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of the Brownian Sheet

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Institute of Statistics Mimeo Series #1221

April 1979

DEPARTMENT OF STATISTICS  
Chapel Hill, North Carolina

A HÖLDER CONDITION FOR THE LOCAL TIME  
OF THE BROWNIAN SHEET

by

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Abstract

$L(x, \tilde{t})$ ,  $x \in \mathbb{R}^1$ ,  $\tilde{t} \in \mathbb{R}_+^N$  is the local time of a real valued,  $N$ -parameter, Brownian sheet. It is shown that if  $B$  is any cube lying within the set  $[\varepsilon, 1]^N$  for some  $\varepsilon > 0$ , and  $L(x, B)$  is the increment of the local time over  $B$ , then as long as the sides of  $B$  are of sufficiently small length  $|L(x, B)|$  is always bounded above by  $\xi[\lambda_N(B)]^\gamma$  for some a.s. finite random variable  $\xi$ , if  $\gamma < 1 - (2N)^{-1}$ . This Hölder result for the local time was previously derived using an inequality of Tran, the proof of which is currently in doubt. The proof given here is effectively self-contained.

<sup>1</sup>Research supported in part by a Queen Elizabeth II Research Fellowship and the Office of Naval Research, Contract N00014-75-C-0809.

AMS 1970 subject classifications: Primary 60J65, 60J55; Secondary 60G15, 60G17.

Key Words and Phrases: Brownian sheets, local time, Hölder conditions, Hausdorff dimension.

Abbreviated Title: Brownian sheet local time

### 1. The problem and its background.

This paper is concerned with sample path properties of the N-parameter Wiener process, or Brownian sheet, that involve its local time. We let  $W^{(N)}$ , or simply  $W$ , denote this process, so that  $W^{(N)}$  is a real valued Gaussian random field with zero mean and covariance

$$E\{W^{(N)}(\underline{s})W^{(N)}(\underline{t})\} = \prod_{i=1}^N \min(s_i, t_i) ,$$

where  $\underline{s} = (s_1, \dots, s_N)$  and  $\underline{t} = (t_1, \dots, t_N)$  belong to the parameter space  $\mathbb{R}_+^N$  of points in  $\mathbb{R}^N$  with all components nonnegative. If we use  $\lambda_N$  to denote N-dimensional Lebesgue measure, and let  $B$  be a Borel set of  $\mathbb{R}^1$ , we can define an occupation measure  $\mu(\underline{t}, B, \omega)$  for the Brownian sheet by

$$\mu(\underline{t}, B, \omega) = \lambda_N(\underline{s}: 0 \leq s_i \leq t_i, W^{(N)}(s, \omega) \in B) .$$

From this measure, we can define a local time function  $L(x, \underline{t}, \omega)$  to be any function  $L$  satisfying

$$\mu(\underline{t}, B, \omega) = \int_B L(x, \underline{t}, \omega) dx$$

for almost every  $\omega$ .

It is now well established that such a local time exists, and, furthermore, is jointly continuous in both its parameters. (Cairoli and Walsh (1975), Theorem 6.4, for  $N = 2$ , and Davydov (1978) for general  $N$ .) For certain purposes, it is often necessary to have more detailed knowledge of the Brownian sheet local time other than mere joint continuity, and in Adler (1978), Lemma 4, we obtained a Hölder-type bound for it. However, in the derivation used there, use was made

of some inequalities on the moments of  $L$  derived by Tran (1976). The validity of these inequalities has now been thrown into doubt following the discovery of an error in Tran's work, noted in the review [8] by Pruitt. In this paper we present a new derivation of the aforementioned Hölder condition, using a method based on a result of Davydov (1978). The new result, Theorem 1, can then be used to replace Lemma 4 in Adler (1978) in the study of the Hausdorff dimension of the level sets of the Brownian sheet presented there, as noted in the correction note [2].

