C_{n(t)}^2(\hat{\Delta}_{U,n(t)} - \Delta^o) - C_{n(s)}^2(\hat{\Delta}_{U,n(s)} - \Delta^o)\}| \xrightarrow{p} 0 \quad (\text{as } \delta \downarrow 0). \tag{4.33}$$

By virtue of (4.20), (4.32) and (4.33), as $d \downarrow 0$,

$$C_{n_d}^{-1}|\{C_{N_d}^2(\hat{\Delta}_{U,N_d} - \hat{\Delta}_{L,N_d}) - C_{n_d}^2(\hat{\Delta}_{U,n_d} - \hat{\Delta}_{L,n_d})\}| \xrightarrow{p} 0, \tag{4.34}$$

where by (4.30) - (4.31), as $d \downarrow 0$,

$$C_{n_d}(\hat{\Delta}_{U,n_d} - \hat{\Delta}_{L,n_d}) \xrightarrow{p} 2\tau_{\alpha/2}\nu(\psi, F). \tag{4.35}$$

Also, from (4.20), (4.28) and (4.29), as $d \downarrow 0$,

$$C_{N_d}(\hat{\Delta}_{N_d} - \Delta^o)/\nu(\psi, F) \overset{p}{\sim} C_{n_d}(\hat{\Delta}_{n_d} - \Delta^o)/\nu(\psi, F) \xrightarrow{\mathcal{D}} Z_{n_d}(1), \tag{4.36}$$

where $Z_{n_d}(1)$ is asymptotically (as $d \downarrow 0$) $N(0, 1)$. Finally, by (4.32) - (4.33), (4.20), as $d \downarrow 0$,

$$C_{n_d}^{-1}|\{C_{N_d}^2(\hat{\Delta}_{U,N_d} - \hat{\Delta}_{N_d}) - C_{n_d}^2(\hat{\Delta}_{U,n_d} - \hat{\Delta}_{n_d})\}| \xrightarrow{p} 0. \tag{4.37}$$

Thus, $\hat{\nu}_{N_d} \xrightarrow{p} \nu(\psi, F)$ as $d \downarrow 0$ and hence, by (4.36) and (4.37),

$$\lim_{d \downarrow 0} P\{\Delta^o \in I_{N_d}\} = 1 - \alpha . \tag{4.38}$$

This conpletes the proof of the theorem.

Note that by (4.13) and (4.30) - (4.31), for every $t_0 > 0$, as $n \to \infty$,

$$\sup_{t_0 \le t \le k_1^*} |C_n^{-1} C_{n(t)}\{\hat{\nu}_{n(t)} - \nu(\psi, F)\}| \xrightarrow{p} 0 . \tag{4.39}$$

We now appeal to Theorem 2.3 to present an invariance principle relating to estimators of $\gamma_1(\psi, F)$. Note that by (4.6), (4.13) and (4.17), we have

$$B_n = s_n/\hat{\nu}_n = 2\tau_{\alpha/2}s_n/C_n(\hat{\Delta}_{U,n} - \hat{\Delta}_{L,n})$$

$$\xrightarrow{p} \sigma_0/\nu(\psi, F) = \gamma_1(\psi, F), \quad \text{as } n \to \infty, \tag{4.40}$$

We intend to study the asymptotic behavior of $\{B_{n(t)} - \gamma_1(\psi, F),\ t \in K_1^*\}$.

For this, we note that by (4,14), (4,15), (4,16) and (4,27), we may improve (4,17) to the following: for every $t_0 > 0$, as $n \to \infty$,

$$\sup_{t_0 \le t \le k_1^*} |s_{n(t)}^2 - \sigma_0^2| \xrightarrow{P} 0, \tag{4.41}$$

where $n(t)$ is defined by (2.12). [Actually, in (4.15) - (4,16), we may replace ω_n^* by $\sup\{\omega_{n(t)}: t_0 \le t \le k_1^*\}$ which by (4.27) $\xrightarrow{P} 0$ as $n \to \infty$]. From (4.39), (4.40) and (4.41), we obtain that as $n \to \infty$,

$$\sup_{t_0 \le t \le k_1^*} |C_n^{-1} C_{n(t)} \{B_{n(t)} - \gamma_1(\psi, F)\}| \xrightarrow{P} 0, \ \forall\ t_0 > 0 . \tag{4.42}$$

