

ASYMPTOTIC THEORY OF SOME TIME-SEQUENTIAL TESTS BASED ON  
PROGRESSIVELY CENSORED QUANTILE PROCESSES\*†

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ABSTRACT

In the context of time-sequential tests, a general class of (parametric as well as nonparametric) testing procedures rests on progressively censored linear combinations of functions of order statistics with stochastic coefficient-vectors. Invariance principles for such quantile processes, developed in Sen (1979), are extended here to more general models, and these are incorporated in the study of the asymptotic properties of the allied time-sequential tests.

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## 1. INTRODUCTION

In *clinical trials* and *life testing problems*, usually, one draws statistical inference from *censored* or *truncated* data. In this context, a *progressive censoring scheme* (PCS) allows a monitoring of the experimentation continuously from the beginning with the objective of an *early termination* whenever the accumulating outcome provokes so; hence, a PCS involves a *time-sequential procedure*. For a *simple regression model* (containing the classical two-sample problem as a special case), Chatterjee and Sen (1973) have studied the general theory of PCS testing procedures based on *linear rank statistics*. For some parametric models, Sen (1976b) and Gardiner and Sen (1978) have employed the *likelihood ratio principle* in a PCS and the resulting test statistics are based on certain *linear combinations of functions of order statistics*. Sen (1979) has also shown that for a general class of location, scale and regression models, *progressively censored likelihood ratio statistics* (PCLRS) are based on some *quantile processes* which involve partial sequences of linear combinations of functions of order statistics with stochastic coefficients depending progressively on the censoring stages; along with some martingale characterizations of these PC quantile processes, some invariance principles for them have been derived there and these have been incorporated in the study of the asymptotic properties of some proposed tests.

Majumdar and Sen (1978) have extended the results of Chatterjee and Sen (1973) for the *multiple regression model* (containing the several sample problem as a particular case); see also Sinha and Sen (1979b) in this context. A multiparameter extension of the results of Sen (1976b) has been considered by Sen and Tsong (1980). The object of the present investigation is to formulate suitable multiparameter generalizations of the model considered in Sen (1979) and to develop the asymptotic theory of PCS tests relating to these models.

Thus, the results in the current paper are the multiparameter extensions of those in Sen (1979). In view of this, in the sequel, whenever possible, we shall attempt to minimize the technical manipulations by suitable cross reference to the earlier paper.

Along with the preliminary notions, the proposed progressively censored quantile processes (PCQP) are introduced in Section 2. Invariance principles for these PCQP's are considered in Section 3. Section 4 is devoted to the proposal and study of some PCS tests based on these PCQP's with special emphasis on their asymptotic properties. In particular, the multiple linear regression model and the location-scale model are treated elaborately in this context.

## 2. THE PROPOSED PCQP'S

Let  $X_1, \dots, X_n$  be the *survival times* (with distribution functions (d.f.)  $F_1, \dots, F_n$  respectively) of  $n (\geq 1)$  items under life testing and let  $Z_n = (Z_{n1}, \dots, Z_{nn})$  be the vector of *order statistics* corresponding to  $X_n = (X_1, \dots, X_n)$ . The  $F_i$  are all assumed to be continuous (and defined on the real line  $(-\infty, \infty)$ ), so that ties among them are neglected, in probability. Let  $Q_n = (Q_{n1}, \dots, Q_{nn})$ , the vector of *anti-ranks* be defined by

$$Z_{nj} = X_{Q_{nj}} \quad \text{for } j = 1, \dots, n. \quad (2.1)$$

From time and cost considerations, often, the experimentation is terminated at the  $r$ -th failure  $Z_{nr}$ , where

$$r = [np] + 1, \quad \text{for some } p \in (0, 1); \quad (2.2)$$

( $[s]$  being the largest integer contained in  $s$ ), so that for such a *censored case*, the observable random vectors (r.v.) are

$$\tilde{Z}^{(r)} = (Z_{n1}, \dots, Z_{nr}) \quad \text{and} \quad \tilde{Q}^{(r)} = (Q_{n1}, \dots, Q_{nr}), \quad (2.3)$$

and a test statistic  $L_{nr} = (\tilde{Z}_n^{(r)}, \tilde{Q}_n^{(r)})$  depends on  $\tilde{Z}_n^{(r)}$  and  $\tilde{Q}_n^{(r)}$ .

In a PCS, the experiment is monitored from the beginning, so that at each failure  $Z_{nk}$ ,  $L_n(Z_n^{(r)}, Q_n^{(k)})$  is constructed; if  $L_{nk}$  favors a clear cut terminal decision, experimentation is curtailed at that time-point and if no such  $k(\leq r)$  exists then experimentation is stopped at the preplanned  $r$ -th failure  $Z_{nr}$ ; here,  $r$  may even be taken as  $n$ . Thus, in a PCS, one confronts a *time-sequential* testing procedure based on the partial sequence  $\{Z_n^{(k)}, Q_n^{(k)}; k \leq r\}$ , where neither the  $Z_{ni}$  nor the  $Q_{ni}$  are mutually independent, and hence, the  $L_{nk}$  may not have independent increments.