To state the result we require some notation. For each  $x \in [0, \infty]$ ,  $\underline{t}, \underline{k} \in \mathbb{R}_+^N$ , and real valued, continuous,  $F: \mathbb{R}^{N+1} \rightarrow \mathbb{R}^1$  we use  $F(\langle x, \underline{t} \rangle, \langle x+h, \underline{t}+\underline{k} \rangle)$  to denote the usual increment of  $F$  over the  $(N+1)$ -dimensional rectangle  $[x, x+h] \times \prod_{i=1}^N [t_i, t_i+k_i]$  obtained by taking the  $(N+1)$ -th difference of  $F$  around the corner points of the rectangle. Similarly, for fixed,  $x$ ,  $F(x, \langle \underline{t}, \underline{t}+\underline{k} \rangle)$  denotes the increment of  $F$  (considered as a function of  $N$  variables only) over  $\prod_{i=1}^N [t_i, t_i+k_i]$ . If  $\underline{t}, \underline{k} \in \mathbb{R}_+^N$  we write  $\langle \underline{t}, \underline{t}+\underline{k} \rangle$  to denote the rectangle  $\prod_{i=1}^N [t_i, t_i+k_i]$ . Clearly  $\lambda_N(\langle \underline{t}, \underline{t}+\underline{k} \rangle) = \prod_{i=1}^N k_i$ . Finally, for any  $\underline{t} \in \mathbb{R}_+^N$ , set  $\Delta(\underline{t}) = \{ \underline{s} \in \mathbb{R}_+^N : s_i \leq t_i \text{ for all } i \}$ , while  $\Delta(1)$  denotes the unit cube  $\Delta((1, \dots, 1))$  and, for any  $\varepsilon > 0$ ,  $\Delta_\varepsilon(1)$  denotes the set  $\{ \underline{t} \in \Delta(1) : t_i \geq \varepsilon \text{ for all } i \}$ .

In the following section we shall prove

Theorem 1.

There is a version of the local time  $L(x, \underline{t}, \omega)$ , jointly continuous in all  $N+1$  variables, such that for any  $\varepsilon \in (0, 1]$  and every interval  $[a, b] \in \mathbb{R}_+^1$  and  $\gamma < 1 - (2N)^{-1}$ , there exist random variables  $\eta$  and  $\xi$  which are almost surely positive and finite for which

$$(1.1) \quad |L(x, \langle \underline{t}, \underline{t+k} \rangle)| \leq \xi |k|^{N\gamma} = \xi [\lambda_N(\langle \underline{t}, \underline{t+k} \rangle)]^\gamma$$

whenever  $x \in [a, b]$ ,  $\underline{k} = (k, \dots, k)$ ,  $|k| < \eta$ , and the cube  $\langle \underline{t}, \underline{t+k} \rangle$  lies in  $\Delta_\varepsilon(1)$ .

For the sake of completeness we note that Walsh (1978) has also obtained a Hölder type condition for the local time of a two-dimensional Brownian sheet. However, since his concern is with increments in the space parameter  $x$ , rather than the time parameter,  $\underline{t}$ , as in (1.1), his results do not overlap with those presented here.

## 2. Proof of Theorem 1.

The proof of Theorem 1 is built on two lemmas. The first is essentially Lemma 2 of Adler (1978), except whereas that lemma yields a final Hölder condition like (2.4) below for increments over sets of the form  $\prod_{i=1}^N [t_i, t_i+k_i]$ , the result below concerns only increments over cubes. This allows some relaxation in the conditions required for the lemma to hold. With the exception of replacing rectangles by cubes throughout, the proof of the following lemma is, verbatim, that of the original result. Throughout Lemma 1,  $\underline{k} = (k, \dots, k)$ .

### Lemma 1.

Let  $Y(x, \underline{t})$ ,  $x \in [0, 1]$ ,  $\underline{t} \in \Delta(1)$ , be an  $(N+1)$ -dimensional random field. Let  $U \subset \Delta(1)$  be compact. Suppose there are positive constants  $r, b, c, d$  such that

$$(2.1) \quad E|Y(\langle x, \underline{t} \rangle, \langle x+h, \underline{t+k} \rangle)|^r \\ \leq b|h|^{1+c} |k|^{N(1+d)} \quad \text{for } x, x+h \in [0, 1], \underline{t}, \underline{t+k} \in U,$$

$$(2.2) \quad E|Y(x+h, \underline{t}) - Y(x, \underline{t})|^r \\ \leq b|h|^{1+c} \quad \text{for } x, x+h \in [0, 1], \underline{t} \in U,$$

$$(2.3) \quad E|Y(x, \langle \underline{t}, \underline{t+k} \rangle)|^r \\ \leq b|k|^{N(1+d)} \quad \text{for } x \in [0,1), \underline{t}, \underline{t+k} \in U .$$

Then for every  $\gamma < d/r$  there exists a version of  $Y$ , and almost surely positive, finite, random variables  $\xi$  and  $\eta$ , such that for all  $x \in [0,1)$ ,  $\underline{t}, \underline{t+k} \in U$  for which  $|k| < \eta$ , we have

$$(2.4) \quad |Y(x, \langle \underline{t}, \underline{t+k} \rangle)| \leq \xi|k|^{N\gamma} .$$

The second lemma is a consequence of Theorem 5 of Davydov (1978). To state it, we set, for  $\underline{t} \in \mathbb{R}_+^N$ ,  $V_{\underline{t}}$  to be the  $(N-1)$ -dimensional rectangle  $\prod_{i=1}^{N-1} [0, t_i]$ , while  $W_{\underline{t}}(s)$  denotes the scaled, one-dimensional Brownian motion defined by

$$(2.5) \quad W_{\underline{t}}(s) = W(t_1, \dots, t_{N-1}, s), \quad s \geq 0 .$$

Lemma 2.

