ON NONPARAMETRIC SEQUENTIAL POINT ESTIMATION OF LOCATION BASED ON GENERAL RANK ORDER STATISTICS*

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ABSTRACT

A nonparametric sequential procedure for the point estimation of location of an unspecified (symmetric) distribution based on a general class of one-sample (signed) rank order statistics is considered and its asymptotic theory is developed. The asymptotic risk-efficiency of the proposed procedure is established and certain almost sure and moment convergence results on rank based estimators are studied in this context.

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1. INTRODUCTION

Sequential point estimation of the mean of a normal distribution and its *asymptotic risk-efficiency* have been considered by Robbins (1959), Starr (1966), Starr and Woodroofe (1969) and Woodroofe (1977), among others. Recently, Ghosh and Mukhopadhyay (1979) have studied an asymptotically risk-efficient sequential procedure for point estimation of the mean of an unspecified distribution (having a finite eighth order moment). For the dual problem of *bounded-length confidence interval* for the mean of an unspecified distribution, Chow and Robbins (1965) have developed an elegant sequential procedure which is asymptotically both consistent and efficient. Sen and Ghosh (1971) have employed a general class of one-sample rank order statistics for *nonparametric sequential interval estimation* of location. Nothing parallel is done for the point estimation problem. The asymptotic theory of nonparametric sequential point estimation of location, based on a general class of rank statistics and derived estimates, is developed in the current paper. The asymptotic risk-efficiency of the proposed procedure is established and a comparison is made with other procedures.

Along with the preliminary notions, the proposed sequential procedure for the point estimation of location is formulated in Section 2. Certain moment-convergence results on rank order estimates of location are also studied in this section. These results have an important role to play in the development of the asymptotic theory in the subsequent sections. The main theorems of the paper are presented in Section 3 and their proofs are considered in Section 4. Unlike the case of Woodroofe (1977), our *stopping variable* does not involve a sum of independent random variables (or martingales/reverse martingales), and hence, a different approach is needed here. The concluding section deals with the asymptotic relative performances of the proposed and some other competing procedures and the superiority of the nonparametric appraoch is stressed in this context.

2. THE PROPOSED PROCEDURE

Let $\{X_i, i \geq 1\}$ be a sequence of independent and identically distributed random variables (i.i.d.rv) with a continuous distribution function (df) $F_\theta(x)$, $x \in E = (-\infty, \infty)$, θ real (unknown), F_θ unknown and where

$$F_\theta(x) = F(x - \theta), \quad F \text{ symmetric about } 0. \tag{2.1}$$

For nonparametric estimation of θ, we proceed as follows. Let $\phi^* = \{\phi^*(u), \ 0 < u < 1\}$ be a non-decreasing, skew-symmetric (i.e., $\phi^*(u) + \phi^*(1-u) = 0, \ \forall \ 0 < u < 1$), non-constant and square-integrable *score function*, $\phi = \{\phi(u) = \phi^*((1+u)/2), \ 0 < u < 1\}$ and, for every $n(\geq 1)$, let

$$a_n(i) = E\phi(U_{ni}) \quad \text{or} \quad \phi(i/(n+1)), \quad i = 1,\ldots,n, \tag{2.2}$$

where $U_{n1} \leq \cdots \leq U_{nn}$ are the ordered rv's of a sample of size n from the uniform (0, 1) df. Further, let $\underset{\sim}{X}_n = (X_1,\ldots,X_n)$, $\underset{\sim}{1}_n = (1,\ldots,n)$ and for every real b, let $\underset{\sim}{X}_n(b) = \underset{\sim}{X}_n - b\underset{\sim}{1}_n$. Consider then the usual one-sample rank order statistics

$$S_n(b) = S(\underset{\sim}{X}_n(b)) = \sum_{i=1}^n \text{sgn}(X_i - b) a_n(R^+_{ni}(b)), \ b \in E, \ n \geq 1,$$

where $R^+_{ni}(b)$ is the rank of $|X_i - b|$ among $|X_1 - b|,\ldots,|X_n - b|$, for $i = 1,\ldots,n$. Then, for every $n(\geq 1)$,

$$S_n(b) \text{ is } \downarrow \text{ in } b: -\infty < b < \infty,$$

while, under H_0: $\theta = 0$ and (2.1), $S_n(0)$ has a distribution symmetric about 0 and for large n,

$$n^{-\frac{1}{2}}S_n(0)/A_n \xrightarrow[\mathcal{D}]{} N(0, 1) \quad , \tag{2.5}$$

where

$$A_n^2 = n^{-1}\sum_{i=1}^n a_n^2(i) \to A^2 = \int_0^1 \phi^2(u)du \quad (< \infty) \quad . \tag{2.6}$$

We define

$$\hat{\theta}_n^{(1)} = \sup\{b: S_n(b) > 0\}, \quad \hat{\theta}_n^{(2)} = \inf\{b: S_n(b) < 0\}; \tag{2.7}$$

$$\hat{\theta}_n = (\hat{\theta}_n^{(1)} + \hat{\theta}_n^{(2)})/2. \tag{2.8}$$

Then, $\hat{\theta}_n$ is a translation invariant, median-unbiased, robust and consistent estimator of θ, and, as $n \to \infty$,

$$n^{\frac{1}{2}}(\hat{\theta}_n - \theta) \xrightarrow[\mathcal{D}]{} N(0, \nu^2) , \tag{2.9}$$

where

$$\nu^2 = A^2/B^2 \quad \text{and} \quad B = B(\phi, F) = \int_{-\infty}^{\infty} \frac{d}{dx} \phi^*(F(x))dF(x) \quad (> 0) . \tag{2.10}$$