To introduce the PCQP's, we consider a multiparameter extension of the model considered in Sen (1979) and assume (for the time being) that the d.f.  $F_i$  has a continuous probability density function (p.d.f.)  $f_i$  (almost everywhere), where

$$f_i(x) = f(x; \Delta(\underline{c}_i - \bar{c}_n)'), \quad i = 1, \dots, n, \quad x \in R = (-\infty, \infty), \quad (2.4)$$

$\Delta$  is an  $m \geq q$  matrix of unknown parameters, the  $\underline{c}_i$  are specified  $q$ -vectors ( $m \geq 1, q \geq 1$ ) and  $\bar{c}_n = n^{-1} \sum_{i=1}^n \underline{c}_i$ . The two (as well as several) samples location-scale models and the multiple regression models are special cases of (2.4).

Suppose now that we intend to test for

$$H_0: \Delta = \underline{0} \quad \text{against} \quad H_1: \Delta \neq \underline{0}. \quad (2.5)$$

The likelihood function of  $(Z_n^{(k)}, Q_n^{(k)})$  is given by

$$L_{nk}(Z_n^{(k)}, Q_n^{(k)}) = \prod_{i=1}^k f(Z_{ni}; \Delta(\underline{c}_{Q_{ni}} - \bar{c}_n)'), \prod_{i=k+1}^n [1 - F(Z_{nk}; \Delta(\underline{c}_{Q_{ni}} - \bar{c}_n)')] \quad (2.6)$$

Let  $\Theta$  be an  $mq$ -dimensional open rectangle containing  $\underline{0}$  as an inner point,  $\Theta = ((\theta_{j\ell}))_{j=1, \dots, m; \ell=1, \dots, q}$  and let  $\{f(x; \underline{\theta}), \underline{\theta} \in \Theta\}$  be a family of absolutely continuous p.d.f.'s. For every  $x \in R$  and  $j (= 1, \dots, m)$ , we let

$$g_j(x) = -(\partial/\partial\theta_{j1}) \log f(x; \underline{\theta})|_{\underline{0}}, \quad (2.7)$$

$$\bar{G}_j(x) = [1 - F(x; \underline{0})]^{-1} \int_x^\infty g_j(z) dF(z; \underline{0}), \quad (2.8)$$

$$\underline{g}(x) = (g_1(x), \dots, g_m(x))' \quad \text{and} \quad \bar{\underline{G}}(x) = (\bar{G}_1(x), \dots, \bar{G}_m(x))'. \quad (2.9)$$

Then, by (2.4) through (2.9), we obtain that

$$\begin{aligned} T_{nk} &= -(\partial/\partial\Delta)\log L_{nk}(Z_n^{(k)}, Q_n^{(k)})|_{\Delta=\tilde{\Delta}} \\ &= \sum_{i=1}^k [g(Z_{ni}) - \bar{g}(Z_{nk})] (c_{Q_{ni}} - \bar{c}_n)', \quad 1 \leq k \leq n, \end{aligned} \quad (2.10)$$

and  $T_{n0} = 0$ . Proceeding as in (2.8) through (2.12) of Sen (1979), we may consider the related sequence

$$T_{nk}^* = \begin{cases} 0, & k = 0 \\ \sum_{i=1}^k g(Z_{ni}) [(c_{Q_{ni}} - \bar{c}_n) + \frac{1}{n-k} \sum_{s=1}^k (c_{Q_{ns}} - \bar{c}_n)]', & 1 \leq k \leq n-1 \\ T_{nn-1}, & k = n. \end{cases} \quad (2.11)$$

We shall find it more convenient to study the asymptotic properties of  $\{T_{nk}^*\}$ .

Let  $\{d_i, 1 \leq i \leq n\}$  be  $q$ -vectors for which  $\sum_{i=1}^n d_i = 0$  and

$$D_n = \sum_{i=1}^n d_i d_i' \text{ is positive definite (p.d.)} \quad (2.12)$$

Also, let  $Q = (Q_1, \dots, Q_n)$  assume each permutation of  $(1, \dots, n)$  with the common probability  $(n!)^{-1}$ . Note that for every  $k: 1 \leq k \leq n$ ,

$$\begin{aligned} & \sup_{\ell \neq 0} (\ell' D_n \ell)^{-\frac{1}{2}} \left| \sum_{i=1}^k \ell' d_{Q_i} \right| \\ &= \left[ \left( \sum_{i=1}^k d_{Q_i} \right)' D_n^{-1} \left( \sum_{i=1}^k d_{Q_i} \right) \right]^{\frac{1}{2}} = U_{nk}^*, \text{ say,} \end{aligned} \quad (2.13)$$

where

$$\begin{aligned} EU_{nk}^{*2} &= \text{Trace} \{ D_n^{-1} [ \left( \sum_{i=1}^k d_{Q_i} \right) \left( \sum_{i=1}^k d_{Q_i} \right)' ] \} \\ &= \text{Trace} \{ D_n^{-1} [ D_n k(n-k)/n(n-1) ] \} = qk(n-k)/n(n-1), \quad 1 \leq k \leq n. \end{aligned} \quad (2.14)$$

Further, proceeding as in the proof of Lemma 2.1 of Sen (1979), we claim that

$\{U_{nk} = (n-k)^{-1} U_{nk}^*, 1 \leq k < n\}$  is a non-negative submartingale. Hence, by a

direct vector-extension of the proof of Lemma 2.1 of Sen (1979), we arrive at the following:

