

RATE OF CONSISTENCY OF ONE SAMPLE TESTS OF LOCATION

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ABSTRACT

Let  $X_1, \dots, X_n$  be a sample from a population with continuous distribution function  $F(x - \theta)$  such that  $F(x) + F(-x) = 1$  and  $0 < F(x) < 1$ ,  $x \in \mathbb{R}^1$ . It is shown that the power-function of a monotone test of  $H: \theta = \theta_0$  against  $K: \theta > \theta_0$  cannot tend to 1 as  $\theta - \theta_0 \rightarrow \infty$  more than  $n$  times faster than the tails of  $F$  tend to 0. Some standard as well as robust tests are considered with respect to this rate of convergence.

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## 1. INTRODUCTION

Let  $X_1, \dots, X_n$  be independent random variables, identically distributed according to a distribution function  $F(x - \theta)$ , where  $F$  is a continuous distribution function satisfying

$$\begin{aligned} F(x) + F(-x) &= 1 \\ 0 < F(x) < 1 \end{aligned} \quad x \in \mathbb{R}^1; \quad (1.1)$$

$\theta$  is an unknown parameter. The problem is that of testing the hypothesis  $H: \theta = \theta_0$  against the alternative  $K: \theta > \theta_0$ .

The power-function of a consistent test tends to 1 as  $\theta - \theta_0 \rightarrow \infty$ . We shall consider the rate of this convergence and show that the rate at which the tails of the distribution  $F$  tend to 0 form a natural upper bound on the rate of convergence of the power-function of any monotone test. More precisely, we shall show that the probability of the error of the second kind tends to 0 at most  $n$ -times faster than the tails of the distribution.

Noting this, we are interested in the behavior of some well-known one-sided tests from this point of view. It turns up that the behavior of the standard  $t$ -test strongly depends on the underlying distribution. While its rate of convergence is near to the upper bound in the case of normal distribution, this convergence becomes very slow if the normal distribution is contaminated by a heavy-tailed distribution. The rate of convergence of the sign-test is always slightly below the middle of the range, whatever is the basic distribution, and similar is the situation of the robust version of the probability ratio test. The rate of convergence of the power-function of the one-sample Wilcoxon test is bounded from below as well as from above.

2. THE UPPER BOUND ON THE RATE OF CONSISTENCY

Let  $X_1, \dots, X_n$  be independent and identically distributed (i.i.d.) random variables with continuous distribution function  $F(x - \theta)$  such that  $F$  satisfies (1.1). Consider the tests of the hypothesis  $H: \theta = \theta_0$  against the alternative  $K: \theta > \theta_0$  which have the form

$$\psi_n(X) = \begin{cases} 1 & \text{if } T_n(X_1 - \theta_0, \dots, X_n - \theta_0) > C_\alpha^{(n)} \\ \gamma_n & \text{if } T_n = C_\alpha^{(n)} \\ 0 & \text{if } T_n < C_\alpha^{(n)} \end{cases}, \quad (2.1)$$

where  $C_\alpha^{(n)}$  and  $\gamma_n$  are determined by the condition

$$E_{\theta_0} \psi_n(X) = \alpha, \quad \text{where } \alpha \in (0, \frac{1}{2}). \quad (2.2)$$

Suppose that the statistic  $T_n(X_1, \dots, X_n)$  satisfies the conditions

$$T_n(X_1 - t, \dots, X_n - t) \text{ is nonincreasing in } t \quad (2.3)$$

and

$$[X^{(n)} < \theta_0] \implies T_n(X_1 - \theta_0, \dots, X_n - \theta_0) < C_n^{(\alpha)}, \quad (2.4)$$

where  $X^{(n)}$  is the  $n$ -th order statistic in the vector  $X^{(1)} \leq \dots \leq X^{(n)}$  of order statistics corresponding to  $X_1, \dots, X_n$ .

The following theorem gives an upper bound on the rate of convergence of the power-functions of tests satisfying (2.1) - (2.4).