We let

$$\nu_n^2 = E(\hat{\theta}_n - \theta)^2 \quad . \tag{2.11}$$

We shall see [viz., Theorem 2.1] that ν_n^2 exists ($< \infty$) under very mild regularity conditions and [viz., Theorem 2.2] under additional conditions,

$$n\nu_n^2 \to \nu^2 \quad \text{as} \quad n \to \infty \quad . \tag{2.12}$$

To motivate and propose the sequential procedure, suppose that the *loss* incurred in estimating θ by $\hat{\theta}_n$ is

$$L_n(\underset{\sim}{c}) = c_1(\hat{\theta}_n - \theta)^2 + c_2 n; \ \underset{\sim}{c} = (c_1, c_2) > \underset{\sim}{0}, \tag{2.13}$$

where c_1 and c_2 (cost per unit sample) are specified constants. Then the *risk* (for a given $\underset{\sim}{c}$) is

$$\lambda_n(\underset{\sim}{c}) = EL_n(\underset{\sim}{c}) = c_1\nu_n^2 + c_2 n \tag{2.14}$$

and we like to minimize this risk by a proper choice of n. By virtue of (2.12) and (2.14), if c_2/c_1 is small, then $\lambda_n(\underset{\sim}{c})$ is minimized at $n = n_o(\underset{\sim}{c})$, where

$$n_o(\underset{\sim}{c}) \sim \nu(c_1/c_2)^{\frac{1}{2}} \text{ and } \nu^2_{n_o(\underset{\sim}{c})} \sim \nu(c_2/c_1)^{\frac{1}{2}}; \tag{2.15}$$

here, $a(\underset{\sim}{c}) \sim b(\underset{\sim}{c})$ means that $a(\underset{\sim}{c})/b(\underset{\sim}{c}) \to 1$ as $c_2/c_1 \to 0$. Thus, even in this asymptotic setup, the minimum risk estimation of θ demands the knowldege of ν, which, by (2.10), depends on the unknown df F (through the functional $B(\phi, F)$). Hence, we take recourse to a sequential procedure based on a sequential estimator of ν.

As has been mentioned after (2.4), there exists a sequence of known constants $\{C_{n,\alpha}\}$ (for any $0 < \alpha < 1$), such that

$$P\{|S_n(0)| \geq C_{n,\alpha} | \theta = 0\} \geq \alpha > \{|S_n(0)| > C_{n,\alpha} | \theta = 0\}; \tag{2.16}$$

$$n^{-\frac{1}{2}} C_{n,\alpha} \to A\tau_{\alpha/2} \text{ as } n \to \infty , \tag{2.17}$$

where τ_β is the upper $100\beta\%$ point of the standard normal df. Let then

$$\hat{\theta}_{L,n} = \sup\{b: S_n(b) > C_{n,\alpha}\} \quad \hat{\theta}_{U,n} = \inf\{b: S_n(b) < -C_{n,\alpha}\}; \tag{2.18}$$

$$V_n = nA(\hat{\theta}_{U,n} - \hat{\theta}_{L,n})/2C_{n,\alpha}. \tag{2.19}$$

Then, it follows from Sen and Ghosh (1971) that V_n is a translation-invariant, robust and strongly consistent estimator of ν. Hence, motivated by (2.14), we consider the following sequential procedure.

Let n_0 be an initial sample size and $\gamma(> 0)$ be a positive constant (to be defined more precisely in Section 3). Define the *stopping variable* by

$$N(\underset{\sim}{c}) = \min n \geq n_0: \ n \geq (c_1/c_2)^{\frac{1}{2}}(V_n + n^{-\gamma})\} \tag{2.20}$$

Then, $\hat{\theta}_{N(\underset{\sim}{c})}$ is the proposed sequential point estimator of θ and the risk for this estimator is

$$\lambda^*(\underset{\sim}{c}) = EL_{N(\underset{\sim}{c})}(\underset{\sim}{c}) = c_1 E(\hat{\theta}_{N(\underset{\sim}{c})} - \theta)^2 + c_2 EN(\underset{\sim}{c}). \tag{2.21}$$

We are primarily concerned with the asymptotic behavior of $N(\underset{\sim}{c})$, $\hat{\theta}_{N(\underset{\sim}{c})}$ and $\lambda^*(\underset{\sim}{c})$ (when $c_2/c_1 \downarrow 0$) and these will be reported in Section 3. In the remaining of this section, we consider certain additional results on rank statistics and estimates, which yield (2.12) and have a useful role in the derivations in Section 4.

Let $X_{n,1} \le \cdots \le X_{n,n}$ be the ordered rv's corresponding to $X_1,\ldots,X_n$. Then, it is known [viz., Sen (1959)] that for any positive p $(<\infty)$,

$$E|X_1|^p < \infty \Longrightarrow E|X_{n,r}|^k < \infty, \ \forall\ k/p \le r \le n - k/p + 1 . \tag{2.22}$$

Also, if $0 < \alpha_1 \le \liminf_{n\to\infty} r/n \le \limsup_{n\to\infty} r/n \le 1 - \alpha_1 < 1$, then

$$E|X_1|^p < \infty \Longrightarrow \limsup_{n\to\infty} E|X_{n,r}|^k < \infty \ , \quad \text{for every} \quad k \ge 0. \tag{2.23}$$

Further, note that the scores $a_n(i)$ in (2.2) are nonnegative and nondecreasing (not all equal to 0) and hence, $\bar{\phi} = \int_0^1 \phi(u)du > 0$ (or $A > 0$). This insures that there exists a sequence $\{k_n\}$ of positive numbers, such that $k_n = \min\{k \le n\}$ for which

$$\textstyle\sum_{\{i<k_n\}} a_n(i) > \sum_{\{i>k_n\}} a_n(i) \tag{2.24}$$

and there exists an α: $\frac{1}{2} < \alpha < 1$, such that

$$n^{-1}k_n \to \alpha \quad \text{as} \quad n \to \infty \ . \tag{2.25}$$

[For $k_n = n$ (or 1), $\{i > k_n\}$ (or $\{i < k_n\}$) is an empty set, so that the corresponding sum in (2.24) is null. Also, note that $\int_{1-\varepsilon}^1 \phi(u)du \to 0$ as $\varepsilon \downarrow 0$ and hence $\bar{\phi} > 0$ insures (2.25).] Then, we have the following

<u>Theorem 2.1</u> *Under* (2.22), *for every* k (> 0), *there exists an* $n_0 = n_0(k) = \min\{n\colon n - k_n + 1 \ge k/p\}$ *where* k_n *is defined by* (2.24) - (2.25), *such that* $E|\hat{\theta}_n|^k < \infty$ *for every* $n \ge n_0$ *and*

$$\limsup_{n\to\infty} E|\hat{\theta}_n - \theta|^k < \infty, \quad \textit{for every} \quad k \ge 0. \tag{2.26}$$

In particular, if $p \ge k$, *then* $E\hat{\theta}_n$ *exists for every* $n \ge 1$.