This, in turn, insures [by (4.20)] that

$$B_{N_d} - \gamma_1(\psi, F) \xrightarrow{P} 0, \quad \text{as} \quad d \downarrow 0 . \tag{4.43}$$

To obtain results deeper than (4.41) and (4.43), we note that by virtue of Theorems 2.1 and 2.3 (where as in (4.24) - (4,37), we take $c_{ki} = c_i$ and $d_{ki} = c_{ki}/C_k$, $1 \le i \le k$; $k \ge 1$) and (4.30) - (4.31), for every $t_0 > 0$,

$$\sup_{t_0 \le t \le k_1^*} \{|\{W_n^*(t, a_{n(t)}) - W_n^*(t, b_{n(t)})\} - \{W_n^*(t, a_{n(t)}) - W_n^*(t, a_{n(t)} + 2c)\}|\} \xrightarrow{P} 0 , \tag{4.44}$$

where $a_{n(t)} = C_{n(t)}(\hat{\Delta}_{L,n(t)} - \Delta^o)$, $b_{n(t)} = C_{n(t)}(\hat{\Delta}_{U,n(t)} - \Delta^o)$ and $c = \tau_{\alpha/2}\nu(\psi, F)$. Also, by Theorem 2.3, as $n \to \infty$,

$$\{W_n^*(t, a_{n(t)} - W_n^*(t, a_{n(t)} + 2c), t_0 \le t \le k_1^*\} \xrightarrow{\mathcal{D}} 2c\{\xi(t), t_0 \le t \le k_1^*\}. \tag{4.45}$$

From (4.44) and (4.45), we have [by (4.11)]

$$\frac{1}{\sigma_1 A_n}\{[2\tau_{\alpha/2} C_{n(t)} s_{n(t)} - \gamma_1(\psi, F) C_{n(t)}^2 (\hat{\Delta}_{U,n(t)} - \hat{\Delta}_{L,n(t)})], t_0 \le t \le k_1^*\} \xrightarrow{\mathcal{D}} 2\tau_{\alpha/2}\nu(\psi, F)\{\xi(t), t_0 \le t \le k_1^*\} , \tag{4.46}$$

where we assume the regularity conditions of Theorem 2,2 and further that defining h_n by (2.14),

$$\lim_{n\to\infty} h_n(t) = 0, \quad \forall \quad t_0 \le t \le k_1^* . \tag{4.47}$$

By using (4.39), (4.40) and (4.46), we have

$$\{[C_{n(t)}\hat{\nu}_{n(t)}/\nu(\psi,\ F)\sigma_1 A_n][B_{n(t)} - \gamma_1(\psi,\ F)],\ t_0 \le t \le k_1^*\} \xrightarrow{\mathcal{D}} \{\xi(t),\ t_0 \le t \le k_1^*\} \tag{4.48}$$

where $\sup\{|\hat{\nu}_{n(t)}/\nu(\psi,\ F) - 1|:\ t_0 \le t \le k_1^*\} \xrightarrow{p} 0$ and

$$C_{n(t)}^2/A_n^2 = C_n^2 C_{n(t)}^2/\sum_{i=1}^n c_i^4 \tag{4.49}$$

so that from (4.46) and (4.48), we obtain that as $n \to \infty$,

$$\{C_n[\sum_{i=1}^n c_i^4]^{-\frac{1}{2}} C_{n(t)}(B_{n(t)} - \gamma_1(\psi,\ F)),\ t_0 \le t \le k_1^*\} \xrightarrow{\mathcal{D}} \{\sigma_1\xi(t),\ t_0 \le t \le k_1^*\}, \tag{4.50}$$

for every $0 < t_0 < k_1^* < \infty$. From (4.21) and (4.48), we conclude that as $d \downarrow 0$

$$C_{n_d}^2 [\sum_{i=1}^{n_d} c_i^4]^{-\frac{1}{2}}\{B_{N_d} - \gamma_1(\psi,\ F)\}/\sigma_1 \longrightarrow N(0,\ 1). \tag{4.51}$$

