

RANK ANALYSIS OF COVARIANCE UNDER PROGRESSIVE CENSORING, II*

by

Pranab Kumar Sen

Department of Biostatistics
University of North Carolina at Chapel Hill

Institute of Statistics Mimeo Series No. 1275

MARCH 1980

RANK ANALYSIS OF COVARIANCE UNDER PROGRESSIVE CENSORING, II*

by

PRANAB KUMAR SEN

Department of Biostatistics
University of North Carolina, Chapel Hill, N.C., U.S.A.

SUMMARY

For some general analysis of covariance models, a class of progressively censored nonparametric tests based on suitable rank order statistics is considered. The proposed procedure is a generalization of an one (for a simpler model) in Sen (1979b). Along with some invariance principles for progressively censored (multivariate) linear rank statistics, asymptotic properties of the proposed tests are studied.

1. INTRODUCTION

In *clinical trials* or *life testing problems*, a *progressive censoring scheme* (PCS) incorporates a *continuous monitoring* of experimentation from the beginning with the objective of an *early termination* (contingent on the accumulating statistical evidence) without increasing the margin of the *type I error*. Chatterjee and Sen (1973) have formulated a general class of PCS *nonparametric tests* for a simple regression model (which includes the classical two-sample problem as a

*Work supported by the National Heart, Lung and Blood Institute, Contract No. NIH-NHLBI-71-2243 from the National Institutes of Health.

special case); their theory rests on some *invariance principles* for some PCS linear rank statistics. Sen (1979a) has developed some invariance principles for some related quantile processes arising in PCS; some generalizations of these are due to Sen (1980). Sen (1979b) has incorporated *concomitant variates* in the simple regression model and studied some PCS *analysis of covariance* (ANOCOVA) tests; references to other relevant works are cited in these papers. PCS nonparametric tests for *multiple regression* (containing the several sample problem as a special case) have been considered by Majumdar and Sen (1978). The object of the present investigation is to incorporate (stochastic) concomitant variates in some general multiparameter models and to formulate appropriate PCS rank tests for such general ANOCOVA models.

Section 2 deals with the basic ANOCOVA model and the preliminary notions. The proposed PCS tests are then developed in Section 3. Section 4 is devoted to the asymptotic distribution theory of the allied test-statistics, both under the null and some local alternative hypotheses. Since our proposed procedure is a natural extension of Sen (1979b), in the sequel, often, to minimize the technical manipulations, we shall omit some details by suitable cross references to the earlier paper. Some general remarks are made in the concluding section.

2. PRELIMINARY NOTIONS

Let $X_i^* = (X_{0i}, X_{1i}, \dots, X_{pi})' = (X_{0i}, X_i')'$, $i = 1, \dots, n$ be independent random vectors (r.v.) with continuous $(p+1)$ -variate distribution functions (d.f.) F_i^* , $i = 1, \dots, n$, where the X_{0i} are the *primary variates* with (marginal) d.f. F_{0i} , defined on $E = (-\infty, \infty)$ and the X_i' are the *concomitant variates* with marginal (joint) d.f.

F_i , defined on E^p , for some $p \geq 1$. Let $F_i^0(y|\underline{x})$ be the conditional d.f. of X_{0i} , given $\underline{x}_i = \underline{x}$, $i = 1, \dots, n$, and, as is usually the case in an ANOCOVA model, we assume that $F_1 \equiv \dots \equiv F_n \equiv F$ (unknown). Our basic problem is to test for the null hypothesis

$$H_0: F_1^0 \equiv \dots \equiv F_n^0 \equiv F^0 \text{ (unknown)}, \quad (2.1)$$

against an alternative that they are not all the same. Generalizing the model in Sen (1979b), we let (for every $y \in E$, $\underline{x} \in E^p$):

$$F_i^0(y|\underline{x}) = F^0(y; \underline{\Delta}(\underline{c}_i - \bar{\underline{c}}_n) | \underline{x}), \quad i = 1, \dots, n, \quad (2.2)$$

where $\underline{\Delta}$ is an $m \times q$ matrix ($m \geq 1$, $q \geq 1$) of unknown parameters, the \underline{c}_i are specified q -vectors and $\bar{\underline{c}} = n^{-1} \sum_{i=1}^n \underline{c}_i$. The classical ANOCOVA model relating to the one way layout is a special case of (2.2)

where $m = 1$, $q \geq 1$ and $F^0(y; \underline{\Delta}(\underline{c}_i - \bar{\underline{c}}_n) | \underline{x}) \equiv F^0(y - \underline{\Delta}(\underline{c}_i - \bar{\underline{c}}_n) | \underline{x})$,

$\forall i \geq 1$. Further, in this special case, we have $k (= q + 1)$ samples of sizes n_1, \dots, n_k , respectively ($n = \sum_{i=1}^k n_i$), $\underline{c}_i = \dots = \underline{c}_{n_1} = \underline{0}$, $\underline{c}_{n_1+1} = \dots = \underline{c}_{n_1+n_2} = (1, 0, \dots, 0)'$, \dots , $\underline{c}_{n_1+\dots+n_{q-1}+1} = \dots = \underline{c}_{n_1+\dots+n_k} = (0, \dots, 0, 1)'$ and $\underline{\Delta}$ stands for the vector of treatment-effects.