PROOF: By (2.4), (2.7) and (2.8), we have

$$[\hat{\theta}_n^{(2)} > X_{n,k_n}] \Longleftrightarrow [S_n(X_{n,k_n}) \geq 0], \ [\hat{\theta}_n^{(1)} > X_{n,k_n}] \Longleftrightarrow [S_n(X_{n,k_n}) > 0]. \tag{2.27}$$

Let $\tilde{R}_i$ be the rank of $|X_{n,i} - X_{n,k_n}|$ among $|X_{n,1} - X_{n,k_n}|, \ldots, |X_{n,n} - X_{n,k_n}|$, for $i = 1, \ldots, n$. Then, by (2.3) and the monotonicity of the scores in (2.2),

$$\begin{aligned} S_n(X_{n,k}) &= \left\{ -\sum_{\{i<k_n\}} a_n(\tilde{R}_i) + \sum_{\{i>k_n\}} a_n(\tilde{R}_i) \right\} \\ &\leq \left\{ \sum_{\{i>k_n\}} a_n(i) - \sum_{\{i<k_n\}} a_n(i) \right\} < 0, \quad \text{by (2.23)}. \end{aligned} \tag{2.28}$$

Hence, by (2.7), (2.8), (2.6), (2.27) and (2.28), for every $n \geq 1$,

$$\hat{\theta}_n < X_{n,k_n}, \quad \text{with probability 1.} \tag{2.29}$$

Similarly, for every $n \geq 1$,

$$\hat{\theta}_n > X_{n,n-k_n+1}, \quad \text{with probability 1.} \tag{2.30}$$

Then, (2.25) follows from (2.29), (2.30) and (2.22) - (2.25). Q.E.D.

It follows from Theorem 2.1 that unlike the case of $\bar{X}_n = n^{-1}\sum_{i=1}^n X_i$, $n \geq 1$, the existence of $E|X_n|^k$ does not necessarily require that $E|X_1|^k < \infty$; (2.22) and n adequately large suffice for $E|\hat{\theta}_n - \theta|^k < \infty$. We need to strengthen (2.25) to (2.12), under appropriate regularity conditions. Since $\hat{\theta}_n$ is not a martingale or reverse martingale, the usual theorems are not applicable here. We proceed to exploit the asymptotic linearity (in b) of $n^{-\frac{1}{2}}S_n(b)$ for this purpose.

First, we note that by virtue of (3.4) of Sen and Ghosh (1973a), for every $t \geq 0$ and $n \geq 1$,

$$E\{\exp(tn^{-\frac{1}{2}}S_n(0)) | H_0: \theta = 0\} \leq \exp\{\tfrac{1}{2} t^2 A_n^2\}, \tag{2.31}$$

where A_n^2 is defined by (2.6). Hence, for every $c > 0$ and $d < 0$, there exists an $n_0 (= n_0(c, d)$, such that

$$P\{n^{-\frac{1}{2}}|S_n(0)| > c \log n | H_0: \theta = 0\} \le n^{-d}, \ \forall\ n \ge n_0 .$$

Now, we denote by $\phi^{(r)}(u) = (d^r/du^r)\phi(u)$, $r = 0, 1, 2$, $0 < u < 1$ and assume that the $\phi^{(r)}$ exists almost everywhere and there exist positive constants K_0 and $\delta(< \frac{1}{2})$, such that

$$|\phi^{(r)}(u)| \le K_0(1-u)^{-\delta-r}, \ 0 < u < 1, \ \ r = 0, 1, 2. \tag{2.32}$$

Further, we assume that the df F is symmetric, absolutely continuous and possesses an absolutely continuous density function $f(x)$ with a first derivative $f'(x)$ for almost all x (a.a.x), such that (i) for δ, defined by (2.32),

$$\sup_x f(x)\{F(x)[1-F(x)]\}^{-\delta-\eta} < \infty \quad \text{for some} \ \ \eta > 0, \tag{2.34}$$

and (ii) $f'(x)$ is continuous a.a.x. and

$$\sup_x |f'(x)| < \infty . \tag{2.35}$$

Define $B = B(\phi, F)$ as in (2.10) and let

$$\omega_n = \sup\{n^{-\frac{1}{2}}|S_n(b) - S_n(0) + nbB| : |b| \le n^{-\frac{1}{2}}(\log n)\} . \tag{2.36}$$

Then, following the lines of the proof of Theorem 2 of Sen (1980), we arrive at the following:

Suppose that (2.33) - (2.34) holds for some $\delta < (4+2\tau)^{-1} (< \frac{1}{4})$, $\tau > 0$ and (2.34) holds. Then, there exist positive numbers d^* and q and an integer n_0 (possibly dependent on τ), such that under H_0: $\theta = 0$,

$$P\{\omega_n > n^{-d^*}(\log n) | H_0\} \le qn^{-1-\tau}, \ \forall\ n \ge n_0. \tag{2.37}$$

Now, by virtue of (2.3), (2.4) and (2.7) - (2.8), we have

$$P\{n^{\frac{1}{2}}(\hat{\theta}_n - \theta) > \log n\} = P\{n^{-\frac{1}{2}}S_n((\log n)/\sqrt{n}\,) \ge 0 | \theta = 0\}$$

$$= P\{n^{-\frac{1}{2}}[S_n\left(\frac{\log n}{\sqrt{n}}\right) - S_n(0) + B\sqrt{n}\log n] + n^{-\frac{1}{2}}[S_n(0) - B\sqrt{n}\log n] \ge 0 | \theta = 0\}$$

$$\le P\{\omega_n > \frac{1}{2} B \log n | \theta = 0\} + P\{n^{-\frac{1}{2}}S_n(0) > \frac{1}{2} B \log n | \theta = 0\}$$

$$\le q^* n^{-1-\tau}, \quad q^* < \infty, \ \forall\ n \ge n_0,$$

by (2.32) and (2.37), where we let $d = 1+\tau$, $c = \frac{1}{2}B$. A similar bound holds for $P\{n^{\frac{1}{2}}(\hat{\theta}_n - \theta) < -\log n\}$. Then, we proceed to prove the following