Theorem 1. *Under the assumptions made above, it holds*

$$\lim_{\theta - \theta_0 \rightarrow \infty} B(T_n, \theta, \alpha) \leq n \quad (2.5)$$

where

$$B(T_n, \theta, \alpha) = \frac{-\log E_\theta (1 - \psi_n(X))}{-\log(1 - F(\theta - \theta_0))}. \quad (2.6)$$

Proof: It follows from the assumption (2.4) that

$$\begin{aligned} E_\theta (1 - \psi_n(X)) &\geq P_\theta \{T_n(X - \theta_0) < C_\alpha\} \\ &\geq P_\theta \{X^{(n)} < \theta_0\} = (1 - F(\theta - \theta_0))^n \end{aligned}$$

which implies (2.5). □

### 3. ONE-SAMPLE t-TEST

The standard one sample t-test has the form

$$\psi_n(\tilde{X}) = 1 \quad \text{if} \quad T_n(\tilde{X} - \theta_0) > t_\alpha^{(n-1)}, \quad (3.1)$$

where

$$T_n(\tilde{X} - \theta_0) = \sqrt{n-1}(\bar{X}_n - \theta_0)/S_n, \quad (3.2)$$

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i, \quad S_n^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X}_n)^2 \quad (3.3)$$

and  $t_\alpha^{(n-1)}$  is the upper  $\alpha$ -percentile of  $t$  distribution with  $(n-1)$  degrees of freedom.

The following theorem shows that, if the sample comes from a heavy-tailed distribution, the power-function of the t-test tends to 1 only as slowly as the tails of  $F$  tend to 0.

Theorem 2. Let  $X_1, \dots, X_n$  be a sample from a population with the continuous distribution function  $F(x - \theta)$  such that  $F$  satisfies (1.1) and

$$\lim_{x \rightarrow \infty} \frac{-\log(1 - F(x))}{m \cdot \log x} = 1 \quad \text{for some } m > 0. \quad (3.4)$$

Then

$$\lim_{\theta - \theta_0 \rightarrow \infty} \overline{B}(T_n, \theta, \alpha) \leq 1, \quad (3.5)$$

holds for any  $\alpha \in (0, \frac{1}{2})$  and for any fixed  $n \geq 3$ .

Proof: It holds

$$P_\theta \left\{ \frac{\bar{X}_n - \theta_0}{S_n} \sqrt{n-1} < t_\alpha \right\} = P_0 \left\{ \frac{S_n t_\alpha}{\sqrt{n-1}} - \bar{X}_n > \theta - \theta_0 \right\} \geq P_0 \left\{ -\bar{X}_n > \theta - \theta_0 \right\},$$

and it follows from Theorem 2.2, part (ii) of Jurečková (1979b) and from (3.4) that

$$\begin{aligned} & \overline{\lim}_{\theta - \theta_0 \rightarrow \infty} B(T_n, \theta, \alpha) \\ & \leq \overline{\lim}_{\theta - \theta_0 \rightarrow \infty} \left[ \frac{-(n-1) \log F(\theta - \theta_0)}{-\log(1 - F(\theta - \theta_0))} + \frac{-\log[1 - F((2n-1)(\theta - \theta_0))]}{-\log(1 - F(\theta - \theta_0))} \right] \leq 1. \quad \square \end{aligned}$$

The next theorem shows that, in the case of normal population, the rate of convergence of the t-test is near to the upper bound given in Theorem 1.

Theorem 3. Let  $X_1, \dots, X_n$  be a sample from the normal distribution  $N(\theta, \sigma^2)$  and let  $T_n$  be given by (3.2) and (3.3). Then

$$\lim_{\theta - \theta_0 \rightarrow \infty} B(T_n, \theta, \alpha) \geq n \cdot \left( 1 + \frac{t_{\alpha}^{(n-1)}}{\sqrt{n-1}} \right)^{-2}, \quad (3.6)$$

holds for any  $\alpha \in (0, \frac{1}{2})$  and  $n \geq 3$ .