More general models may be conceived by allowing the location and scales to vary under alternatives. Now, under (2.2), we may recast (2.1) as

$$H_0: \underline{\Delta} = \underline{0} \text{ against } H_1: \underline{\Delta} \neq \underline{0}. \quad (2.3)$$

In a life testing situation, though the concomitant variates $\underline{x}_1, \dots, \underline{x}_n$ may be observable at the beginning of the experimentation, the primary variates are not so. If $Z_{n1}^0 < \dots < Z_{nn}^0$ be the ordered rv's corresponding to X_{01}, \dots, X_{0n} (ties neglected, with probability 1, by virtue of the assumed continuity of F_0) and if we define the anti-ranks S_1, \dots, S_n by

$$Z_{nj}^0 = X_{0S_j}, \quad \text{for } j = 1, \dots, n, \quad (2.4)$$

then, at the k -th failure Z_{nk}^0 , the observable rv's are

$$\underline{Q}_i = (S_i, Z_{ni}^0, \underline{X}_{S_i}), \quad i=1, \dots, k, \quad \text{for } k=1, \dots, n. \quad (2.5)$$

[Though the complementary sets of concomitant variates are known, their anti-ranks are not specified, in advance.]

Nonparametric ANOCOVA tests (based on the entire set $\{\underline{X}_1^*, \dots, \underline{X}_n^*\}$) have been considered by Quade (1967), Puri and Sen (1969a) and Sen and Puri (1970), among others. Under PCS, a special case of a simple regression model, has been studied by Sen (1979b). In this paper, the general model in (2.2) (for $m \geq 1, q \geq 1$) will be considered. We introduce the following notations. Let R_{ji} be the rank of X_{ji} among X_{j1}, \dots, X_{jn} , for $i=1, \dots, n; j=0, 1, \dots, p$. These yield the *rank-collection matrix* $\underline{R}_n = ((R_{ji}))$ (of order $(p+1) \times n$), and permuting the columns of \underline{R}_n , so that the top row is in the natural order, we obtain the *reduced rank-collection matrix* $\underline{R}_n^* = ((R_{ji}^*))$, where by (2.4),

$$R_{0S_i} = R_{0i}^* = i \quad \text{and} \quad R_{jS_i} = R_{ji}^*, \quad \text{for } 1 \leq j \leq p, \quad i=1, \dots, n, \quad (2.6)$$

For each $j (=0, \dots, p)$, let $\{a_{n,j}(i) = (a_{n,j}^{(1)}(i), \dots, a_{n,j}^{(m)}(i))\}'$, $i=1, \dots, n$ be a set of *scores* (vectors), which are chosen with the model in (2.2) in mind. For example, if we restrict ourselves to the multiple linear regression model, we have then $m=1$ and we may choose $a_{n,j}(i) = i/(n+1)$ [Wilcoxon scores] or $\Phi^{-1}(\frac{i}{n+1})$ [normal scores], $1 \leq i \leq n$, where Φ is the standard normal d.f. If we have the joint location/scale model, then $m=2$, and besides the above scores (suitable for location alternatives), we need to choose [for $a_{n,j}^{(2)}(i)$] some scores suitable for scale alternatives. These scores will be defined more formally later on. As in Puri and Sen (1969b), we consider the (multivariate)

linear rank statistics

$$T_n = \sum_{i=1}^n (c_i - \bar{c}_n) [a'_{n,0}(R_{0i}), \dots, a'_{n,p}(R_{pi})] \quad (2.7)$$

where $\bar{c}_n = n^{-1} \sum_{i=1}^n c_i$. By (2.6), we may rewrite T_n as

$$T_n = \sum_{i=1}^n (c_{S_i} - \bar{c}_n) [a'_{n,0}(i), a'_{n,1}(R_{1i}^*), \dots, a'_{n,p}(R_{pi}^*)]. \quad (2.8)$$

Keeping in mind the observable r.v.'s in (2.5), as in Sen (1979b), we let P_n be the conditional (permutational) probability measure generated by the $n!$ (conditionally) equally likely column permutations of R_n (given R_n^*) and let $S_{n,k} = (S_1, \dots, S_p)$, for $k = 1, \dots, n$. Then, we let

$$\begin{aligned} T_{n,k} &= E_{Q_n} (T_n | S_{n,k}) \\ &= \sum_{i=1}^k (c_{S_i} - \bar{c}_n) [a'_{n,0}(i) - a^*_{n,0}(k), \dots, a'_{n,p}(R_{pi}^*) - a^*_{n,p}(k)] \end{aligned} \quad (2.9)$$

where

$$a^*_{n,j}(k) = \begin{cases} (n-k)^{-1} \{n\bar{a}'_{n,j} - \sum_{i=1}^k a'_{n,j}(R_{ji}^*)\}, & 1 \leq k \leq n-1 \\ 0, & k = n, \end{cases} \quad (2.10)$$

and

$$\bar{a}'_{n,j} = n^{-1} \sum_{i=1}^n a'_{n,j}(i), \quad \text{for } j = 0, 1, \dots, p. \quad (2.11)$$

Conventionally, we let $T_{n,0} = 0$, $\forall n \geq 1$. The proposed PCS tests rest on the partial sequence $\{T_{n,k}; k \leq n\}$ and are formulated in the next section.

3. THE PROPOSED PCS TESTS

Let us define

$$C_n = \sum_{i=1}^n (c_i - \bar{c}_n)(c_i - \bar{c}_n)' \quad (3.1)$$

and assume that C_n is positive definite (p.d.). Further, the statistics in (2.9) are location invariant, and hence, without any loss of generality, we may set

$$\bar{a}'_{n,j} = 0, \quad \text{for } j = 0, 1, \dots, p. \quad (3.2)$$

For every $k: 1 \leq k \leq n$, and $j, \ell (=0, 1, \dots, p)$, we define

$$\tilde{V}_{n,j\ell}^{(k)} = n^{-1} \left\{ \sum_{i=1}^k a_{n,j}^{(R_{ji}^*)} a'_{n,\ell} (R_{li}^*) + (n-k) a_{n,j}^{*} a_{n,\ell}^{*} \right\}, \quad (3.3)$$