<u>*Theorem 2.2*</u> *Under* (2.22), (2.35) *and* (2.34) *for some* $\delta < (4+2\tau)^{-1}$, $\tau > 0$, *for every* $k < 2(1+\tau)$,

$$\lim_{n\to\infty} E\{n^{k/2}|\hat{\theta}_n - \theta|^k\} = \nu^k E|Z|^k, \tag{2.40}$$

where Z *has the standard normal* df.

PROOF: Let $I(A)$ be the indicator function of the set A. Then,

$$\begin{aligned} E\{n^{\frac{1}{2}}(\hat{\theta}_n - \theta)|^k &= E\{|n^{\frac{1}{2}}(\hat{\theta}_n - \theta)|^k I(n^{\frac{1}{2}}|\hat{\theta}_n - \theta| \le \log n)\} \\ &+ E\{|n^{\frac{1}{2}}(\hat{\theta}_n - \theta)|^k I(n^{\frac{1}{2}}|\hat{\theta}_n - \theta| > \log n)\} = J_{n1} + J_{n2}, \text{ say.} \end{aligned} \tag{2.41}$$

For $q > k$, by the Hölder-inequality,

$$J_{n2} \le n^{k/2}\{E|\hat{\theta}_n - \theta|^q\}^{k/q}\{P(n^{\frac{1}{2}}|\hat{\theta}_n - \theta| > \log n)\}^{1-k/q} . \tag{2.42}$$

By Theorem 2.1, for every $q > 0$, there exists an n_q, such that $E|\hat{\theta}_n - \theta|^q < \infty$, for every $n \ge n_q$, while by (2.39), for n adequately large,

$$\begin{aligned} & n^{k/2}\{P(n^{\frac{1}{2}}|\hat{\theta}_n - \theta| > \log n)\}^{1-k/q} \\ & = O(n^{k/2-(1+\tau)(1-k/q)}) . \end{aligned} \tag{2.43}$$

Since, $k < 2(1+\tau)$, by choosing q adequately large, the right-hand side of (2.43) can be made to converge to 0 as $n \to \infty$. Hence,

$$J_{n2} \to 0 \quad \text{as} \quad n \to \infty . \tag{2.44}$$

On the other hand, letting $Y_n = n^{\frac{1}{2}}(\hat{\theta}_n - \theta)$, we have (on letting $\theta = 0$)

$$\begin{aligned} J_{n1} &= E\{|Y_n|^k I(|Y_n| \le \log n)\} \\ &= E\{|Y_n|^k I(|Y_n| \le \log n) I(\omega_n \le n^{-d^*}(\log n))\} \\ &+ E\{|Y_n|^k I(|Y_n| \le \log n) I(\omega_n > n^{-d^*}(\log n))\} \\ &= J_{n11} + J_{n12}, \text{ say,} \end{aligned} \tag{2.45}$$

where by (2.37) and (2.45), for n adequately large,

$$J_{n12} \leq (\log n)^k P\{\omega_n > n^{-d^*}(\log n)\}$$
$$= 0(n^{-1-\tau}(\log n)^k) \to 0 \quad \text{as} \quad n \to \infty \tag{2.46}$$

Further, for $|Y_n| \leq \log n$ and $\omega_n \leq n^{-d^*}(\log n)$, by (2.7), (2.8) and (2.36),

$$BY_n = n^{-\frac{1}{2}} S_n(0) + R_n; \; |R_n| \leq n^{-d^*}(\log n) . \tag{2.47}$$

Finally, note that $|n^{-\frac{1}{2}}S_n(0)| \leq n^{\frac{1}{2}}A_n = 0(n^{\frac{1}{2}})$, with probability 1, (2.31) holds and $n^{-\frac{1}{2}}S_n(0)/A$ is asymptotically $N(0, 1)$. Thus, by some routine steps, it follows that under H_0: $\theta = 0$, for every fixed k (≥ 0),

$$E\{|n^{-\frac{1}{2}}S_n(0)|^k I(|Y_n| \leq \log n) I(\omega_n \leq n^{-d^*}(\log n))\} \to A^k E|Z|^k. \tag{2.48}$$

Hence, from (2.45), (2.47) and (2.48), we have

$$\lim_{n\to\infty} J_{n1} = B^{-k}Z^k E|Z|^k = \nu^k E|Z|^k, \; \forall \; k \geq 0 \; . \tag{2.49}$$

Thus, (2.40) follows from (2.41), (2.44) - (2.46) and (2.49). Q.E.D.

Remarks. For the particular case of the Wilcoxon Scores (i.e., $\phi(u) = u$: $0 \leq u \leq 1$), Inagaki (1974) has considered an almost sure (a.s.) representation for $n^{\frac{1}{2}}(\hat{\theta}_n - \theta)$ interms of a sum of i.i.d.r.v's. Our Theorem 2.2 provides analogous results for a broad class of rank order estimators. Following the lines of the proof of Theorem 2.2, we have

$$n(\hat{\theta}_n - \theta) - B^{-1}S_n(\theta) = \xi_n, \quad \text{say}, \tag{2.50}$$

where under the hypothesis of Theorem 2.2, as $n \to \infty$,

$$n^{-\frac{1}{2}}\xi_n \to 0 \text{ a.s. and } E|n^{-\frac{1}{2}}\xi_n|^k \to 0, \; \forall \; k < 2(1+\tau), \; \tau > 0. \tag{2.51}$$