Let  $\omega = \{\omega(t), 0 < t < 1\}$  be a continuous, non-negative, U-shaped and square integrable function [inside  $(0, 1)$ ] and  $\{d_i\}, Q$  be defined as in above. Then,

$$P\left\{\sup_{\ell \neq 0} \max_{1 \leq k < n} \omega\left(\frac{k}{n}\right) (\ell' D_{\sim n} \ell)^{-1/2} \left| \sum_{i=1}^k \ell' d_{\sim n} Q_i \right| \geq 1\right\} \leq q \int_0^1 \omega^2(t) dt. \quad (2.15)$$

Note that Lemma 2.2 of Sen (1979) also directly extends to the case where both  $g$  and  $\bar{G}$  are  $q$ -vectors (as is the case here), and hence, by using (2.15) and some standard inequalities, we obtain that under (2.2) and  $H_0$  in (2.5), as  $n \rightarrow \infty$ ,

$$\max_{k \leq r} \sup_{\substack{a: a' a = 1 \\ \sim}} \sup_{\ell \neq 0} |a' (T_{\sim nk} - T_{\sim nk}^*) \ell| (\ell' C_{\sim n} \ell)^{1/2} \xrightarrow{p} 0, \quad (2.16)$$

where

$$C_{\sim n} = \sum_{i=1}^n (c_i - \bar{c}_n)(c_i - \bar{c}_n)' \text{ is assumed to be p.d.} \quad (2.17)$$

(2.16) provides the justification for replacing  $\{T_{\sim nk}^*\}$  by  $\{T_{\sim nk}^*\}$ . Also, to be more general, we consider a class of  $\underline{h}(x) = (h_1(x), \dots, h_m(x))'$ ,  $x \in R$ , (quite arbitrary and need not be  $g$ , defined by (2.7)), where (i) each  $h_j(x)$  is assumed to be expressible as a difference of two nondecreasing functions and (ii)

$$\underline{v} = \int_{-\infty}^{\infty} [\underline{h}(x)][\underline{h}(x)]' dF(x) \text{ is assumed to be p.d. and finite;} \quad (2.18)$$

here  $F(x) = F(x; 0)$ . Further, let  $\xi_\alpha: F(\xi_\alpha) = \alpha$ ,  $0 < \alpha < 1$  and let

$$\underline{v}_\alpha = \int_{-\infty}^{\xi_\alpha} [\underline{h}(x)][\underline{h}(x)]' dF(x) + (1 - \alpha)^{-1} \left( \int_{-\infty}^{\xi_\alpha} \underline{h}(x) dF(x) \right) \left( \int_{-\infty}^{\xi_\alpha} \underline{h}(x) dF(x) \right)'. \quad (2.19)$$

Now, in (2.11), we replace the  $g(Z_{ni})$  by  $\underline{h}(Z_{ni})$  and post-multiply the matrix by  $C_{\sim n}^{-1/2}$ . The resulting matrix is denoted by  $T_{\sim nk}^{**}$  i.e.,

$$T_{\sim nk}^{**} = (T_{\sim nk}^* |_{g=\underline{h}}) C_{\sim n}^{-1/2}, \text{ for } k = 0, 1, \dots, n. \quad (2.20)$$

We roll out  $T_{\sim nk}^{**}$  into an  $m$ -vector and denote it by  $\tilde{T}_{\sim nk}^{**}$ ,  $0 \leq k \leq n$ .

### 3. INVARIANCE PRINCIPLES FOR $\{\tilde{T}_{\sim nk}^{**}; 0 \leq k \leq r\}$

First, we consider the case of the null hypothesis  $H_0: X_1, \dots, X_n$  are independent and identically distributed (i.i.d.) random variables with an absolutely continuous d.f.  $F$  (and p.d.f.f.). Then, one denoting by  $E_0$

the expectation under  $H_0$ , we have

$$E_0(\tilde{T}_{nk}^{**}) = 0, \quad \forall 0 \leq k \leq 1, \quad (3.1)$$

and proceeding as in the proof of Lemma 3.1 of Sen (1979), we obtain that

$$[k/n \rightarrow \alpha] \Rightarrow E_0\{(\tilde{T}_{nk}^{**})(\tilde{T}_{nk}^{**})'\} \rightarrow \underset{\sim}{V}_\alpha \times \underset{\sim}{I}_q, \quad (3.2)$$

for every  $0 < \alpha < 1$ , where  $\underset{\sim}{V}_\alpha$  is defined by (2.19).

For every  $n (\geq 1)$  and  $r$ , satisfying (2.2), we introduce an  $mq$ -variate stochastic process  $\underset{\sim}{W}_n = \{\underset{\sim}{W}_n(t), t \in [0, 1]\}$  by letting

$$\underset{\sim}{W}_n(t) = \tilde{T}_{nn}^{**}(t), \quad n(t) = \max\{k: k/n \leq pt\}, \quad t \in [0, 1]. \quad (3.3)$$

Thus,  $\underset{\sim}{W}_n$  belongs to the  $D^{mq}[0, 1]$  space, endowed with the (extended) Skorokhod  $J_1$ -topology. Also, let  $\underset{\sim}{W} = \{\underset{\sim}{W}(t), t \in [0, 1]\}$  be an  $mq$ -variate Gaussian function on  $[0, 1]$ , such that  $E\underset{\sim}{W}(t) = 0, \forall t \in [0, 1]$  and

$$E[(\underset{\sim}{W}(t))(\underset{\sim}{W}(s))'] = \underset{\sim}{V}_p(s \wedge t) \times \underset{\sim}{I}_q, \quad \forall s, t \in [0, 1]. \quad (3.4)$$

Then, the main theorem of this section is the following, where we impose the following condition: as  $n \rightarrow \infty$ ,

$$\max_{1 \leq k \leq n} (\underset{\sim}{c}_k - \bar{\underset{\sim}{c}}_n)' \underset{\sim}{C}^{-1} (\underset{\sim}{c}_k - \bar{\underset{\sim}{c}}_n) \rightarrow 0. \quad (3.5)$$

[(3.5) may be regarded as the vector-version of the classical Noether condition.]