Proof: The tails of the normal distribution are exponentially decreasing, more precisely, it holds

$$\lim_{x \rightarrow \infty} \frac{-\log(1 - F(x))}{\frac{x^2}{2\sigma^2}} = 1. \quad (3.7)$$

Using Markov inequality, we can write for any  $\varepsilon$ ,  $0 < \varepsilon < 1$ ,

$$\begin{aligned} E_{\theta}(1 - \psi_n(X)) &= P_{\theta} \left\{ S_n \frac{t_{\alpha}}{\sqrt{n-1}} - \bar{X}_n > \theta - \theta_0 \right\} \\ &\leq P_{\theta} \left\{ S_n \frac{t_{\alpha}}{\sqrt{n-1}} + |\bar{X}_n| > \theta - \theta_0 \right\} \\ &\leq P_{\theta} \left\{ \left( \frac{1}{n} \sum_{i=1}^n X_i^2 \right)^{\frac{1}{2}} \left( 1 + \frac{t_{\alpha}}{\sqrt{n-1}} \right) > \theta - \theta_0 \right\} \end{aligned}$$

$$\begin{aligned}
 &\leq \exp \left\{ -\frac{(\theta - \theta_0)^2}{2\sigma^2} n(1 - \varepsilon) \left( 1 + \frac{t_\alpha}{\sqrt{n-1}} \right)^{-2} \right\} \cdot E_0 \left[ \exp \left\{ \frac{n(1 - \varepsilon)}{2\sigma^2} \cdot \frac{1}{n} \sum_{i=1}^n X_i^2 \right\} \right] \\
 &= \exp \left\{ -\frac{(\theta - \theta_0)^2}{2\sigma^2} n(1 - \varepsilon) \left( 1 + \frac{t_\alpha}{\sqrt{n-1}} \right)^{-2} \right\} \cdot 2^{\frac{n}{2}} \Gamma \left( \frac{n}{2} \right) \int_0^\infty e^{-\frac{\varepsilon y}{2}} y^{\frac{n}{2} - 1} dy \\
 &= \varepsilon^{-\frac{n}{2}} \exp \left\{ -\frac{(\theta - \theta_0)^2}{2\sigma^2} n(1 - \varepsilon) \left( 1 + \frac{t_\alpha}{\sqrt{n-1}} \right)^{-2} \right\},
 \end{aligned}$$

so that

$$\lim_{\theta - \theta_0 \rightarrow \infty} \left\{ -\log[E_\theta(1 - \psi_n(\tilde{X}))] \cdot 2\sigma^2(\theta - \theta_0)^{-2} \right\} \geq n(1 - \varepsilon) \left( 1 + \frac{t_\alpha}{\sqrt{n-1}} \right)^{-2}, \quad (3.8)$$

holds for any  $\varepsilon$ ,  $0 < \varepsilon < 1$ ; this implies (3.6). □

Remark. Suppose that  $X_1, \dots, X_n$  is a sample from the population with distribution function  $F(x - \theta)$  such that

$$F(x) = (1 - \lambda)\Phi\left(\frac{x}{\sigma}\right) + \lambda G(x), \quad x \in \mathbb{R}^1, \quad (3.9)$$

where  $\Phi$  is the standard normal distribution function,  $\sigma > 0$ ,  $\lambda$  is a fixed number,  $0 < \lambda < 1$ , and  $G$  is a heavy-tailed absolutely continuous distribution function satisfying (3.4). We could then interpret  $F$  as a normal distribution function contaminated by a heavy-tailed distribution, and it follows from Lemma 2.1 of Jurečková (1979b), that  $F$  is again heavy-tailed. It means that, for a distribution satisfying (3.9), t-test has a slowly increasing power function in the sense of (3.5).

#### 4. SIGN TEST

Let  $X_1, \dots, X_n$  be i.i.d. random variables and let  $T_n(X_1 - \theta_0, \dots, X_n - \theta_0)$  denote the number of positive components among  $X_1 - \theta_0, \dots, X_n - \theta_0$ . If distribution function  $F$  is continuous and symmetric then, under

H:  $\theta = \theta_0$ ,  $T_n(X_1 - \theta_0, \dots, X_n - \theta_0)$  has the binomial distribution  $b(\frac{1}{2}, n)$ .