Firther, it follows from Sen and Ghosh (1973a) that under H_0: $\theta = 0$, $\{S_n(0), n \geq 1\}$ is a (zero-mean) martingale sequence and under conditions less restrictive than (2.33) - (2.34), there exists an $\eta > 0$, such that under H_0,

$$A^{-1}S_n(0) = W(n) + O(n^{\frac{1}{2}-\eta}) \quad \text{a.s., as} \quad n \to \infty, \tag{2.52}$$

where $W = \{W(t),\ t \in [0, \infty)\}$ is a standard Wiener process on $[0, \infty)$. From (2.50), (2.51) and (2.52), we conlcude that the *Skorokhod-Strassen embedding of Wiener process* holds for $\{\nu^{-1}n(\hat{\theta}_n - \theta)\}$ under the hypothesis of Theorem 2.2. Sen and Ghosh (1973b) have also condidered an a.s. representation for one-sample rank order statistics. As a special case of their theorem, we have under H_0: $\theta = 0$,

$$S_n(0) = \sum_{i=1}^{n} \phi^*(F(X_i)) + \xi_n^*, \quad n \geq 1,$$

where, under conditions less restrictive than the ones in Theorem 2.2,

$$n^{-\frac{1}{2}}\xi_n^* = O(n^{-\eta}) \ \text{a.s.,} \quad \text{as} \quad n \to \infty \quad (\text{for some } \eta > 0). \tag{2.54}$$

From (2.50), (2.51), (2.53) and (2.54), we conclude that

$$n^{-\frac{1}{2}}|n(\hat{\theta}_n - \theta) - B^{-1}\sum_{i=1}^{n}\phi^*(F_\theta(X_i))X \to 0 \ \text{a.s., as} \ n \to \infty, \tag{2.55}$$

and this extends Inagaki's theorem to a broad class of signed rank statistics where ϕ^* need not be bounded.

3. THE MAIN THEOREMS

The asymptotic behavior of $N(\underset{\sim}{c})$, $\hat{\theta}_{N(\underset{\sim}{c})}$ and $\lambda^*(\underset{\sim}{c})$ (as $c_2/c_1 \downarrow 0$) will be studied in this section. The results to follow depend on c_1, c_2 through $c = c_2/c_1$ only, and hence, for notational simplicity, we let $c_1 = 1$, $c_2 = c$, $N(\underset{\sim}{c}) = N_c$, $\hat{\theta}_{N(\underset{\sim}{c})} = \hat{\theta}_{N_c}$, $\lambda^*(\underset{\sim}{c}) = \lambda_c^*$, $n_o(\underset{\sim}{c}) = n_c^o$ and $\lambda_{n_o(\underset{\sim}{c})}(\underset{\sim}{c}) = \lambda_c^o$. Then, we have the following.

<u>*Theorem 3.1*</u> *Under* (2.22) *and* (2.33) - (2.34) *for some* $\delta < \frac{1}{4}$, *for every* $\gamma > 0$ [*in* (2.20)], *as* $c \downarrow 0$

$$N_c/n_c^o \xrightarrow{p} 1, \quad E(N_c/n_c^o)^k \longrightarrow 1, \ \forall\ k \in [0, 1], \tag{3.1}$$

$$\sqrt{n_c^o}(\hat{\theta}_{N_c} - \theta)/\nu \xrightarrow[\mathcal{D}]{} \mathcal{N}(0, 1). \tag{3.2}$$

Theorem 3.2 *Under* (2.22), (2.35) *and* (2.33) - (2.34), *when*

$$\delta \le (4+2\tau)^{-1}, \quad \tau > 1; \; 1+2\gamma < \tau, \quad \gamma > 0, \tag{3.3}$$

where γ *is defined in* (2.20), *then*

$$\lim_{c \downarrow 0} (\lambda_c^* / \lambda_c^o) = 1 \; . \tag{3.4}$$

It may be remarked that (3.4) asserts that the *risk* involved in the proposed sequential procedure is asymptotically (as $c \downarrow 0$) equal to the risk of the corresponding *optimal* fixed-sample size procedure. Thus, for all F and ϕ satisfying the hypothesis of Theorem 3.2, the proposed sequential procedure is *asymptotically risk-efficient*. In particular, (3.3) demands that $\int_0^1 |\phi(u)|^r du < \infty$ for some $r > 6$ and this is true for the Wilcoxon, normal scores and the other commonly used rank statistics and is less restrictive than $\int_0^1 \exp\{t\phi(u)\} du < \infty$ (for some $t > 0$), as employed in Sen and Ghosh (1971) for the confidence interval problem.

The asymptotic normality of $(n_c^o)^{-\frac{1}{2}}(N_c - n_c^o)$ depends on the asymptotic normality and uniform continuity, in probability, of $\{n^{\frac{1}{2}}(V_n - \nu)\}$. For Wilcoxon scores, the asymptotic normality of $n^{\frac{1}{2}}(V_n - \nu)$ has been studied by Jurečková (1973) and her treatment holds generally for bounded and continuously differentiable score functions; however, for unbounded scores, this remains as a challenging open problem. The following theorem presents the impact of this on the asymptotic normality of stopping times.

Theorem 3.3 *Suppose that in* (2.20) $\gamma > \frac{1}{2}$, $N_c / n_c^o \xrightarrow{P} 1$ *as* $c \downarrow 0$ *and*

$$n^{\frac{1}{2}}(V_n - \nu)/\beta \xrightarrow[\mathcal{D}]{} N(0,1), \quad \sup_{m: |m-n| \le \delta n} \left\{ n^{\frac{1}{2}} |V_m - V_n| \right\} \xrightarrow{P} 0 \text{ as } \delta \downarrow 0, \tag{3.5}$$

where β *is a finite positive number. Then, as* $c \downarrow 0$,

$$(n_c^o)^{-\frac{1}{2}}(N_c - n_c^o) \xrightarrow[\mathcal{D}]{} N(0, \beta^2/\nu^2). \tag{3.6}$$

Note that (3.5) may be replaced by the weak convergence of the partial sequence $\{n^{-\frac{1}{2}}(v_k - \nu)/\beta: k \le n\}$ to a Gaussian function. The proofs of the theorems are presented in the next section.