Let us now define

$$s_n^{o2} = n^{-1}\sum_{i=1}^n \psi^2(X_i - \Delta^o c_i) - (\tfrac{1}{n}\sum_{i=1}^n \psi(X_i - \Delta^o c_i))^2,\quad n \ge 1. \tag{4.52}$$

By (4.1) and assuming that $\int_{-\infty}^{\infty}\psi^4(x)dF(x) < \infty$, we may repeat the proof of Theorem 2.1 and show that for every $\varepsilon > 0$ and $K < \infty$,

$$\max_{[n\varepsilon] \le m \le [nK]} \sqrt{n}|s_m^2 - s_m^{02}| \xrightarrow{p} 0 \text{ as } n \to \infty . \tag{4.53}$$

Also, from (4.46), as $n \to \infty$

$$\{(\gamma_1(\psi,\ F)/\sigma_1 A_n)\{C_{n(t)}[s_{n(t)}|\sigma_0 - \hat{\nu}_{n(t)}|\nu(\psi,\ F)]\},\quad t_0 \le t \le k_1^*\} \xrightarrow{\mathcal{D}} \{\xi(t):\ t_0 \le t \le k_1^*\}\ , \tag{4.54}$$

By (4.1), (4.53) and (4.54),

$$\{(\gamma_1(\psi,\ F)/\sigma_1 A_n)\{C_{n(t)}[s_{n(t)}^o/\sigma_0 - \hat{\nu}_{n(t)}/\nu(\psi,\ F)]\},\ t_0 \le t \le k_1^*\} \xrightarrow{\mathcal{D}} \{\xi(t),\ t_0 \le t \le k_1^*\} \tag{4.55}$$

Now $\{n(n-1)^{-1}s_n^{02},\ n \ge 2\}$ is a sequence of U-statistics of degree 2 and the weak convergence of partial sequences to Wiener processes follows from Miller and Sen (1972). If we let

$$\sigma_2^2 = \int_{-\infty}^{\infty} \psi^4(x)dF(x) - (\int_{-\infty}^{\infty} \psi^2(x)dF(x))^2, \quad (0 < \sigma_2 < \infty), \tag{4.56}$$

then using the decomposition (2.18) and (3.1) - (3.5) of Miller and Sen (1972) along with the proof of our Theorem 2.1, it follows that (when (4.1) holds) $\{(C_{n(t)}^2[s_{n(t)}^{o2}/\sigma_0^2 - 1]/C_n\sigma_2, Z_n(t), \ t \in K_1^*\}$ weakly converges to $\{(\xi_1(t), \xi_2(t)), \ t \in K_1^*\}$ where Z_n is defined after (4.23) and ξ_1 and ξ_2 are independent copies of a standard Wiener process. Thus, if we assume that

$$\lim_{n\to\infty}\left(\sum_{i=1}^n c_i^4\right)/C_n^2 = C_0^2 \quad (< \infty), \tag{4.57}$$

then from (4.55), and the above discussion, it follows that an $n \to \infty$

$$\{(\gamma_1(\psi, F)/\sigma_1 C_0)\{C_{n(t)}[\hat{\nu}_{n(t)}/\nu(\psi, F) - 1], \quad t_0 \le t \le k_1^*\}$$

$$\xrightarrow{\mathcal{D}} \{\xi_1(t) + (\gamma_1(\psi, F)\sigma_2/2\sigma_1 C_0)t^{-\frac{1}{2}}\xi_2(t), \ t_0 \le t \le k_1^*\}, \tag{4.58}$$

for every $0 < t_0 < k_1^* < \infty$. By (4.20) and (4.58),

$$C_{n_d}[\hat{\nu}_{N_d}/\nu(\psi, F) - 1] \sim N(0, \sigma^{*2}), \tag{4.59}$$

where

$$\sigma^{*2} = \frac{C_0^2\sigma_1^2}{\gamma_1^2(\psi, F)} [1 + \tfrac{1}{4}\sigma_2^2] \ . \tag{4.60}$$