Let  $C_\alpha^{(n)}$  denote the largest integer satisfying

$$\left(\frac{1}{2}\right)^n \sum_{i=0}^{C_\alpha^{(n)}-1} \binom{n}{i} \leq 1 - \alpha; \quad (4.1)$$

the sign test then rejects H if  $T_n(\tilde{X} - \theta_0) > C_\alpha^{(n)}$ .

The following theorem shows that the power-function of the sign test tends to one approximatively  $\left\{n - C_\alpha^{(n)}\right\}$  times faster than the tails of F tend to zero, whatever is the basic distribution.

Theorem 4. Let  $X_1, \dots, X_n$  be a sample from a population with distribution function  $F(x - \theta)$  such that F is continuous and satisfies

(1.1). Then

$$n - C_\alpha^{(n)} \leq \lim_{\theta - \theta_0 \rightarrow \infty} B(T_n, \theta, \alpha) \leq \overline{\lim}_{\theta - \theta_0 \rightarrow \infty} B(T_n, \theta, \alpha) \leq n - C_\alpha^{(n)} + 1 \quad (4.2)$$

holds for any  $\alpha \in (0, \frac{1}{2})$  and any fixed n.

Proof: If  $\theta$  is the true parameter value then  $T_n(\tilde{X} - \theta_0)$  has the binomial distribution  $b(p, n)$  with  $p = F(\theta - \theta_0)$ , so that

$$\begin{aligned} E_\theta(1 - \psi_n(\tilde{X})) &\geq \sum_{i=0}^{C_\alpha^{(n)}-1} \binom{n}{i} (F(\theta - \theta_0))^i (1 - F(\theta - \theta_0))^{n-i} \\ &\geq \left[ C_\alpha^{(n)} - 1 \right] (F(\theta - \theta_0))^{C_\alpha^{(n)}-1} (1 - F(\theta - \theta_0))^{n - C_\alpha^{(n)} + 1} \end{aligned}$$

which implies

$$\overline{\lim}_{\theta - \theta_0 \rightarrow \infty} \frac{-\log E_\theta(1 - \psi_n(\tilde{X}))}{-\log(1 - F(\theta - \theta_0))} \leq n - C_\alpha^{(n)} + 1. \quad (4.3)$$

Analogously,



$$\begin{aligned}
 E_{\theta}(1 - \psi_n(\tilde{X})) &\leq \sum_{i=0}^{C_{\alpha}^{(n)}} \binom{n}{i} (F(\theta - \theta_0))^i (1 - F(\theta - \theta_0))^{n-i} \\
 &\leq (F(\theta - \theta_0))^{C_{\alpha}^{(n)}} (1 - F(\theta - \theta_0))^{n - C_{\alpha}^{(n)}} \sum_{i=0}^{C_{\alpha}^{(n)}} \binom{n}{i}
 \end{aligned} \tag{4.4}$$

holds for  $\theta - \theta_0 > K$ , where the last inequality follows from the fact that, for sufficiently large  $\theta$ ,

$$1 < \left[ \frac{F(\theta - \theta_0)}{1 - F(\theta - \theta_0)} \right]^i \leq \left[ \frac{F(\theta - \theta_0)}{1 - F(\theta - \theta_0)} \right]^{C_{\alpha}^{(n)}} \tag{4.5}$$

holds for  $0 \leq i \leq C_{\alpha}^{(n)}$ . (4.4) then implies

$$\lim_{\theta - \theta_0 \rightarrow \infty} \frac{-\log E_{\theta}(1 - \psi_n(\tilde{X}))}{-\log(1 - F(\theta - \theta_0))} \geq n - C_{\alpha}^{(n)}. \quad \square$$

## 5. WILCOXON ONE-SAMPLE TEST

Denote

$$T_n(\tilde{X} - \theta_0) = \sum_{i=1}^n \text{sign}(X_i - \theta_0) R_i^+(|X_i - \theta_0|) \tag{5.1}$$

where  $R_i^+(|X_i - \theta_0|)$  is the rank of  $|X_i - \theta_0|$  among  $|X_1 - \theta_0|, \dots, |X_n - \theta_0|$ .