4. PROOFS OF THEOREMS 3.1, 3.2 and 3.3

We let $n_c^o = [c^{-\frac{1}{2}}\nu]$ (see (2.15)), and, for every $0 < \varepsilon < 1$, we let

$$n_{1c} = [c^{-1/2(1+\gamma)}],\ n_{2c} = [(1-\varepsilon)n_c^o] \text{ and } n_{3c} = [(1+\varepsilon)n_c^o], \quad (4.1)$$

where, we choose c so small that $n_0 \le n_{1c} < n_{2c} < n_{3c}$. Then, by definition in (2.20), $N_c \ge n_{1c}$, with probability 1, so that

$$\begin{aligned} P\{N_c \le n_{2c}\} &= P\{V_n < c^{\frac{1}{2}}n, \text{ for some } n: n_{1c} \le n \le n_{2c}\} \\ &\le P\{|V_n - \nu| \ge \varepsilon\nu, \text{ for some } n_{1c} \le n \le n_{2c}\}, \end{aligned} \quad (4.2)$$

as, for n n_{2c}, $c^{\frac{1}{2}}n - \nu \le c^{\frac{1}{2}}n_c^0(1-\varepsilon) - c^{\frac{1}{2}}n_0^0 \sim -\varepsilon\nu$. Now, by (2.16) - (2.19) and proceeding as in (2.38), it follows that, for n adequately large,

$$P\{n^{\frac{1}{2}}(\hat{\theta}_{u,n} - \theta) \not> \log n\} = 0(n^{-1-\tau}). \quad (4.3)$$

$$P\{n^{\frac{1}{2}}(\hat{\theta}_{L,n} - \theta) < -\log n\} = 0(n^{-1-\tau}), \quad (4.4)$$

so that by (2.17), (2.19), (4.3), (4.4), (2.36) and (2.37), we conclude that for n adequately large,

$$P\{|V_n - \nu| \ge \varepsilon\nu\} \le 3q^* n^{-1-\tau}, \quad q^* < \infty. \quad (4.5)$$

Hence, by (4.2) and (4.5), as $c \downarrow 0$,

$$\begin{aligned} P\{N_c \le n_{2c}\} &\le \sum_{n_{1c} \le n \le n_{2c}} \{3q^* n^{-1-\tau}\} = 0(n_{1c}^{-\tau}) \\ &= 0(c^{\tau/2(1+\gamma)}), \text{ by (4.1)}. \end{aligned} \quad (4.6)$$

In a similar manner, for $n \ge n_{3c}$,

$$\begin{aligned} P\{N_c > n\} &= P\{k < c^{-\frac{1}{2}}(V_k + k^{-\gamma}),\ \forall\ k \in [n_0, n]\} \\ &\le P\{n < c^{-\frac{1}{2}}(V_n + n^{-\gamma})\} = P\{\nu_n > c^{\frac{1}{2}}n - n^{-\gamma}\} \\ &= P\{V_n - \nu > c^{\frac{1}{2}}n - c^{\frac{1}{2}}n_c^0 - n^{-\gamma}\} \\ &\le P\{|V_n - \nu| > \eta\} \text{ where by (4.1), } \eta > 0. \end{aligned} \quad (4.7)$$

Thus, by (4.5) and (4.7), $P\{N_c > n_{3c}\} \to 0$ as $c \downarrow 0$. Since $\varepsilon(>0)$ is arbitrary, by (4.6) and the above, $P\{|N_c/n_c^o - 1| > \varepsilon\} \to 0$ as $c \downarrow 0$. Also, by (4.5) and (4.7), $\sum_{n \geq n_{3c}} P(N > n) = O(n_{3c}^{-\tau}) \to 0$ as $c \downarrow 0$, and hence, $E\{N_c I(N_c \geq n_{3c})\}/n_c^o \to 0$ as $c \downarrow 0$. Also, $E\{N_c I(N_c \leq n_{2c})\}/n_c^0 \leq (1-\varepsilon)P\{N_c \leq n_{2c}\} \to 0$ by (4.6), while by (4.5), (4.6) and (4.7), $E\{N_c I(n_{2c} < n < n_{3c})\}/n_c^o$ can be made arbitrarily close to 1 by choosing ε small. Hence, $E(N_c/n_c^o) \to 1$ as $c \downarrow 0$. A similar proof holds for $E(N_c/n_c^o)^k \to 1$, $\forall \; 0 \leq k < 1$. This completes the proof of (3.1). Now, it follows from Sen and Ghosh (1973a) that under H_0: $\theta = 0$, $\{S_n(0)\}$ is a martingale sequence, and hence, using the Kolmogorov inequality (for submartingales) and some standard analysis, we obtain that as $n \uparrow \infty$,

$$\lim_{\delta \downarrow 0} \left\{ \max_{m: |m-n| \leq \delta n} n^{-\frac{1}{2}} |S_m(0) - S_n(0)| \right\} = 0, \quad \text{in probability.} \tag{4.8}$$

By (2.39), (2.47) and (4.8), we obtain that as $n \to \infty$,

$$\lim_{\delta \downarrow 0} \left\{ \max_{m: |m-n| \leq \delta n} n^{\frac{1}{2}} |\hat{\theta}_m - \hat{\theta}_n| \right\} = 0, \quad \text{in probability,} \tag{4.9}$$

so that (2.9), (4.9) and (3.1) insure (3.2). Hence the proof of Theorem 3.1 is complete.