Let  $L(x, \underline{t})$  be the local time of an  $N$ -parameter Brownian sheet. Then for each  $x$  and  $\underline{t}$   $L(x, \underline{t})$  can be expressed as an integral of lower dimensional local times; viz

$$(2.6) \quad L(x, \underline{t}) = \int_{V_{\underline{t}}} L_{\underline{t}}(x, t_N) dt_1 \dots dt_{N-1} ,$$

where  $L_{\underline{t}}(x, \cdot)$  is the local time of the one-dimensional process  $W_{\underline{t}}$ .

The importance of this lemma lies in the fact that it is comparatively simple to obtain bounds on the moments of  $L_{\underline{t}}(x, s)$ , which, via (2.6), can be used to obtain bounds of the form required in Lemma 1. As a first step, we prove

Lemma 3.

Let  $\varepsilon \in (0, 1]$ , and  $n \geq 2$  be even. Then there exists a finite constant  $K > 0$ , depending only on  $\varepsilon$  and  $n$ , such that if  $\delta \in [0, \frac{1}{2}]$ ,  $\underline{k} = (k, \dots, k)$  and  $\underline{t}, \underline{t} + \underline{k} \in \Delta_\varepsilon(1)$ , then the following three inequalities hold:

$$(2.7) \quad E |L(\langle(x, \underline{t}), (x+h, \underline{t} + \underline{k})\rangle)|^n \leq K |h|^{n\delta} |\underline{k}|^{nN[1-(1+\delta)/(2N)]} ,$$

$$(2.8) \quad E |L(x+h, \underline{t}) - L(x, \underline{t})|^n \leq K |h|^{n\delta} ,$$

$$(2.9) \quad E |L(x, \langle\underline{t}, \underline{t} + \underline{k}\rangle)|^n \leq K |\underline{k}|^{nN[1-1/(2N)]} .$$

Proof.

As the proofs of each of these inequalities are very much alike, we shall only give a detailed proof of (2.7). We commence by noting that if for  $k > 0$  we write  $V_{\underline{t}}(k)$  to denote the set  $\prod_{i=1}^{N-1} [t_i, t_i+k]$  then it follows from Lemma 2 that

$$L(\langle(x, \underline{t}), (x+h, \underline{t} + \underline{k})\rangle) = \int_{V_{\underline{t}}(k)} L_{\underline{t}}(\langle(x, t_N), (x+h, t_N+k)\rangle) dt_1 \dots dt_{N-1} .$$

Substituting this into the left hand side of (2.7) yields that the expectation there is bounded above by

$$E \int \dots \int \prod_{i=1}^n |L_{\underline{t}}^{(i)}(\langle(x, t_N), (x+h, t_N+k)\rangle)|$$

where the  $n$ -fold integral is over  $n$  copies of  $V_{\underline{t}}(k)$  and we have left out the  $n \times (N-1)$  differentials.

However, this expression is clearly bounded above by

$$(2.10) \quad \int \dots \int \prod_{i=1}^n [E |L_{\underline{t}}^{(i)}(\langle(x, t_N), (x+h, t_N+k)\rangle) |^n]^{1/n} .$$

Consider the expectation here. Since we are now dealing with the local time of a one-dimensional process, we can use known results for such processes. In fact, since  $L_{\underline{t}}(x, s)$  is the local time for  $W_{\underline{t}}(s)$ , it has incremental variance function

$$(2.11) \quad \begin{aligned} \sigma_{\underline{t}}^2(s) &= E |W_{\underline{t}}(s) - W_{\underline{t}}(0)|^2 \\ &= s \times \prod_{i=1}^{N-1} t_i . \end{aligned}$$