and consider then the $m(p+1) \times m(p+1)$ matrices

$$\tilde{V}_{n,k} = \begin{pmatrix} \tilde{V}_{n,00}^{(k)} & \dots & \tilde{V}_{n,0p}^{(k)} \\ \vdots & & \\ \tilde{V}_{n,p0}^{(k)} & \dots & \tilde{V}_{n,pp}^{(k)} \end{pmatrix} = \begin{pmatrix} \tilde{V}_{n,00}^{(k)} & \tilde{V}_{n,0*}^{(k)} \\ m \times m & m \times mp \\ \tilde{V}_{n,*0}^{(k)} & \tilde{V}_{n,**}^{(k)} \\ mp \times m & mp \times mp \end{pmatrix}, \text{ say,} \quad (3.4)$$

for $k=1, \dots, n$. Also, we rewrite $\tilde{T}_{n,k}$ in (2.9) as

$$\tilde{T}_{n,k} = (\tilde{T}_{n,k}^{(0)}, \tilde{T}_{n,k}^{(1)}, \dots, \tilde{T}_{n,k}^{(p)}) = (\tilde{T}_{n,k}^{(0)}, \tilde{T}_{n,k}^{(*)}), \text{ say.} \quad (3.5)$$

Then, following the line of attack of Sen (1979b), we have by (2.9),

$$E_p \tilde{T}_{n,k} = E\{E_p(\tilde{T}_{n,k} | S_{n,k})\} = E_p(\tilde{T}_{n,k}) = \tilde{0}, \quad (3.6)$$

$$V_p(\tilde{T}_{n,k}) = C_n \otimes V_{n,k}, \quad \forall 1 \leq k \leq n. \quad (3.7)$$

To eliminate the effects of the concomitant variates, we take into account the asymptotic multinormality of $\tilde{T}_{n,k}$ (insuring the asymptotic linearity of regression of $\tilde{T}_{n,k}^{(0)}$ on $\tilde{T}_{n,k}^{(*)}$) and work with the residuals:

$$\begin{aligned} \tilde{T}_{n,k}^0 &= \tilde{T}_{n,k}^{(0)} - (\text{fitted value of } \tilde{T}_{n,k}^{(0)} \text{ on } \tilde{T}_{n,k}^{(*)}) \\ &= \tilde{T}_{n,k}^{(0)} - \tilde{V}_{n,0*}^{(k)} (\tilde{V}_{n,**}^{(k)})^{-1} \tilde{T}_{n,k}^{(*)}, \quad (1 \leq k \leq n), \end{aligned} \quad (3.8)$$

where A^- stands for the generalized inverse of A . Note that

$$E_p \tilde{T}_{n,k}^0 = \tilde{0} \quad \text{and} \quad V_p(\tilde{T}_{n,k}^0) = C_n \otimes [\tilde{V}_{n,00}^{(k)} - \tilde{V}_{n,0*}^{(k)} (\tilde{V}_{n,**}^{(k)})^{-1} \tilde{V}_{n,*0}^{(k)}]. \quad (3.9)$$

We let

$$\tilde{V}_{n,00**}^{(k)} = \tilde{V}_{n,00}^{(k)} - \tilde{V}_{n,0*}^{(k)} (\tilde{V}_{n,**}^{(k)})^{-1} \tilde{V}_{n,*0}^{(k)}, \quad i \leq k \leq n; \quad (3.10)$$

$$G_{nk} = \tilde{T}_{n,k}^0 \tilde{T}_{n,k}^{0'} \quad \text{and} \quad H_{nk} = C_n \otimes \tilde{V}_{n,00**}^{(k)}, \quad 1 \leq k \leq n; \quad (3.11)$$

$$\xi_{nk} = \text{Trace}[G_{nk} H_{nk}^-], \quad 1 \leq k \leq n. \quad (3.12)$$

Then, ξ_{nk} is the covariate-adjusted rank test statistic (for testing H_0 in (2.1)) based on Q_1, \dots, Q_k , for $k=1, \dots, n$. Suppose now that $r(=r_n)$ is some prefixed positive integer, such that $r/n \rightarrow \tau$, $0 < \tau \leq 1$. Then, the proposed PCS can be formulated as follows:

Continue experimentation as long as $k \geq n_0$ and $\xi_{nk} \leq \ell_n^{(\alpha)}$; if, for the first time, for some $k = N(\leq r)$, ξ_{nN} is $> \ell_n^{(\alpha)}$ then experimentation is curtailed at the N -th failure Z_{nN}^0 along with the rejection of H_0 . If no such $k(\leq r)$ exists, then experimentation is stopped at the preplanned r -th failure Z_{nr}^0 along with the acceptance of H_0 . Here $\alpha(0 < \alpha < 1)$ is the desired *level of significance* of the PCS test and the critical value $\ell_n^{(\alpha)}$ (and the initial number n_0) are to be chosen in such a way that

$$P\{\xi_{nk} > \ell_n^{(\alpha)} \text{ for some } k: n_0 \leq k \leq r | H_0\} \leq \alpha. \quad (3.13)$$

Note that N is the *stopping number* and Z_{nN}^0 is the *stopping time*. Our task is to develop theory leading to the choice of $\ell_n^{(\alpha)}$ (and n_0) satisfying (3.13) and to study the (asymptotic) properties of the proposed tests. This is done in the next section.