To prove (3.4), we make use of (2.15), (2.21) and (3.1) (i.e., $\lim_{c \downarrow 0} E(N_c/n_c^o) = 1$), and hence, it suffices to show that

$$\lim_{c \downarrow 0} (\nu c^{\frac{1}{2}})^{-1} E(\hat{\theta}_{N_c} - \theta)^2 = 1 . \tag{4.10}$$

Now, by Theorem 2.2, for every $k \varepsilon (2, 2+2\tau)$,

$$\begin{aligned} E\{(\hat{\theta}_{N_c} - \theta)^2 I(N_c \leq n_{2c})\} &= \textstyle\sum_{n=n_{1c}}^{n_{2c}} E\{(\hat{\theta}_n - \theta)^2 I(N_c = n)\} \\ &\leq \textstyle\sum_{n=n_{1c}}^{n_{2c}} (E|\hat{\theta}_n - \theta|^k)^{2/k} (P\{N_c = n\})^{1-2/k} \\ &\leq \left(\textstyle\sum_{n=n_{1c}}^{n_{2c}} E|\hat{\theta}_n - \theta|^k \right)^{2/k} (P\{N_c \leq n_{2c}\})^{1-2/k} \\ &= (O(n_{1c}^{-(k-2)/2}))^{2/k} (O(c^{\tau/2(1+\gamma)}))^{1-2/k} \quad \text{[by (4.6)]} \\ &= O(c^{(k-2)(1+\tau)/2k(1+\gamma)}). \quad \text{[by (4.1)]} \end{aligned} \tag{4.11}$$

Since, by (3.3), $(1+\tau) > 2(1+\gamma)$, while for $\tau = 1+\xi$ and $k = 2(1+\tau)-\eta$, $\xi > 0$, $\eta > 0$, $(k-2)/k = \frac{1}{2} + \frac{2\xi - \eta}{8+4\xi-2\eta} > \frac{1}{2}$ for $\eta > 2\xi$ we obtain from (4.11) that by a proper choice of $k \varepsilon (2, 2(1+\tau))$, under (3.3),

$$\lim_{c\downarrow 0}(\nu c^{\frac{1}{2}})^{-1} E\{(\hat\theta_{N_c} - \theta)^2 I(N_c \le n_{2c})\} = 0. \qquad (4.12)$$

Similarly, as $c \downarrow 0$,

$$E\{(\hat\theta_{N_c} - \theta)^2 I(N_c \ge n_{3c})\} \le \left(\sum\nolimits_{n\ge n_{3c}} E|\hat\theta_n - \theta|^k\right)^{2/k} (P(N_c \ge n_{3c}))^{1-2/k}$$
$$= 0(c^{(2+\tau)(k-2)/2k}), \qquad (4.13)$$

so that noting that by (3.3), $\tau > 1$ and then taking $k = 4$, we obtain from (4.13) that

$$\lim_{c\downarrow 0}(\nu c^{\frac{1}{2}})^{-1} E\{(\hat\theta_{N_c} - \theta)^2 I(N_c \ge n_{3c})\} = 0. \qquad (4.14)$$

Thus, to prove (4.10), it suffices to show that

$$\lim_{c\downarrow 0}(\nu c^{\frac{1}{2}})^{-1} E\{(\hat\theta_{N_c} - \theta)^2 I(n_{2c} < N_c < n_{3c})\} = 1. \qquad (4.15)$$

Now, by Theorem 2.2 and the definition of n_c^o, we have

$$\lim_{c\downarrow 0}(\nu c^{\frac{1}{2}})^{-1} E(\hat\theta_{n_c^o} - \theta)^2 = 1, \quad \lim_{c\downarrow 0}(\nu c^{\frac{1}{2}})^{-1}\{E(\hat\theta_{n_c^o} - \theta)^4\}^{\frac{1}{2}} < \infty, \qquad (4.16)$$

while, $P\{n_{2c} < N_c < n_{3c}\} \to 1$ as $c \downarrow 0$. Hence, to prove (4.15), it is enough to show that

$$\lim_{c\downarrow 0}(\nu c^{\frac{1}{2}})^{-1} E\{[(\hat\theta_{N_c} - \hat\theta_{n_c^o})^2 I(n_{2c} < N_c < n_{3c})\} = 0. \qquad (4.17)$$

Let us then write (as in the proof of Theorem 2.2)

$$Bn^{\frac{1}{2}}(\hat\theta_n - \theta) = n^{-\frac{1}{2}} S_n(\theta) + R_n, \qquad (4.18)$$

so that writing $ER_n^2 = ER_n^2 I(n^{\frac{1}{2}}|\hat\theta_n - \theta| \le \log n) + ER_n^2 I(n^{\frac{1}{2}}|\hat\theta_n - \theta| > \log n)$ $\le ER_n^2 I(n^{\frac{1}{2}}|\hat\theta_n - \theta| \le \log n) + 2B^2 E\{n(\hat\theta_n - \theta)^2 I(n^{\frac{1}{2}}|\hat\theta_n - \theta| > \log n)\} +$ $2E\{n^{-1} S_n^2(\theta) I(n^{\frac{1}{2}}|\hat\theta_n - \theta| > \log n)\}$ and then proceeding as in (2.42) through (2.47) (where we take $k = 2$ and $\tau > 1 + 2\gamma$, $\gamma > 0$), we obtain by some standard steps that

$$ER_n^2 = 0(n^{-1-\gamma}), \tag{4.19}$$

so that

$$E\left\{\max_{n_{2c}\le n\le n_{3c}} R_n^2\right\} \le \sum_{n=n_{2c}}^{n_{3c}} ER_n^2 = 0\left(n_{2c}^{-\gamma}\right) = 0\left(c^{\gamma/2}\right). \tag{4.20}$$

Further, for $n_{2c} < N_c < n_{3c}$, $N_c^{-1} \le n_{2c}^{-1} = 0(c^{\frac{1}{2}})$. Hence, by virtue of (4.18) and (4.20), it suffices to show that

$$\lim_{c\downarrow 0}\left\{(n_c^o)^{-1}E[(S_{N_c}(0) - S_{n_c^o}(0))^2 I(n_{2c} < N_c < n_{3c})\,|\,\theta = 0]\right\} = 0. \tag{4.21}$$