Note that by (4.7) and (4.18), as $d \downarrow 0$, $\forall \ n \ge n_0$

$$P\{N_d > n\} = P\{\hat{\nu}_m/\nu(\psi, F) \ge C_m/C_{n_d}, \ \forall \ n_0 \le m \le n\} \ , \tag{4.61}$$

and by (4.21), (4.61) converges to 1 or 0 according as n/n_d converges to a limit less than or greater than 1. If we let $n = n_d + O(n_d^{\frac{1}{2}})$, the right hand side will have a limit different from 0 and 1. In fact, using (4.32), (4.33), (4.56), (4.59) and (4.60), it follows from the above that as $d \downarrow 0$,

$$P_{\Delta^o}\{d^{-1}(\sqrt{N_d/n_d} - 1) \le y\} \to \Phi(yC_0/\sigma^*), \ \forall\ y \in R \tag{4.62}$$

where Φ is the standard normal df and C_0 and σ^* are defined by (4.57) and (4.60). This leads us to the following

Theorem 4.2. Under (4.1), (4.2), (4.4), (4.12), (4.56) *and* (4.57),

$$\lim_{d\downarrow 0} P_{\Delta^o}\{d^{-1}(\sqrt{N_d/n_d} - 1) \le y\} = \Phi(yC_0/\sigma^*), \ \forall\ y \in R,$$

where n_d, N_d *are defined by* (4.7) *and* (4.12) *and* C_0, σ^* *by* (4.57) *and* (4.60).

5. SEQUENTIAL TESTS FOR Δ^o BASED ON M-ESTIMATORS

As in Section 4, we consider the regression model: $X_i = \Delta^o c_i + X_i^o$ where Δ is specified. [In (5.1), we may let H_0: $\Delta^o = \Delta_0$ for any specified Δ_0. But, then, working with $X_i - \Delta_0 c_i$, $i \ge 1$, we reduce it to (5.1).] Some sequential test for (5.1) based on linear rank statistics and derived estimators have been considered by Ghosh and Sen (1977). Because of the close relationship of M- and R-estimators, led by their motivation, we may consider the following *sequential M-test*.

As in (4.3), let $S_n(a) = \sum_{i=1}^n c_i \psi(X_i - ac_i)$, $a \in R^1$, $n \ge 1$ and define s_n^2 and B_n as in (4.10) and (4.40). Let then $(0 <)\ \alpha_1, \alpha_2\ (< 1)$ $(0 < \alpha_1 + \alpha_2 < 1)$ be the desired type I and type II error probabilities. Consider two numbers $B(\ge \alpha_2/(1 - \alpha_1))$ and $A(\le (1 - \alpha_2)/\alpha_1)$ (so that $0 < B < 1 < A < \infty$) and define $a = \log A$, $b = \log B$ $(-\infty < b < a < \infty)$. Then, we start with an initial sample of size $n_0(=n_0(\Delta))$ and continue sampling as long as

$$bs_n^2 < \Delta B_n S_n(\tfrac{1}{2}\Delta) < as_n^2 , \quad n \ge n_0(s) . \tag{5.2}$$

Define the *stopping variable* $N(=N(\Delta))$ by

$$N(\Delta) = \min\{n \geq n_0(s);\ \ \Delta B_n S_n(\tfrac{1}{2}\Delta)/s_n^2 \notin (b,\ a)\} \tag{5.3}$$

(we allow $N(\Delta) = +\infty$ if (5.2) holds for all $n \geq n_0(\Delta)$). Then, we stop sampling after having $N(\Delta)$ observations and accept H_0: $\Delta^o = 0$ or H_1: $\Delta^o = \Delta(>0)$, according as $\Delta B_{N(\Delta)} S_{N(\Delta)}(\frac{1}{2}\Delta)$ is $\leq bs^2_{N(\Delta)}$ or $\geq as^2_{N(\Delta)}$.