4. ASYMPTOTIC PROPERTIES OF THE PROPOSED TESTS

Note that if in (2.7) - (2.9), we replace the c_i by $d_i = Dc_i$ where D is any p.d. matrix and proceed as in (3.11) - (3.12) (with \tilde{c}_n replaced by $\tilde{Dc}_n D'$), then the ξ_{nk} remain invariant under any choice of D . Hence, without any loss of generality, in (2.7) - (2.9), and elsewhere, we may replace $(c_i - \bar{c}_n)$ by \tilde{c}_{ni} , $1 \leq i \leq n$, where

$$\sum_{i=1}^n \tilde{c}_{ni} = 0, \quad \sum_{i=1}^n \tilde{c}_{ni} \tilde{c}'_{ni} = \tilde{C}_n = I_q \quad (4.1)$$

and we assume that

$$\max_{1 \leq i \leq n} (\tilde{c}'_{ni} \tilde{c}_{ni}) \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (4.2)$$

Secondly, defining $r = r_n$, as in after (3.12), we assume that

$$\lim_{n \rightarrow \infty} n^{-1} r_n = \tau: 0 < \tau \leq 1. \quad (4.3)$$

Concerning the scores $\{a_{n,j}(i)\}$, we assume that for each $j (=0, 1, \dots, p)$,

$$a_{n,j}(i) = E\phi_j(U_{ni}) \text{ or } \phi_j(i/(n+1)), \quad 1 \leq i \leq n, \quad (4.4)$$

where $U_{n1} < \dots < U_{nn}$ are the ordered r.v.'s of a sample of size n

from the rectangular $(0, 1)$ d.f. and $\phi_j = (\phi_{1j}, \dots, \phi_{mj})'$ is expressible

as

$$\phi_{sj}(u) = \phi_{sj,1}(u) - \phi_{sj,2}(u), \quad 0 < u < 1, \quad 1 \leq s \leq m, \quad 0 \leq j \leq p, \quad (4.5)$$

where the $\phi_{sj,k}$ are all nondecreasing, absolutely continuous and

square integrable inside $(0, 1)$.

Let now $F_{[j]}$, $F_{[j\ell]}$ and $F_{[0j\ell]}$ be respectively the marginal d.f. of X_{ji} , the joint d.f. of $(X_{ji}, X_{\ell i})$ and the trivariate (if $p \geq 2$)

d.f. of $(X_{0i}, X_{ji}, X_{\ell i})$, for $0 \leq j \leq p$, $0 \leq j \neq \ell \leq p$ and $1 \leq j \neq \ell \leq p$,

respectively, all under H_0 in (2.1). Further, we assume that

$F_{[0]}(x) = t$ has a unique solution ζ_t^0 , $\forall t \in (0, 1)$, and let

$\zeta_0^0 = -\infty$ and $\zeta_1^0 = +\infty$. Let then

$$\bar{\phi}_{0t} = (1-t)^{-1} \int_t^1 \phi_0(u) du, \quad 0 \leq t < 1, \quad \bar{\phi}_{00} = \bar{\phi}_{01} = \bar{\phi}_{0t} = \bar{\phi}_{0t}; \quad (4.6)$$

$$\bar{\phi}_{jt} = \frac{1}{1-t} \int_{\zeta_t^0}^{\infty} \int_{-\infty}^{\infty} \phi_j(F_{[j]}(y)) dF_{[0j]}(x, y), \quad 1 \leq j \leq p, \quad t \in [0, 1], \quad (4.7)$$

so that $\bar{\phi}_{j0} = \int_0^1 \phi_j(u) du = \bar{\phi}_j = \bar{\phi}_{j0}$ (by assumption), and let $\bar{\phi}_{j1} = 0$, $1 \leq j \leq p$.

Also, let

$$\lambda_{00}(t) = \int_0^1 [\phi_0(u)] [\phi_0(u)]' du + (1-t) \bar{\phi}_{0t} \bar{\phi}_{0t}', \quad t \in [0, 1], \quad (4.8)$$

$$\begin{aligned} \lambda_{0j}(t) = [\lambda_{j0}(t)]' &= \int_{\zeta_t^0}^{\infty} \int_{-\infty}^{\infty} [\phi_0(F_{[0]}(x))] [\phi_j(F_{[j]}(y))] dF_{[0j]}(x, y) \\ &+ (1-t) \bar{\phi}_{0t} \bar{\phi}_{0t}', \quad 1 \leq j \leq p, \quad t \in [0, 1], \end{aligned} \quad (4.9)$$

$$\begin{aligned} \nu_{j\ell}(t) &= \int_{-\infty}^{\zeta_t^0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\phi_j(F_{[j]}(y))] [\phi_\ell(F_{[\ell]}(z))] {}^{dF}_{[0j\ell]}(x, y, z) \\ &+ (1-t) \bar{\nu}_{jt} \bar{\nu}_{\ell t}, \quad 1 \leq j, \ell \leq p, \quad t \in [0, 1], \end{aligned} \quad (4.10)$$

$$\nu(t) = \begin{pmatrix} \nu_{00}(t) & \cdots & \nu_{0p}(t) \\ \vdots & & \vdots \\ \nu_{p0}(t) & \cdots & \nu_{pp}(t) \end{pmatrix} = \begin{pmatrix} \nu_{00}(t) & \nu_{0*}(t) \\ \nu_{*0}(t) & \nu_{**}(t) \end{pmatrix}, \quad 0 \leq t \leq 1, \quad (4.11)$$

$$\nu_{00}^*(t) = \nu_{00}(t) - \nu_{0*}(t) [\nu_{**}(t)]^{-1} \nu_{*0}(t), \quad 0 \leq t \leq 1. \quad (4.12)$$