Since, under H_0: $\theta = 0$, $\{S_n(0)\}$ is a (zero-mean) martingale, by (3.1), $P\{n_{2c} < N_c < n_{3c}\} \to 1$ as $c \downarrow 0$ and (2.31) holds, we have

$$\begin{aligned}
&(n_c^o)^{-1}E[(S_{N_c}(0) - S_{n_c^o}(0))^2 I(n_{2c} < N_c < n_{3c})\,|\,\theta = 0] \\
&\le (n_c^o)^{-1}\sum_{n=n_{2c}}^{n_{3c}} (E[(S_n(0) - S_{n_c^o}(0))^4\,|\,\theta = 0])^{\frac{1}{2}}(P\{N_c = n\})^{\frac{1}{2}} \\
&\le (n_c^o)^{-1}\left(\sum_{n=n_{2c}}^{n_{3c}} E[(S_n(0) - S_{n_c^o}(0))^4\,|\,\theta = 0]\right)^{\frac{1}{2}}(P(n_{2c} \le N_c \le n_{3c}))^{\frac{1}{2}} \\
&\le (n_c^o)^{-1}\left(\sum_{n=n_{2c}}^{n_{3c}} 0(|n - n_c^o|)\right)^{\frac{1}{2}} \cdot 1 \\
&= 0((n_{3c} - n_{2c})/n_c^o),
\end{aligned} \tag{4.22}$$

where by (4.11), $(n_{3c} - n_{2c})/n_c^o \sim 2\varepsilon$ and this can be made arbitrarily small by choosing ε so. This proves (4.21) and the proof of Theorem 3.2 is complete.

To prove Theorem 3.3, we note that by definition in (2.20),

$$c^{-\frac{1}{2}}\nu_{N_c} \le N_c \le c^{-\frac{1}{2}}(\nu_{N_c - 1} + (N_c - 1)^{-\gamma}) \quad \text{whenever} \quad N_c > n_0. \tag{4.23}$$

Thus, noting that $n_c^o \sim \nu c^{-\frac{1}{2}}$, we have from (4.23), for $N_c > n_0$,

$$\sqrt{n_c^o}(\nu_{N_c} - \nu)/\nu \le (N_c - n_c^o)/\sqrt{n_c^o} \le \sqrt{n_c^o}(\nu_{N_c - 1} - \nu)/\nu + c^{-\frac{1}{2}}(N_c - 1)^{-\gamma}/\sqrt{n_c^o},$$

where $\gamma > \frac{1}{2}$ and $N_c/n_c^o \xrightarrow{p} 1$ insure that $c^{-\frac{1}{2}}(N_c - 1)^{-\gamma}/\sqrt{n_c^o} \xrightarrow{p} 0$, as $c \downarrow 0$. Hence, (3.6) follows from (3.5) and (4.24). Q.E.D.

5. SOME CONCLUDING REMARKS

Under the hypothesis of Theorem 3.2, the sequential procedure is asymptotically risk-efficient. It may be noted that $\lambda_c^o (\sim 2\nu(c_1c_2)^{\frac{1}{2}})$ depends on the score function through $\nu = A/B$, where A and B are defined by (2.6) and (2.10) and are functions of ϕ (and F). To make this depence clear, we denote

$$\lambda_c^0 = \lambda(c_1, c_2, A_\phi, B_\phi) = 2(A_\phi/B_\phi)(c_1c_2)^{\frac{1}{2}} \qquad (5.1)$$

where defining ϕ^* as in before (2.2),

$$A_\phi^2 = \int_0^1 \phi^2(w)du \quad \text{and} \quad B_\phi = \int_{-\infty}^{\infty} \frac{d}{dx}\phi^*(F(x)dF(x)\,. \qquad (5.2)$$

Thus, if we have two different score functions say, ϕ_1 and ϕ_2, then the corresponding optimal λ_c^o are

$$\lambda(c_1, c_2, A_{\phi_1}, B_{\phi_1}) \quad \text{and} \quad \lambda(c_1, c_2, A_{\phi_2}, B_{\phi_2})\,, \qquad (5.3)$$

and smaller is the quantity, the better is the corresponding procedure. Hence, the relative efficiency of the procedure based on the score function ϕ_2 with respect to the one based on ϕ_1 is

$$\begin{aligned} e(\phi_1, \phi_2) &= \lambda(c_1, c_2, A_{\phi_1}, B_{\phi_1})/\lambda(c_1, c_2, A_{\phi_2}, B_{\phi_2}) \\ &= (A_{\phi_1}B_{\phi_2})/(A_{\phi_2}B_{\phi_1}) \end{aligned} \qquad (5.4)$$

and this agrees with the (square root of the) classical Pitman-efficiency of the rank tests (for location) based on the score function ϕ_2 with respect to the score function ϕ_1. Hence, if we define $\psi(u) = \psi^*((1+u)/2)$, $0 < u < 1$, where

$$\psi^*(u) = -f(F^{-1}(u))/f(F^{-1}(u)), \quad 0 < u < 1,$$

then $A_\psi^2 = \int (f'/f)^2 dF$ is the Fisher information and by (5.4),

$$e(\phi, \psi) = \rho(\phi, \psi) \tag{5.6}$$
$$= \left(\int_0^1 \phi^*(u)\psi^*(u)du)/A_\phi A_\psi\right) \le 1. \tag{5.6}$$

and the equality sign holds *iff* $\phi^* \equiv \psi^*$. Thus, ψ is an optimal score function.

For the procedure based on the sample means and variances, considered by Starr (1966) and Ghosh and Makhopadhyay (1979), the corresponding λ_c^o is

$$2\sigma(c_1 c_2)^{\frac{1}{2}} \text{ where } \sigma^2 = \text{Var}(X_1). \tag{5.7}$$

Thus, the asymptotic relative efficiency of the proposed sequential procedure with respect to the normal theory procedure is

$$e(\phi, N) = \sigma B_\phi / A_\phi . \tag{5.8}$$

In particular, if we use normal scores (i.e., $\phi^*(u)$, the inverse of the standard normal df), then, (5,8) is ≥ 1 for all F, where the equality sign holds only when F is normal. This expalins the asymptotic supremacy of the normal scores procedure over the parametric procedure. Even for Wilcoxon scores, when F is normal, (5.8) reduces to $(3/\pi)^{\frac{1}{2}} \simeq .978$ and it is usually >1 for distributions with heavy tails. For both these scores, conditions for the applicability of Theorems 3.1 and 3.2 hold, while Jurečková's (1973) theorem insure (3.5) for the Wilcoxon scores.

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