Consider the Wilcoxon one-sample test with critical values based on the normal approximation; this test has the form

$$\psi_n(\tilde{X}) = \begin{cases} 1 & \text{if } T_n(\tilde{X} - \theta_0) > \sigma_n \tau_{\alpha} \\ \gamma & \text{if } T_n = \sigma_n \tau_{\alpha} \\ 0 & \text{if } T_n < \sigma_n \tau_{\alpha} \end{cases} \tag{5.2}$$

where

$$\sigma_n^2 = \frac{1}{6}n(n+1)(2n+1), \tag{5.3}$$

$$\tau_{\alpha} = \Phi^{-1}(1 - \alpha) \tag{5.4}$$

and  $\Phi$  is the standard normal distribution function.

We shall show that the rate of convergence of the Wilcoxon test is bounded from below as well as from above; this means that the test is neither too slow nor too fast, whatever is the basic distribution.

Theorem 5. Let  $X_1, \dots, X_n$  be a sample from a population with continuous distribution function  $F(x - \theta)$  such that  $F$  satisfies

$$(1.1). \text{ Then, for } \alpha \leq 0.15 \text{ and } n \geq \left[ \frac{\tau_\alpha^2}{3(3 - 2\sqrt{2})} \right] + 1 = n_0(\alpha), \text{ it holds}$$

$$a_n(\alpha) \leq \lim_{\theta \rightarrow \theta_0} B(T_n, \theta, \alpha) \leq \overline{\lim}_{\theta \rightarrow \theta_0} B(T_n, \theta, \alpha) \leq b_n, \quad (5.5)$$

where

$$a_n(\alpha) = \left[ \frac{1}{\sqrt{2}} \left\{ n(\sqrt{2} - 1) - \sqrt{\frac{n}{3}} \tau_\alpha \right\} \right] \quad (5.6)$$

and

$$b_n = [0.8n] + 1; \quad (5.7)$$

$[x]$  denotes the integer part of  $x$ .

Proof: Let  $S_n$  denote the number of positive components among  $X_1 - \theta_0, \dots, X_n - \theta_0$ . Then

$$E_\theta(1 - \psi_n(\tilde{X})) \leq P_0(S_n \leq d_n) \quad (5.8)$$

where  $d_n = \frac{1}{\sqrt{2}}(n + \sqrt{\frac{n}{3}} \tau_\alpha)$ .

Actually, if  $S_n > d_n$ , then for  $\alpha \leq 0.15$  and  $n \geq n_0(\alpha)$ ,

$$\begin{aligned} T_n(\tilde{X} - \theta_0) &\geq \frac{1}{2} S_n(S_n + 1) - \frac{1}{2} (n - S_n)(S_n + n + 1) \\ &= S_n(S_n + 1) - \frac{n(n + 1)}{2} \end{aligned}$$

$$\geq \frac{\tau_\alpha}{\sqrt{6}} \left[ 2n^3 + (2\sqrt{2} + \frac{\tau_\alpha^2}{6} + \frac{2\tau_\alpha}{\sqrt{3}})n^2 + (1 + \frac{2\tau_\alpha}{\sqrt{6}})n \right]$$

$$\geq \tau_\alpha \sigma_n,$$

which together with (5.2) implies (5.8) and moreover,

$$E_\theta(1 - \psi_n(\underline{X})) \leq P_\theta(X^{([n-dn])} - \theta_0 < 0)$$

and this gives the first inequality in (5.5). On the other hand, we have

$$E_\theta(1 - \psi_n(\underline{X})) \geq P_\theta\{T(\underline{X} - \theta_0) < 0\}$$

and the rest of the proof follows from Theorem 3.3 of Jurečková (1979b).

## 6. A ROBUST TEST

Let  $X_1, \dots, X_n$  be i.i.d. random variables, distributed according to a continuous distribution function  $F(x - \theta)$ , where  $F(x)$  satisfies (1.1).