Note that for every fixed Δ^o and $\Delta(>0)$,

$$P_{\Delta^o}\{N(\Delta) > n\} = P_{\Delta^o}\{bs_m^2 < \Delta B_m S_m(\tfrac{1}{2}\Delta) < as_m^2\ ,\ \forall\ n_0(\Delta) \leq m \leq n\}$$

$$\leq P_{\Delta^o}\{bs_n^2 < \Delta B_n S_n(\tfrac{1}{2}\Delta) < as_n^2\}$$

$$= P_0\{bs_n^2 < \Delta B_n S_n(\tfrac{1}{2}\Delta - \Delta^o) < as_n^2\}\ . \tag{5.4}$$

When $\Delta^o = \frac{1}{2}\Delta$, $S_n(0)/\sigma_0 C_n$ is asymptotically $N(0,\ 1)$ [see Section 4] where $C_n \to \infty$ as $n \to \infty$, while $B_n \xrightarrow{p} \gamma_1(\psi,\ F)$ (finite) and $s_n^2 \xrightarrow{p} \sigma_0^2$. Hence, (5.4) converges to 0 as $n \to \infty$. If $d = -\Delta^o + \frac{1}{2}\Delta > 0$, then [as in Section 4, $\psi \nearrow$ and $S_n(a)$ is $\searrow$ in a] for every $K(0 < K < \infty)$, there exist an n^*, such that

$$d \geq K/C_n,\ \forall\ \ n \geq n^* \tag{5.5}$$

which insures that $S_n(d) \leq S_n(K/C_n)$, $\forall\ n \geq n^*$, and hence, by Theorem 2.1-2.2, $C_n^{-1}[S_n(K/C_n) - S_n(0)] \to -K\gamma_1(\psi,\ F)$ as $n \to \infty$, while $B_n \xrightarrow{p} \gamma_1(\psi,\ F)$ and $s_n^2 \xrightarrow{p} \sigma_0^2$. Since $KC_n\gamma_1(\psi,\ F) \longrightarrow \infty$, the right hand side of (5.4) again converges to 0. A similar case holds for $d < 0$. Thus,

$$\lim_{n\to\infty} P_{\Delta^o}\{N(\Delta) > n\} = 0\ , \tag{5.6}$$

so that the process terminated with probability 1.

We like to study the OC function of the proposed procedure. As in Ghosh and Sen (1977), we take recourse to the asymptotic case where we let $\Delta \to 0$. We assume that

$$\Delta^o = \phi\Delta \quad \text{where} \quad \Phi \in I = \{u: |u| \le K\} \quad \text{for some} \quad K > 1. \tag{5.7}$$

[Note that for fixed Δ^o ($\neq 0$), the OC will approach to 1 or 0 as $\Delta \to 0$, according as Δ^o is < or > 0.] Further, we assume that $n_0(\Delta) \to \infty$ as $\Delta \downarrow 0$, such that

$$\lim_{\Delta \downarrow 0} n_0(s) = \infty \quad \text{but} \quad \lim_{\Delta \downarrow 0} \Delta^2 n_0(\Delta) = 0 \ . \tag{5.8}$$

Finally, all the regularity conditions of Theorem 4.2 are assumed to be true here. Let $L_F(\phi, \Delta)$ be the OC function of the sequential test in (5.2)-(5.3) when F is the underlying df and $\Delta^o = \phi\Delta$. Then, we have the following

Theorem 5.1. For every $\phi \in I$,

$$\lim_{\Delta \downarrow 0} L_F(\phi,\Delta) = L(\phi) = \begin{cases} (A^{1-2\phi} - 1)/(A^{1-2\phi} - B^{1-2\phi}), & \phi \neq \tfrac{1}{2} \\ a/(a-b) \ , & \phi \neq \tfrac{1}{2} \ . \end{cases} \tag{5.9}$$

Thus, the asymptotic strength of the test is $(L(0) = 1 - \alpha_1,\ L(1) = \alpha_2)$ *for all* F.