Theorem 3.1. Under  $H_0$ , (2.2) and (3.5), as  $n \rightarrow \infty$ ,

$$\underset{\sim}{W}_n \xrightarrow{\mathcal{D}} \underset{\sim}{W}, \quad \text{in the extended } J_1\text{-topology on } D^{mq}[0, 1]. \quad (3.6)$$

Proof: Since this theorem is a multiparameter generalization of Theorem 1 of Sen (1979), we shall only provide a sketch of the proof by repeated cross reference to the proof in Sen (1979). We need to show that (i) the finite-dimensional distributions (f.d.d.) of  $\{\underset{\sim}{W}_n\}$  converge to those of  $\underset{\sim}{W}$  and (ii)  $\{\underset{\sim}{W}_n\}$  is tight. Let  $B_{nk} = \mathcal{B}(Z_n^{(k)}, Q_n^{(k)})$  and  $B_{nk}^* = \mathcal{B}(Z_n^{(n)}, Q_n^{(k)})$  be respectively the  $\sigma$ -fields generated by  $(Z_n^{(k)}, Q_n^{(k)})$  and  $(Z_n^{(n)}, Q_n^{(k)})$ , so that both are nondecreasing in  $k (0 \leq k \leq n)$ . Then, repeating the proof of Lemma 3.2



of Sen (1979) for each coordinate of  $T_{nk}^{**}$ , we obtain that

$$E_0\{\tilde{T}_{nk+1}^{**} | \mathcal{B}_{nk}^*\} = \tilde{T}_{nk}^{**} \quad \text{a.e., } \forall 0 \leq k \leq n-1. \quad (3.7)$$

Thus, if we let

$$\hat{T}_{nk}^{**} = \tilde{T}_{nk}^{**} - \tilde{T}_{nk-1}^{**}, \quad 1 \leq k \leq n; \quad \hat{T}_{n0}^{**} = \tilde{T}_{n0}^{**} = 0, \quad (3.8)$$

and if  $\lambda_{nk}$ ,  $0 \leq k \leq n$  are arbitrary (non-stochastic) mq-vectors, then

$$E_0\{\lambda_{nk} \hat{T}_{nk}^{**} | \mathcal{B}_{nk-1}^*\} = E_0\{\lambda_{nk} \hat{T}_{nk}^{**} | \mathcal{B}_{nk-1}^*\} = 0, \quad \text{a.e. } \forall 1 \leq k \leq n. \quad (3.9)$$

Now, to prove (i), for any (fixed)  $d(\geq 1)$ ,  $(0 \leq) t_1 (< \dots <) t_d \leq 1$  and arbitrary  $\lambda_i$ ,  $1 \leq i \leq d$  (all mq-vectors), consider the linear compound

$$\begin{aligned} W_n^* &= \sum_{i=1}^d \lambda_i W_n(t_i) = \sum_{i=1}^d \lambda_i \tilde{T}_{nn}^{**}(t_i) \quad (\text{by (3.3)}) \\ &= \sum_{k \leq n} \left\{ \sum_{i=1}^d I(k \leq n(t_i)) \lambda_i \right\} \hat{T}_{nk}^{**} \quad (\text{by (3.8)}) \\ &= \sum_{k \leq n} \omega_{nk} \hat{T}_{nk}^{**} = \sum_{k \leq n} W_{nk}^*, \quad \text{say} \end{aligned} \quad (3.10)$$

where the  $\omega_{nk}$  depend on  $\lambda_1, \dots, \lambda_d$  and  $n(t_1), \dots, n(t_d)$ , and by (3.9),

$$E_0\{W_{nk}^* | \mathcal{B}_{nk-1}^*\} = 0 \quad \text{a.e., } \forall 1 \leq k \leq n. \quad (3.11)$$

For the  $W_{nk}^*$ , we may readily extend the proof of the Lemma 3.3 of Sen (1979) and show that under (2.18) and (3.5),

$$\sum_{k \leq n} E_0\{W_{nk}^{*2} | \mathcal{B}_{nk-1}^*\} \xrightarrow{p} \sigma_0^2, \quad \text{as } n \rightarrow \infty, \quad (3.12)$$

where  $\sigma_0^2$  (is the variance of  $\sum_{i=1}^d \lambda_i W(t_i)$ , see (3.4), and) is given by

$$\sigma_0^2 = \sum_{i=1}^d \sum_{j=1}^d \lambda_i (\lambda_j) \times I_q \lambda_j. \quad (3.13)$$

Further, by a direct extension (using the  $C_T$ -inequality, of the proof of Lemma 3.4 of Sen (1979), we obtain that for every  $\epsilon > 0$ ,

$$\sum_{k \leq n} E_0\{W_{nk}^{*2} I(|W_{nk}^*| > \epsilon | \mathcal{B}_{nk-1}^*)\} \xrightarrow{p} 0, \quad \text{as } n \rightarrow \infty. \quad (3.14)$$

Hence, the asymptotic normality of  $W_n^*$  follows from (3.10) through (3.14) and Dvoretzky's (1972) dependent central limit theorem. Finally, to prove (ii), we note that the *tightness* of each (of the mq) component (s) of  $\{W_n\}$  follows

from Theorem 1 of Sen (1979) and since  $m_q$  is fixed, the tightness of  $\{W_n\}$  is insured by the tightness of its  $m_q$  components, Q.E.D.