By a direct generalization of Theorem 4.1 of Sen (1979b), we have then under (2.1), (4.1) - (4.5) and for continuous F^* ,

$$\max_{k \leq n} \{ || \tilde{\nu}_{n,k} - \nu(k/n) || \} \xrightarrow{P} 0, \quad \text{as } n \rightarrow \infty, \quad (4.13)$$

where $||A||$ stands for the maximum of the elements of A [and the proof of (4.13) is omitted as it runs parallel to the proof of Theorem 4.1 of Sen (1979b)]. As such, by (3.10), (4.12) and (4.13), we obtain by some standard steps that under (2.1), (4.1) - (4.5) and for continuous F^* ,

$$\max_{k \leq n} \{ || \tilde{\nu}_{n,00}^{(k)} - \nu_{00}^*(k/n) || \} \xrightarrow{P} 0, \quad \text{as } n \rightarrow \infty. \quad (4.14)$$

We assume that

$$\nu_{00}^*(t) \text{ is p.d., for every } t \in (0, 1], \quad (4.15)$$

and denote the reciprocal matrix by

$$\nu_{00}^{*-1}(t) = ((\nu_{*}^{rs}(t)))_{r,s=1,\dots,m}, \quad \text{for } 0 < t \leq 1. \quad (4.16)$$

Note that by (2.9) - (2.11), for every $n(\geq 1)$, under P_n , $\{\tilde{T}_{n,k}, 0 \leq k \leq n\}$ is a martingale sequence, so that by (3.7), for every $0 \leq k \leq n-1$,

$$\tilde{V}_{n,k+1} - \tilde{V}_{n,k} \text{ is positive semi-definite (p.s.d.)} \quad (4.17)$$

Also, using (3.8), (3.10) and the martingale property of

$\{T_{n,k}, 0 \leq k \leq n\}$, we get that

$$E[T_{n,k+1}^0 - T_{n,k}^0][T_{n,k+1}^0 - T_{n,k}^0]' = V_{n,00}^{(k+1)} - V_{n,00}^{(k)} \text{ is p.s.d.} \quad (4.18)$$

for every $0 \leq k \leq n-1$. Now, by virtue of (4.4) - (4.5) and the assumed continuity of F^* , it follows that

$$\sup\{||\underline{v}(t) - \underline{v}(s)|| : 0 \leq s \leq t \leq s + \delta \leq 1\} \rightarrow 0 \text{ as } \delta \downarrow 0, \quad (4.19)$$

$$\sup\{||\underline{v}_{00}^*(t) - \underline{v}_{00}^*(s)|| : 0 \leq s \leq t \leq s + \delta \leq 1\} \rightarrow 0 \text{ as } \delta \downarrow 0, \quad (4.20)$$

so that by (3.10), (4.13), (4.15), (4.19) and (4.20) - (4.21), as $\delta \downarrow 0$,

$$\max\{||V_{n,00}^{(q)} - V_{n,00}|| : |q - k| \leq \delta n\} \xrightarrow{P} 0, \text{ as } n \rightarrow \infty. \quad (4.21)$$

Theorem 4.1. Under (2.1), (4.2), (4.4) and (4.5), for every (fixed)

$s(\geq 1)$, $(0 \leq) t_1 < \dots < t_s (\leq 1)$ and $\{k_1, \dots, k_s\}$, satisfying

$$\lim_{n \rightarrow \infty} n^{-1} k_j = t_j, \quad 1 \leq j \leq s,$$

$$C_n^{-1/2}(T_{n,k_1}, \dots, T_{n,k_s}) \xrightarrow{D} N(0, \underline{\Gamma}_q \otimes \underline{\Gamma}), \quad (4.22)$$

where

$$\underline{\Gamma} = \begin{pmatrix} \underline{\Gamma}_{11} & \dots & \underline{\Gamma}_{1s} \\ \vdots & & \vdots \\ \underline{\Gamma}_{s1} & & \underline{\Gamma}_{ss} \end{pmatrix}; \quad \underline{\Gamma}_{j\ell} = \underline{v}(t_j \wedge t_\ell), \quad j, \ell = 1, \dots, s. \quad (4.23)$$

Outline of the proof. Note that each of the T_{n,k_j} is a $qm(p+1)$ -vector, so to prove (4.22), one may use the Cramér-Wold theorem, whereby one needs to consider an arbitrary linear combination of these $sqm(p+1)$ random variables and to prove the asymptotic normality of the same. Once we consider such a linear combination, the case becomes very similar to the one treated in the proof of Theorem 4.3 of Sen (1979b) (dealing with $q = m = 1$), and the same martingale-approach works out nicely. Therefore, the details are omitted.

Next, by (3.8), (3.10), (4.12), (4.14) and Theorem 4.1, we arrive at the following.

Theorem 4.2. Under the hypothesis of Theorem 4.1, for every (fixed)

$s(\geq 1)$, $(0 \leq) t_1 < \dots < t_s (\leq 1)$ and $\{k_1, \dots, k_s\}$, satisfying

$$\lim_{n \rightarrow \infty} n^{-1} k_j = t_j, \quad 1 \leq j \leq s,$$

$$C_n^{-\frac{1}{2}}(T_{n,k_1}^0, \dots, T_{n,k_s}^0) \xrightarrow{\mathcal{D}} N(0, I_q \otimes \Gamma^*), \quad (4.24)$$

where

$$\Gamma^* = \begin{pmatrix} \Gamma_{11}^* & \dots & \Gamma_{1s}^* \\ \vdots & & \vdots \\ \Gamma_{s1}^* & & \Gamma_{ss}^* \end{pmatrix}; \quad \Gamma_{j\ell}^* = \nu_{00}^*(t_j \wedge t_\ell), \quad j, \ell = 1, \dots, s. \quad (4.25)$$

By virtue of (4.15) and (4.18), we conclude that for every

$\varepsilon(0 < \varepsilon \leq 1)$,

$$P\{V_{n,00^{**}}^{(k)} \text{ is p.d., } \forall k \geq n\varepsilon\} \rightarrow 1, \text{ as } n \rightarrow \infty. \quad (4.26)$$