Proof. For an arbitrary $\varepsilon (> 0)$, we define stopping variables $N_{ij}^{\varepsilon}(\Delta)$ as the smallest positive integer $(\ge n_0)$ for which

$$b(1 + (-1)^i \varepsilon)\sigma_0^2 < \Delta\gamma_1(\psi,\ F) S_n(\tfrac{1}{2}\Delta) < a\sigma_0^2 (1 + (-1^j \varepsilon) \tag{5.10}$$

is not true, for $i, j = 1, 2$ and denote the associated OC functions by $L^*_{ij,F}(\phi,\ \Delta)$, $i, j = 1, 2$. Then, by virtue of (4.41) and (4.42), for every $\varepsilon > 0$, there exist an Δ (say, $\Delta_0 > 0$), such that for all $0 < \Delta \le \Delta_0$,

$$L^*_{21,F}[\phi,\ \Delta) - \varepsilon \le L_F(\phi,\ \Delta) \le L^*_{12,F}(\phi,\ \Delta) + \varepsilon, \quad \forall \quad \phi \in I. \tag{5.11}$$

Thus, if in (5.10) we let $\varepsilon \to 0$ and denote the limiting stopping variable and OC functions by $N_0(\Delta)$ and $L_F^o(\phi,\ \Delta)$, respectively, then it suffices to show that (5.9) holds for $L_F^o(\phi,\ \Delta)$. Towards this, we let $n_\Delta = \min\{n \ge n_0(s): \Delta^2 C_{n_\Delta}^2 \ge 1\}$, $\Delta > 0$ and define $\{Z_n\}$ as in after

(4.23). Then, by steps similar to in (4.24)-(4.29), we conclude that as $\Delta \to 0$, for every $0 > \varepsilon > k_1^* < \infty$, when $\Delta^o = \phi\Delta$,

$$\sup_{\varepsilon \le t \le k_1^*} |\Delta\{S_{\tilde{n}_\Delta(t)}(\tfrac{1}{2}\Delta) - S_{\tilde{n}_\Delta(t)}(\phi\Delta) + (\phi - \tfrac{1}{2})\Delta\gamma_1(\psi, F)C^2_{\tilde{n}_\Delta(t)}\} \xrightarrow{P} 0, \quad (5.12)$$

$$Z_{n_\Delta} = \{Z_{n_\Delta}(t) = S_{\tilde{n}_\Delta(t)}(\phi\Delta)/\sigma_0 C_{n_\Delta}, \ \varepsilon \le t \le k_1^*\} \xrightarrow{\mathcal{D}} \{\xi(t), \ \varepsilon \le t \le k_1^*\} \quad (5.13)$$

where

$$\tilde{n}_\Delta(t) = \max\{k: C_k^2 \le tC_n^2\} \quad , \quad t \in K_1^* . \quad (5.14)$$

Thus, when $\Delta^o = \phi\Delta$,

$$\{\Delta S_{\tilde{n}_\Delta(t)}(\tfrac{1}{2}\Delta)/\sigma_0, \ \varepsilon \le t \le k_1^*\} \xrightarrow{\mathcal{D}} \{Z_{n_\Delta}(t) - (\phi - \tfrac{1}{2})t/\nu(\psi, F), \ \varepsilon \le t \le k_1^*\}. \quad (5.15)$$

By (5.13), (5.15) and the results of Dvoretzky, Kiefer and Wolfowitz (1953), the desired result follows.

Under more restrictive regularity conditions on the score function, for sequential rank tests, Ghosh and Sen (1977) have studied the limit of $\Delta^2 EN(\Delta)$ (as $\Delta \to 0$). Similar limit in our case also demands more restrictive conditions on ψ and the c_i. However, under (4.1), the limiting distribution of $\Delta N(\Delta)$ is the same as that of the first exit time of $\{\xi(t) + (\phi - \frac{1}{2})t/\nu(\psi, F), \ t \ge \varepsilon\}$ when we have two absorbing barriers $b\nu(\psi,F)$ and $a\nu(\psi,F)$.

We conclude with a remark that very recently Carroll(1977) has studied the asymptotic normality of stopping times based on M-estimators where he needs the assumption that the score function is twice boundedly continuously differentiable except at a finite number of points and that F is Lipschitz of order one in neighbourhoods of these points. Our Theorem 4.2 remains valid under weaker conditions and also the conclusions are valid for a more general model.

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