In the above theorem, by (2.2), we have limited  $r = [np] + 1$  for some  $p < 1$ . The remarks following Theorem 1 of Sen (1979) also apply to this more general case (coordinatewise) when we like to take  $p = 1$ .

Let us now consider the non-null case and conceive of a model similar to (2.4); but, we confine ourselves to local alternatives for which the limiting results are nondegenerate. We conceive of a triangular array  $\{d_{ni}, 1 \leq i \leq n; n \geq 1\}$  of  $q$ -vectors, define  $D_n = \sum_{i=1}^n d_{ni} d_{ni}'$  and assume that

$$\sum_{i=1}^n d_{ni} = 0, \forall n, \sup_n \text{Tr}(D_n) < \infty \quad (3.15)$$

$$D_n \text{ is p.d. for every } n (\geq n_0), \quad (3.16)$$

$$\lim_{n \rightarrow \infty} \{ \max_{1 \leq k \leq n} d_{nk}' D_n^{-1} d_{nk} \} = 0. \quad (3.17)$$

Let then  $\{K_n\}$  be a sequence of alternative hypotheses, where

$$K_n: f_i(x) = f(x; \beta d_{ni}), \quad 1 \leq i \leq n, \quad x \in R, \quad (3.18)$$

$\beta$  is an  $m \times q$  matrix of (fixed) unknown parameters (all finite) and  $f$  satisfies all the regularity conditions, stated after (2.6). We also define  $g, \bar{g}$  etc., as in (2.7) - (2.9). Let us also define  $\xi_\alpha$  ( $0 < \alpha < 1$ ) as in after

(2.18) and define

$$\zeta(\alpha) = \int_{-\infty}^{\xi_\alpha} (h(x))(g(x))' dF(x) + (1 - \alpha) \left( \int_{-\infty}^{\xi_\alpha} h(x) dF(x) \right) \left( \int_{-\infty}^{\xi_\alpha} g(x) dF(x) \right)'. \quad (3.19)$$

Further, we assume that the  $d_{ni}$  and  $c_i$  satisfy the condition that

$$\lim_{n \rightarrow \infty} \left( \sum_{i=1}^n d_{ni} (c_i - \bar{c}_n)' \right) C_n^{-1/2} = P \text{ exists} \quad (3.20)$$

Let then

$$M(\alpha) = [\zeta(\alpha)] \beta [P], \quad 0 < \alpha < 1. \quad (3.21)$$

We roll out  $\underline{M}(\alpha)$  into an  $m\alpha$ -vector and denote it by  $\underline{\mu}(\alpha)$ ,  $0 < \alpha < 1$ . Finally, we define  $r$  as in (2.2) with  $0 < p < 1$  and denote by

$$\underline{\mu} = \{\underline{\mu}_t = \underline{\mu}(pt), 0 \leq t \leq 1\}. \quad (3.22)$$

Then, we have the following

Theorem 3.2. Let  $\{\underline{W}_n\}$ ,  $\underline{W}$  and  $\underline{\mu}$  be defined as in (3.3), (3.4) and (3.22). then, under (2.2), (3.5), (3.15), (3.16), (3.17) and (3.20), for  $f$  having a finite Fisher information (matrix)  $\underline{I} = E[g(x)][g(x)]'$ ,

$$\underline{W}_n - \underline{\mu} \xrightarrow{D} \underline{W}, \text{ in the extended } J_1\text{-topology on } D^{mq}[0, 1]. \quad (3.23)$$

Proof: It follows from Hájek and Šidak (1967, Chapter VI, pp. 239-240) that under the assumed regularity conditions,  $P_n^*$ , the probability measure (for  $\underline{Z}_n^{(n)}, \underline{Q}_n^{(n)}$ ) under  $K_n$ , is contiguous to  $P_n^0$ , the same under  $H_0$ . Hence, as in the proof of Theorem 2 of Sen (1979), the tightness of  $\{\underline{W}_n\}$  under  $H_0$ , proved in Theorem 3.1, and the contiguity of  $\{P_n^*\}$  to  $\{P_n^0\}$  insure that tightness of  $\{\underline{W}_n\}$  under  $\{K_n\}$  as well. Hence, to prove the theorem, it suffices to show that the f.d.d.'s of  $\{\underline{W}_n - \underline{\mu}\}$  converge to those of  $\underline{W}$ . Suppose that in (2.8), we replace the  $g_j$  by  $h_j$ , denote the resulting quantities by  $\bar{H}_j$ ,  $1 \leq j \leq m$  and let  $\bar{H}(x) = (H_1(x), \dots, H_m(x))'$ ,  $x \in R$ . Let then

$$\hat{T}_{nk}^* = \sum_{i=1}^n I(x_i \leq \xi_{k/n}) \{h(x_i) + \bar{H}(x_i)\} (c_{i-1} - \bar{c}_n)' c_{i-1}^{-1/2}, 0 \leq k \leq n, \quad (3.22)$$

where the  $\xi_{k/n}$  are defined by (2.19) for  $\alpha = k/n$ . Then, by repeating the proof of Theorem 2 of Sen (1979) for each coordinate of  $\hat{T}_{nk}^* - T_{nk}^{**}$ , we obtain that for any  $k: k/n \rightarrow \alpha: 0 < \alpha < 1$ , as  $n \rightarrow \infty$ ,

$$\hat{T}_{nk}^* - T_{nk}^{**} \xrightarrow{p} 0, \text{ under } \{K_n\}. \quad (3.25)$$