If  $F$  is not fully specified but is supposed to belong to a neighborhood of a given symmetric unimodal distribution (e.g., of the normal distribution) then we may use the following robust test, less sensitive to the gross errors; put

$$T_n(\underline{X} - \theta_0) = \frac{\sum_{i=1}^n h(X_i - \theta_0)}{\left[ \sum_{i=1}^n h^2(X_i - X_n) \right]^{1/2}}, \quad (6.1)$$

where  $h: R^1 \rightarrow R^1$  is a nondecreasing, skew-symmetric and continuous function such that

$$h(x) = h(k) \quad \text{for } x > k$$

where  $k > 0$  is a given constant. The test then has the following form

$$\psi_n(X) = \begin{cases} 1 & \text{if } T_n(X - \theta_0) > \tau_\alpha \\ \gamma_n & \text{if } T_n = \tau_\alpha \\ 0 & \text{if } T_n < \tau_\alpha \end{cases}$$

where  $\tau_\alpha = \Phi^{-1}(1 - \alpha)$ .

The following theorem shows that in the case of heavy tailed distribution as well as in the case of distribution with exponentially decreasing tails, the rate of convergence of the power-function of (6.2) is always slightly below the middle of the range.

Theorem 6. Let  $X_1, \dots, X_n$  be a sample from a population with continuous distribution function  $F(x - \theta)$  such that  $F$  satisfies (1.1) and moreover, it satisfies either (3.4) for some  $m > 0$  or

$$\lim_{x \rightarrow \infty} \frac{-\log(1 - F(x))}{b \cdot x^r} = 1 \text{ for some } b > 0, r > 0. \quad (6.3)$$

Then, under the above assumptions,

$$a'_n(\alpha) \leq \lim_{\theta - \theta_0 \rightarrow \infty} B(T_n, \theta, \alpha) \leq \overline{\lim}_{\theta - \theta_0 \rightarrow \infty} B(T_n, \theta, \alpha) \leq b'_n \quad (6.4)$$

holds for  $\alpha \in (0, \frac{1}{2})$  and  $n > \tau_\alpha^2$ , where

$$a'_n(\alpha) = \left[ \frac{1}{2}(n - \tau_\alpha \sqrt{n}) \right] \quad (6.5)$$

and

$$b'_n = \frac{n+1}{2}. \quad (6.6)$$

Proof: Let  $Q_n$  denote the number of  $i$ 's, for which  $X_i - \theta_0 > k$ ,

$i = 1, 2, \dots, n$ . Then

$$\begin{aligned} E_\theta(1 - \psi_n(X)) &\leq P_\theta \left\{ \sum_{i=1}^n h(X_i - \theta_0) \leq \tau_\alpha \sqrt{n} \cdot h(k) \right\} \\ &\leq P_\theta \left\{ Q_n \leq \frac{n}{2} + \frac{\sqrt{n}}{2} \tau_\alpha \right\}. \end{aligned} \quad (6.7)$$

Actually, if  $Q_n > \frac{1}{2}(n + \sqrt{n} \tau_\alpha)$ , then

$$\sum_{i=1}^n h(X_i - \theta_0) > \frac{1}{2}(n + \tau_\alpha \sqrt{n}) h(k) - \left(\frac{n}{2} - \frac{\tau_\alpha}{2} \sqrt{n}\right) h(k) = \tau_\alpha \sqrt{n} h(k).$$

It follows from (6.7) that

$$E_\theta(1 - \psi_n(\tilde{X})) \leq P_\theta(X^{(s)} - \theta_0 < k)$$

where  $s = \lfloor \frac{1}{2}(n - \tau_\alpha \sqrt{n}) \rfloor$ ; this implies the first part of (6.4). On the other hand,

$$E_\theta(1 - \psi_n(\tilde{X})) \geq P_\theta \left\{ \sum_{i=1}^n h(X_i - \theta_0) \leq 0 \right\}$$

and this gives the second inequality in (6.4) similarly as in Theorem 3.2 of Jurečková (1979b). □

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