Now, for every $\varepsilon(0 < \varepsilon < 1)$ and (r, n) satisfying (4.3), we

consider a m_q -variate stochastic process ${}_{\varepsilon}W_{n,r} = \{W_{n,r}(t), \varepsilon \leq t \leq 1\}$

by letting

$$W_{n,r}(t) = [C_n \otimes V_{n,00^{**}}^{(k)}]^{-\frac{1}{2}} T_{n,k}^0 \quad \text{for } \frac{k}{r} \leq t < \frac{k+1}{r}, \quad t \in [\varepsilon, 1], \quad (4.27)$$

so that it belongs to the space $D^{qm}[\varepsilon, 1]$. Also, let ${}_{\varepsilon}W = \{W(t),$

$\varepsilon \leq t \leq 1\}$ be a qm -variate Gaussian function with no drift and covariance function

$$E\{[W(s)][W(t)]'\} = I_q \otimes ([\nu_{00}^*(\tau(s \wedge t))]^{\frac{1}{2}} [\nu_{00}^*(\tau(s \vee t))]^{-\frac{1}{2}}), \quad (4.28)$$

for $s, t \in [\varepsilon, 1]$. Then, we have the following.

Theorem 4.3. Under the hypothesis of Theorem 4.1, as $n \rightarrow \infty$,

$${}_{\varepsilon}W_{n,r} \xrightarrow{\mathcal{D}} {}_{\varepsilon}W, \quad \text{in the (extended) } J_1\text{-topology on } D^{mq}[\varepsilon, 1], \quad (4.29)$$

for every $\varepsilon \in (0, 1)$.

Outline of the proof. The convergence of the finite dimensional distri-

butions of ${}_{\varepsilon}W_{n,r}$ to those of ${}_{\varepsilon}W$ follows readily from Theorem 4.2,

(4.14) and (4.26). Also, note that for $s, t \in [\varepsilon, 1]$, $k = [ns]$ and $k' = [nt]$,

$$\begin{aligned} ||\tilde{W}_{n,r}(t) - \tilde{W}_{n,r}(s)|| &\leq ||[\tilde{C}_n \otimes \tilde{V}_{n,00..*}^{(k)}]^{-\frac{1}{2}}(T_{n,k}^0 - T_{n,k'}^0)|| \\ &+ ||([\tilde{C}_n \otimes \tilde{V}_{n,00..*}^{(k)}]^{-\frac{1}{2}} - [\tilde{C}_n \otimes \tilde{V}_{n,00..*}^{(k')}]^{-\frac{1}{2}})T_{n,k'}|| \end{aligned} \quad (4.30)$$

where $||\cdot||$ stands for the Euclidean norm. For each coordinate of $T_{n,k}^0$ (or $T_{n,k}^0 - T_{n,k'}^0$), we may proceed as in the proof of Theorem 4.5 of Sen (1979b), and hence, using (4.21), (4.26) and the above, the *tightness* of $\{\tilde{W}_{n,r}\}$ follows. For intended brevity, the details are omitted.

Let us now define

$$W_\varepsilon^* = \sup\{[\tilde{W}(t)]'[\tilde{W}(t)]: \varepsilon \leq t \leq 1\} \quad (4.31)$$

and, in (3.13), we let $n_0 = [\varepsilon r]$, where $0 < \varepsilon < 1$. Then, by virtue of Theorem 4.3, under the null hypothesis (2.1),

$$\max_{n_0 \leq k \leq r} \xi_{nk} \xrightarrow{D} W_\varepsilon^*, \text{ as } n \rightarrow \infty, \quad (4.32)$$

and hence, in (3.13), for large n , $\ell_n^{(\alpha)}$ may be replaced by

$$\omega_{\alpha\varepsilon}^*, \text{ where } P\{W_\varepsilon^* \geq \omega_{\alpha\varepsilon}^*\} = \alpha. \quad (4.33)$$

Thus, the task reduces to that of finding $\omega_{\alpha\varepsilon}^*$ and, toward this, some developments will be reported in the next section.

Let us next proceed to study the non-null distribution theory. We confine ourselves to some local alternatives for which the asymptotic distributions are nondegenerate. For every n , we conceive of a sequence $\{\tilde{X}_{n1}^*, \dots, \tilde{X}_{nn}^*\}$ of $(p+1)$ -vectors, where \tilde{X}_{ni}^* has the d.f. F_{ni}^* , $1 \leq i \leq n$ and keeping in mind (2.1) - (2.2), we consider the model where F_{ni}^* has an absolutely continuous p.d.f. f_{ni}^* and

$$f_{ni}^*(\tilde{x}^*) = f^*(\tilde{x}^*; \underline{\Delta}_{ni}^d), \quad 1 \leq i \leq n, \quad \tilde{x}^* = (x_0, \tilde{x}')' \in E^{p+1} \quad (4.34)$$

and as in Section 2, we have $\int_{\Omega} f_{ni}^*(x^*) dx_{\Omega} = f_{ni}(x) = f(x)$, $\forall x \in E^p$ and $1 \leq i \leq n$. For the triangular array $\{d_{ni}\}$, we let $D_n = \sum_{i=1}^n d_{ni} d_{ni}'$ and assume that

$$\sum_{i=1}^n d_{ni} = 0, \quad \sup_n \text{Tr}(D_n) < \infty, \quad (4.35)$$

$$D_n \text{ is p.d. for every } n (\geq n_0), \quad (4.36)$$

$$\lim_{n \rightarrow \infty} \left\{ \max_{1 \leq k \leq n} d_{nk}' D_n^{-1} d_{nk} \right\} = 0. \quad (4.37)$$