On the other hand,  $\hat{T}_{nk}^*$  involves independent summands (random matrices) and the classical multivariate central limit theorem along with a theorem of Behnen and Neuhaus (1975) yields that under  $\{K_n\}$ , for any (fixed)  $a \geq 1$  and  $0 \leq t_1 \leq \dots \leq t_a \leq 1$ , letting  $k_j = [npt_j] + 1$ ,  $1 \leq j \leq a$ ,  $\hat{T}_{nk_1}^*, \dots, \hat{T}_{nk_a}^*$  have (jointly)

asymptotically a  $m$ -variate normal distribution with means  $M(ptj)$ ,  $1 \leq j \leq a$ , defined by (3.21) and covariance functions conforming to that of  $W$ . Hence, (3.25) and (3.3) along with the above lead us to the desired result. Q.E.D.

We may remark that both Theorems 3.1 and 3.2 remain true if the  $T_{nk}^{**}$  are replaced by  $\tau_{nk} = E_0(T_{nk}^{**} | Q_n^{(k)})$  which amounts to replacing the  $h(Z_{ni})$  by  $E_0 h(Z_{ni}) = h_n(i)$ , say,  $1 \leq i \leq n$ . In this case,  $v_\alpha$  in (2.19) may also be replaced by

$$v_{n\alpha} = n^{-1} \sum_{i \leq n\alpha} [h_n(i)] [h_n(i)]' + n^{-2} (1 - \alpha)^{-1} (\sum_{i \leq n} h_n(i)) (\sum_{i \leq n} h_n(i))' \quad (3.26)$$

for  $0 < \alpha < 1$ ;  $v_{n0} = 0$ . Note that, by definition, the  $\tau_{nk}$  depends only on the  $Q_n^{(k)}$  and therefore are PCS rank statistics. The necessary modifications in the proofs are quite stright-forward, and hence, the details are omitted.

#### 4. ASYMPTOTIC TIME-SEQUENTIAL TESTS

The quantile processes, studies in the previous sections, depend, in general, on both  $Z_n$  and  $Q_n$ . For the simple regression model (i.e., in (2.4),  $m = q = 1$ ), Hájek (1963) considered an invariance principle related to the  $T_{nk}$ , where  $h(t) \equiv 1$ , wherein a *tied-down Wiener process* approximation was developed and the same was incorporated in the study of the asymptotic properties of some Kolmogorov-Smirnov, Cramér-von Mises' and Rényi type statistics (based on  $Q_n$  alone) for testing the hypothesis of no regression. Sinha and Sen (1979a) have developed a reverse-martingale approach to this problem, extended the Hajek results in a PCS setup and also considered the use of some weighted test statistics. Sinha and Sen (1979b) have also extended their theory to the multiple regression model, which is again a special case of (2.4) with  $m = 1$ ,  $q \geq 1$ . Their procedure is based on an invariance principle related to the tied-down Winer process in the vector case. We shall see later on that an alternative formulation of this problem involving the *Bessel processes* can be made by using our results in Sections 2 and 3.

We shall find it convenient to classify the PCQP's into two-types:

(1) *Type A PCQP's*: Here, as in the end of Section 3, we let  $h(Z_{nk})$  depend only on  $(k, n)$  (but not on  $Z_{nk}$ ), so that the  $T_{nk}^*$  becomes censored rank statistics. These statistics have been studied in detail by Chatterjee and Sen (1973), Majumdar and Sen (1978) and others. At the same time, the needed asymptotic theory also follows from our Theorems 3.1 and 3.2. These are generally distribution-free procedures (under  $H_0$ ).

(2) *Type B PCQP's*: Here the  $T_{nk}^*$  depend on both the  $Z_n^{(k)}$  and  $Q_n^{(k)}$ , so that one needs to estimate the  $v_\alpha$  in (2.18) - (2.19), and the test procedures are only asymptotically distribution-free. These procedures will be mainly discussed here.

Note that by (3.2) and the martingale property in (3.7), for every  $0 < \alpha < \alpha' < 1$ ,  $v_{\alpha'} - v_\alpha$  is positive semi-definite (p.s.d.). We consider the case of  $m=1$  first. For a Type A PCQP, for every  $\alpha \in (0, 1)$ ,  $v_\alpha$  is known and we may choose

$$L_{nk} = (T_{nk}^{**})' (v_p \times I_q)^{-1} (T_{nk}^{**}) = v^{-1} [T_{nk}^{**} T_{nk}^{**}]. \quad (4.1)$$

Let then  $L_{nr}^* = \max_{1 \leq k \leq r} L_{nk}$ . It follows from Theorem 3.1 that under  $H_0$ ,

$$L_{nr}^* \xrightarrow{D} \sup\{[W(t)]' [W(t)]: 0 \leq t \leq 1\} = W^{**}, \text{ say,} \quad (4.2)$$

where  $W^* = \{W^*(t) = (W(t))' (W(t)), t \in [0, 1]\}$  is a  $q$ -variate Bessel process.