Thus we can substitute directly into some inequalities of Berman ([3], p. 294-295, [4], p. 1268-1269 or [5], p. 70, 74) to obtain that for any  $\delta \in [0, \frac{1}{2}]$ ,

$$\begin{aligned} E |L_{\underline{t}}(\langle(x, t_N), (x+h, t_N+k)\rangle) |^n \\ \leq K_1 |h|^{n\delta} \int \dots \int_{\{t_N \leq s_1 \leq \dots \leq s_n \leq t_N+k\}} \left[ \prod_{i=1}^n \sigma(s_i - s_{i-1}) \right]^{-(1+\delta)} ds_1 \dots ds_n , \end{aligned}$$

where  $K_1$  depends only on  $n$ . Using (2.11) and elementary integration, it is straightforward to check that the above expression is bounded by

$$K_2 |h|^{n\delta} \left[ \prod_{j=1}^{N-1} t_j \right]^{-\frac{1}{2}n(1+\delta)} |k|^{\frac{1}{2}n(1-\delta)} ,$$

where  $K_2$  also depends only on  $n$ . But now we are almost done, for substituting this back into (2.10) yields that the left hand side of (2.7) is bounded by

$$(2.12) \quad \begin{aligned} K_2 |h|^{n\delta} |k|^{\frac{1}{2}n(1-\delta)} \int \dots \int \prod_{i=1}^n \left[ \left( \prod_{j=1}^{N-1} t_j^{(i)} \right)^{-\frac{1}{2}(1+\delta)} dt_1^{(i)} \dots dt_{N-1}^{(i)} \right] \\ = K_3 |h|^{n\delta} |k|^{\frac{1}{2}n(1-\delta)} \prod_{i=1}^n \left[ \prod_{j=1}^{N-1} \left( |t_j+k|^{\frac{1}{2}(1-\delta)} - |t_j|^{\frac{1}{2}(1-\delta)} \right) \right] . \end{aligned}$$



But since  $t_j \geq \varepsilon$  for all  $j$ , the mean value theorem yields

$$(2.13) \quad |t_{j+k}|^{\frac{1}{2}(1-\delta)} - |t_j|^{\frac{1}{2}(1-\delta)} \leq k \times [\frac{1}{2}(1-\delta)\varepsilon^{-\frac{1}{2}(1+\delta)}]$$

Absorbing  $K_3$  and the expression in brackets into one  $\varepsilon$ -dependent constant,

(2.10)-(2.13) yield

$$E|L(\langle(x, \underline{t}), (x+h, \underline{t}+k)\rangle)|^n \leq K|h|^{n\delta} |k|^{\frac{1}{2}n(1-\delta)} |k|^{n(N-1)},$$

which establishes the validity of (2.7).

To prove (2.8) we again apply Lemma 2 to obtain

$$L(x+h, \underline{t}) - L(x, \underline{t}) = \int_{V_{\underline{t}}} [L_{\underline{t}}(x+h, t_N) - L_{\underline{t}}(x, t_N)] dt_1 \dots dt_{N-1}.$$

Taking moments as before yields

$$(2.14) \quad E|L(x+h, \underline{t}) - L(x, \underline{t})|^n \leq \int \int \prod_{i=1}^n [E|L_{\underline{t}}^{(i)}(x+h, t_N) - L_{\underline{t}}^{(i)}(x, t_N)|^n]^{1/n},$$

where, as in (2.9), we neglect to write in the differentials. Appealing once again to Berman's inequalities for one dimension (specifically (4.5) of [5]) establishes that, since  $t_N \geq \varepsilon$ , the inner expectation is bounded above by a finite,  $\varepsilon$ -dependent, constant times  $|h|^{n\delta}$ . This fact is all that is required to complete the proof of (2.8).

Finally, note that to establish (2.9) it suffices to proceed with the same argument used to establish (2.7) appealing, on this occasion, to the inequality (4.6) of [5]. This completes the proof of the lemma.

We can now turn to the proof of Theorem 1. In fact, all we need do is establish that  $L(x,t)$  satisfies the conditions of Lemma 1 when  $U = \Delta_\varepsilon(1)$ . It is clear that inequalities (2.7)-(2.9) will correspond exactly to the conditions (2.1)-(2.3) if we can choose appropriate  $r, b, c, d$ . To do this, for a given  $\gamma < 1 - (2N)^{-1}$ , let  $\delta$  lie in the interval  $[0, \frac{1}{2}]$ . Then choose  $n$  (even) so large that the following three inequalities hold:

$$n[1 - (1+\delta)/(2N)] > 1,$$

$$n\delta > 1,$$

$$\gamma < [1 - (1+\delta)/(2N)] - n^{-1}.$$

Now put  $r = n$ ,  $b = K$ ,  $c = n\delta - 1$ ,  $d = n[1 - (1+\delta)/(2N)] - 1$ . These are all positive. The above correspondence is thus established, as is, the proof of Theorem 1.

### 3. Acknowledgement.

I am indebted to Professor Donald Geman for originally drawing my attention to the problems discussed in Section 1, as well as for some extremely helpful comments during the preparation of this work.

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