Further, we define the c_{ni} as in (4.1) - (4.2) and assume that

$$\lim_{n \rightarrow \infty} \left(\sum_{i=1}^n d_{ni} c_{ni}' \right) = P \text{ exists.} \quad (4.38)$$

Let J be an m -dimensional rectangle containing Ω as an inner point and for every $\varrho \in J$, we assume that $f^*(x^*; \varrho)$ satisfies the following:

$$(i) \quad f^*(x^*; \varrho) \text{ is absolutely continuous in } \varrho \text{ for almost every } x^*, \quad (4.39)$$

(ii) For $\varrho = (\theta_1, \dots, \theta_m)'$, the limits

$$\dot{f}_j^*(x^*; \varrho) = \lim_{\theta_j \rightarrow 0} \theta_j^{-1} [f^*(x^*; (0, \dots, \theta_j, 0 \dots 0)) - f^*(x^*; \varrho)], \quad 1 \leq j \leq m \quad (4.40)$$

exist for almost every x^* , and

$$(iii) \quad \lim_{\varrho \rightarrow \Omega} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} [\dot{f}^*(x^*; \varrho)] [\dot{f}^*(x^*; \varrho)]' \{f^*(x^*; \varrho)\}^{-1} dx^* \\ = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} [\dot{f}^*(x^*; \Omega)] [\dot{f}^*(x^*; \Omega)]' \{f^*(x^*; \Omega)\}^{-1} dx^* \quad (4.41)$$

is p.d. and finite, where $\dot{f}^* = (\dot{f}_1^*, \dots, \dot{f}_m^*)'$. We denote the sequence of alternatives in (4.34) by $\{K_n\}$ and assume that (4.35) - (4.41) hold. Note that H_0 in (2.1) relates to $\Delta = \Omega$. The contiguity of the sequence of probability measures under $\{K_n\}$ with respect to one under H_0 is insured by (4.35) - (4.41) [c.f., Hájek and Šidák (1967, pp. 238-239) and Patel (1973) in this context] and will not be proved in detail.

With the same notations as in (4.4) through (4.12), we now let

$$\mu_j(t) = \int_{-\infty}^{\zeta_t^0} \dots \int_{E^p} \phi_j(F_{[j]}(x_j)) [f^*(x^*; 0)]' dx^* + f_{[0]}(\zeta_t^0) \bar{\phi}_{jt} \quad (4.42)$$

for $j = 0, 1, \dots, p$ and $0 < t \leq \tau$ and let

$$\mu_{\tau}^*(t) = [I_q \times \nu_{00}^*(\tau)]^{-\frac{1}{2}P} \Delta' \{ \mu_{00}(\tau) - \nu_{0*}(\tau) \nu_{**}^-(\tau) \mu(\tau) \} \quad (4.43)$$

for $0 < t \leq 1$, where $\mu(\alpha)$ is the $pm \times m$ matrix with components

$\mu_j(\alpha)$, $1 \leq j \leq p$. Then, we have the following.

Theorem 4.4. Under $\{K_n\}$ in (4.34), (4.1) - (4.5) and (4.35) through (4.41),

$$\varepsilon W_{n,r} \xrightarrow{\mathcal{D}} \varepsilon W + \varepsilon \mu^*, \text{ in the } J_1\text{-topology on } D^{qm}[\varepsilon, 1], \quad (4.44)$$

for every $\varepsilon > 0$, where $\varepsilon \mu^* = \{\mu_{\tau}^*(t), \varepsilon \leq t \leq 1\}$.

Outline of the proof. Note that the tightness of $\{W_{\varepsilon n,r}\}$ (under H_0), established in Theorem 4.3 and the contiguity of the sequence of probability measures under $\{K_n\}$ to that under H_0 insure the tightness of $\{W_{\varepsilon n,r}\}$ under $\{K_n\}$ as well [c.f., Theorem 2 of Sen (1976)].

Hence, to prove (4.44), it suffices to establish the convergence of finite dimensional distributions of $\{W_{\varepsilon n,r}\}$ to those of $\varepsilon W + \varepsilon \mu^*$. For this, we may readily extend the proof of Theorem 5.1 of Sen (1979b) [to the case of $q \geq 1$, $m \geq 1$] by considering an arbitrary linear compound of the $W_n(t_j)$, and hence, for intended brevity, the details are omitted.

By virtue of (3.12), (3.13), (4.33) and Theorem 4.4, we conclude that under $\{K_n\}$ in (4.34) - (4.41), the asymptotic power of the proposed PCS test is given by

$$\begin{aligned} \lim_{n \rightarrow \infty} P\{ \max_{[n\varepsilon] \leq k \leq r} |\xi_{nk}| > \ell_n^{(\alpha)} | K_n \} \\ = P\{ [W(t) + \mu_{\tau}^*(t)]' [W(t) + \mu_{\tau}^*(t)] > \omega_{\alpha\varepsilon}^* \text{ for some } t \in [\varepsilon, 1] \}. \end{aligned} \quad (4.45)$$

Also, we have for $k/n \rightarrow u$: $\tau \varepsilon < u \leq \tau$,

$$\lim_{n \rightarrow \infty} P\{N > k | K_n\} = P\{[W(t) + \mu_{\tau}^*(t)]' [W(t) + \mu_{\tau}^*(t)] \leq \omega_{\alpha \varepsilon}^*, \forall \varepsilon \leq t \leq u | \tau\}, \quad (4.46)$$

while $P\{N > n\tau | K_n\} = 0$, by definition in (3.13) and (4.3). Further,

noting that $Z_{n[n\alpha]}^0 \rightarrow \zeta_{\alpha}^0$, in the s -th mean ($s > 0$), $\forall 0 \leq \alpha < 1$ and that $n[\zeta_{k/n}^0 - \zeta_{(k-1)/n}^0] - [f_{[0]}(\zeta_{k/n}^0)]^{-1} \rightarrow 0$, $\forall k \leq r$, we obtain from (4.45) and some routine steps that

$$\lim_{n \rightarrow \infty} E(Z_{nN}^0 | K_n) = \tau \int_0^1 \frac{1}{f_{[0]}(\zeta_{\tau u}^0)} P\{[W(t) + \mu_{\tau}^*(t)]' [W(t) + \mu_{\tau}^*(t)] \leq \omega_{\alpha \varepsilon}^*, \forall \varepsilon \leq t \leq u\} du. \quad (4.47)$$