Thus, if  $W_\alpha^{**}$  be the upper  $100\alpha\%$  point of the d.f. of  $W^{**}$ , then we have the following PCS testing procedure:

*At each failure (k), compute  $L_{nk}$ ; if, for the first time, for some  $k = N(\leq r)$ ,  $L_{nN}$  exceeds  $W_\alpha^{**}$ , stop experimentation along with the rejection of  $H_{01}$  and, if no such  $N$  exists, stop experimentation at the preplanned  $r$ -th failure along with the acceptance of  $H_0$ .*

It follows from (4.2) that the Type I error for this PCS procedure is asymptotically equal to  $\alpha$  ( $0 < \alpha < 1$ ). Also, from Theorem 3.2, we conclude that under the hypothesis of Theorem 3.2, the asymptotic power of the test is given by

$$P\{[\underline{W}(t) + \underline{\mu}_t]'[\underline{W}(t) + \underline{\mu}_t] > W_{\alpha}^{**}, \text{ for some } t \in [0, 1]\} \quad (4.3)$$

Further, it follows from Theorems 3.1 and 3.2 that for  $k/n \rightarrow \gamma$ ,  $0 < \gamma \leq p$ ,

$$P\{N > k | K_n\} \rightarrow P\{[\underline{W}(t) + \underline{\mu}_t]'[\underline{W}(t) + \underline{\mu}_t] \leq W_{\alpha}^{**}, \forall 0 \leq t \leq p^{-1}\gamma\}. \quad (4.4)$$

Thus, noting that  $Z_{nk} \rightarrow \xi_{\alpha}$  (a.s. as well as in the  $t$ -th mean,  $r \geq 1$ ) and that  $n(\xi_{k/n} - \xi_{(k-1)/n}) \rightarrow [f(\xi_{k/n})]^{-1}$  for  $k/n \rightarrow \gamma$ ,  $0 < \gamma \leq p$ , we obtain from (4.4) that the average stopping time is asymptotically equal to

$$\lim_{n \rightarrow \infty} E(Z_{nN} | K_n) = \int_0^1 \frac{p}{f(\xi_{p\theta})} P\{[W(t) + \mu_t]'[W(t) + \mu_t] \leq W_{\alpha}^{**}, 0 \leq t \leq \theta\} d\theta \quad (4.5)$$

A second type of test statistics (more common in RST procedures) may also be considered where we take

$$\tilde{L}_{nk} = v_{k/n}^{-1} [T_{nk}^{**} | T_{nk}]; \quad k \leq r. \quad (4.6)$$

Note that by (4.1) and (4.6),  $\tilde{L}_{nk} = L_{nk}(v_p/v_{k/n})$ . Let then  $\tilde{L}_{nr}^* = \max_{n_1 \leq k \leq r} \tilde{L}_{nk}$  where  $r_1 = [r\epsilon]$ , for some  $\epsilon > 0$ . Parallel to (4.2), we have for every  $0 < \epsilon < 1$  and under  $H_0$ ,

$$\tilde{L}_{nr}^* \xrightarrow{D} \sup\{t^{-1}W^*(t) : \epsilon' \leq t \leq 1\} = \tilde{W}_{\epsilon}^{**}, \text{ say,} \quad (4.7)$$

where  $\epsilon' = v_{p\epsilon}/v_p (> 0)$ . Hence, if  $\tilde{W}_{\epsilon, \alpha}^{**}$  be the upper 100 $\alpha$ % point of the d.f. of  $\tilde{W}_{\epsilon}^{**}$ , then, we may proceed as in the case of  $L_{nr}^*$  but replacing the  $L_{nk}$  and  $W_{\alpha}^{**}$  by  $\tilde{L}_{nk}$  and  $\tilde{W}_{\epsilon, \alpha}^{**}$ , respectively and starting the RST only when  $Z_{nr_1}$  is observed. The reason for choosing an  $\epsilon > 0$  is quite clear: as  $t \downarrow 0$ ,  $t^{-1}W^*(t)$  (or  $t^{-1}[\underline{W}_n(t)]'[\underline{W}_n(t)]$ ) does not behave regularly; in fact,  $t^{-1}W^*(t) \rightarrow \infty$  a.s., as  $t \downarrow 0$ . However, a small exclusion  $[0, \epsilon)$  eliminates this problem. Sinha and Sen (1979a) have discussed (for  $q=1$ ) the choice of  $\epsilon (> 0)$  in this context and a very similar picture holds for general  $q (\geq 1)$ . The asymptotic power of the test based on  $\tilde{L}_{nr}^*$  (for the local alternative

treated in Theorem 3.2) is given by

$$P\{t^{-1} [W(t) + \mu_t]' [W(t) + \mu_t] > \tilde{W}_{\epsilon, \alpha}^{**}, \text{ for some } t \in [\epsilon', 1]\}, \quad (4.8)$$

where  $\mu(t)$  and  $\epsilon'$  are defined as in before. Finally, the asymptotic value of the average stopping time is

$$\epsilon_{p\epsilon} + p \int_{\epsilon}^1 \frac{1}{f(\epsilon_p \theta)} P\{t^{-1} [W(t) + \mu_t]' [W(t) + \mu_t] \leq \tilde{W}_{\epsilon, \alpha}^{**}, \forall \epsilon' \leq t \leq \theta\} d\theta. \quad (4.9)$$

Let us now consider RST based on Type B PCQP's. Since  $v_{\alpha}$  is not known in advance, we shall find it convenient to use the second type of tests, described above, with the  $v_{k/n}$  being estimated by

$$V_{nk} = n^{-1} \sum_{i=1}^k h^2(Z_{ni}) + \frac{1}{n(n-k)} \left( \sum_{i=1}^k h(Z_{ni}) \right)^2, \quad 1 \leq k < n, \quad (4.10)$$

where the  $Z_{ni}$  are defined by (2.1). Thus, in (4.6), we need to replace  $v_{k/n}$  by  $V_{nk}$ , while the rest of the discussions in (4.7) through (4.9) remains true for this case too.

As has been noted earlier, the classical multiple regression model is a special case of (2.4) where  $m=1$  and  $q \geq 1$ . In this case, we have

$$f_i(x) = f(x - \Delta' (c_i - \bar{c}_n)), \quad x \in R, \quad i=1, \dots, n, \quad (4.11)$$

where  $\Delta$  is a  $q$ -vector of regression coefficients and the  $c_i$  are the known regression constants (vectors). Both the Type A and Type B procedures described before are applicable here to test for  $H_0: \Delta = 0$  against  $H_1: \Delta \neq 0$  (under progressive censoring).

We may further add that this multiple regression model includes as a special case the several sample location model: Suppose that there are  $k(=q+1)$  samples of sizes  $n_1, \dots, n_k$  respectively (where  $n = \sum_{i=1}^k n_i$ ) drawn from distributions  $F_1, \dots, F_k$  where

$$F_i(x) = F(x - \theta_i), \quad i=1, \dots, k, \quad x \in R, \quad (4.12)$$

and let  $\Delta_i = \theta_i - \theta_1$ , for  $i=2, \dots, k$ . Then, if we let  $c_1 = \dots = c_{n_1} = 0, c_{n_1+1} = \dots = c_{n_1+n_2} = (1, 0, \dots, 0)', \dots, c_{n-n_k+1} = \dots = c_n = (0, \dots, 0, 1)'$ , (4.12) corresponds to (2.4). Thus, the proposed Type A and Type B tests are applicable for testing the identity (under PCS) of  $F_1, \dots, F_k$  (against shift alternatives); PCS rank tests for this problem are due to Majumdar and Sen(1978). It may be remarked that instead of

the location/regression model in (4.11)-(4.12), one could have considered the regression model in the scale parameter which would have the several sample scale model as a special cases. The proposed procedures remain applicable for this scale model too.

We now proceed to the case of  $m > 1$ . The most notable case pertaining to this model [in (2.4)] is the so-called locatio-scale model where we allow both the location and scale parameters to **vary** (under the alternative hypothesis), retaining the identity of the d.f.'s under the null hypothesis. Though the theory developed earlier extends readily to the case of  $m \geq 1$ ,  $q \geq 1$ , for the Gaussian process  $\tilde{W}$  in Theorem 3.1 [see (3.4)],  $\{[\tilde{W}(t)]' [\tilde{v}_p \otimes \tilde{I}_q]^{-1} [\tilde{W}(t)], 0 \leq t \leq 1\}$  is, in general, not a Bessel process, and for this process, the distribution of the supremum may depend on the sequence  $\{\tilde{v}_\alpha : 0 \leq \alpha \leq p\}$  in a rather involved way, so that the computation of the critical values may pose a serious problem. The situation, however, becomes quite manageable, if we assume that

$$\tilde{v}_\alpha = \gamma_p(\alpha) \tilde{v}_p, \quad \forall 0 \leq \alpha \leq p, \quad (4.13)$$

where  $\gamma_p(\alpha)$  is a nondecreasing function of  $\alpha$  ( $0 \leq \alpha \leq p$ ), for every fixed  $p$  ( $0 < p < 1$ ). In such a case, in (4.1), we may replace  $v_p$  by  $\tilde{v}_p$ , while, in (4.6), we need to take  $\tilde{L}_{nk} = T_{nk}^{**'} [\tilde{v}_{k/n} \times \tilde{I}_q]^{-1} T_{nk}^{**}$ ; for Type B PCQP's,  $v_{k/n}$  need to be replaced by

$$\tilde{v}_{nk} = \frac{1}{n} \sum_{i=1}^k [h(z_{ni})][h(z_{ni})]' + \frac{1}{n(n-k)} \left[ \sum_{i=1}^k h(z_{ni}) \right] \left[ \sum_{i=1}^k h(z_{ni}) \right]', \quad (4.14)$$

for  $1 \leq k \leq n$ , [The Bessel process appearing in (4.2) [or (4.7)] will involve  $mq$  processes, instead of the  $q$  ones in the earlier cases.] The rest of the discussion will be the same. We conclude this section with the remark that Majumdar (1977), Majumdar and Sen (1978) and DeLong (1980) have considered the evaluation of the critical points  $W_\alpha^{**}$  and  $\tilde{W}_{\epsilon, \alpha}^{**}$ , for the  $k$ -parameter Bessel process for some typical  $k$  and these may be used for the actual RST based on the proposed PCQP's.



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