Therefore the *asymptotic power* as well as the *expected stopping time* of the proposed PCS test depends on the boundary crossing probability of $\tilde{W} + \tilde{\mu}^*$. We shall discuss more on this in the next section.

5. SOME GENERAL REMARKS

As has been noted after (2.2), the classical ANOCOVA model is a special case of (2.2) where $m = 1$ and $q \geq 1$. In this case, by virtue of (4.18), (4.28) and (4.31), we can also write

$$W_{\varepsilon}^* = \sup\{t^{-1} [\underline{Y}(t)]' [\underline{Y}(t)]: \varepsilon \leq t \leq 1\} \quad (5.1)$$

where $\underline{Y} = \{\underline{Y}(t), 0 \leq t \leq 1\}$ is a vector of q independent copies of a standard Wiener process, so that $\{[\underline{Y}(t)]' [\underline{Y}(t)], 0 \leq t \leq 1\}$ is a q -parameter Bessel process. Some tabulation of the critical values of W_{ε}^* is due to Majumdar (1977) and a more analytical approach to this is due to DeLong (1980). Thus, for this special case, we have no problem in applying the proposed PCS tests. A very similar case holds when we have an ANOCOVA model relating to the scale parameter when we have (2.2).

$$F^0(y; \underline{\Delta}(\underline{c}_i - \bar{c}_n) | \underline{x}) = F^0(y(1 + \underline{\Delta}(\underline{c}_i - \bar{c}_n)) | \underline{x}) \quad (5.2)$$

and the c_i and Δ satisfy the condition that $1 + \Delta(c_i - \bar{c}) > 0, \forall i$. Though this scale model has been considered for the analysis of variance models [c.f., Hájek and Šidák (1967)], it has not been considered explicitly for the ANOCOVA model, even in the case of no censoring.

In the general case of $m \geq 2$ (e.g., joint location-scale model), the computation of the (simulation or analytical) critical values of $W_{\alpha \epsilon}^*$ seems to be quite involved (because it depends on the covariance structure in (4.28)). Simplification is of course possible when we have for some scalar function $\gamma_t(s)$,

$$\gamma_{00}^*(s) = \gamma_t(s) \gamma_{00}^*(t) \text{ for every } s \leq t, \quad (5.3)$$

where $\gamma_t(s)$ is \uparrow in s . In this case, again we have the solution given by (5.1) with q being replaced by qm .

Study of the asymptotic power properties of the PCS tests [viz. (4.45) - (4.47)] demands the knowledge of the boundary crossing probabilities for drifted Bessel processes. The prospect for this rests heavily on the simulation techniques (as the analytical tools are not yet properly available). For the analysis of variance model, some studies are due to Majumdar (1977) and DeLong (1980), among others. There is some pressing needs for such studies for the ANOCOVA problem.

Further, it is usually difficult to employ the concept of Pitman-efficiency in PCS tests. Hence, some alternative, meaningful measures of asymptotic efficiency should be explored and, if needed, numerical studies should be made.

Finally, throughout the paper, we have considered the case of PCS rank statistics for the ANOCOVA problem. Cox(1972,1975) has considered some quasi-nonparametric regression models for covariate-adjustments in survival analysis. Though Cox has not considered the case of repeated

significance tests arising in PCS models, his theory can be extended to cover such repeated significance testing. In fact, for a class of simple hypotheses relating to the Cox regression model, such repeated significance tests are considered in Sen(1981). The theory of such tests rests on some invariance principle similar to the ones considered in this paper. It is intended to extend the theory to the case of composite hypotheses (which requires a somewhat different approach as the simple unweighted random sampling theory from a finite population will no longer be applicable for this complicated situation) and the same will be taken up in a subsequent communication. Unlike the Cox model, the models in the current paper do not require the *proportional hazard* assumption and therefore remain valid for a wider class of situations. If, however, the proportional hazard model can be justified, then, one would expect that the Cox regression model based theory would perform better.

R E F E R E N C E S

1. CHATTERJEE, S.K. and SEN, P.K. (1973). Nonparametric testing under progressive censoring. *Calcutta Statist. Assoc. Bull.* 22, 13-50.
2. COX, D.R. (1972). Regression models and life tables. *Jour. Royal Statist. Soc. Ser. B.* 34, 187-220.
3. COX, D.R. (1975). Partial likelihoods. *Biometrika*, 62, 269-276.
4. DELONG, D.M. (1980). Some asymptotic properties of a progressively censored nonparametric test for multiple regression. *Jour. Multivar. Anal.* 10, to appear.
5. HÁJEK, J. and SIDÁK, Z. (1967). *Theory of Rank Tests*. Academic Press, New York.
6. MAJUMDAR, H. (1977). Rank order tests for multiple regression for grouped data under progressive censoring. *Calcutta Statist. Assoc. Bull.* 26, 1-16.
7. MAJUMDAR, H. and SEN, P.K. (1978). Nonparametric tests for multiple regression under progressive censoring. *Jour. Multivar. Anal.* 8, 73-95.

8. PATEL, K.M. (1973). Hájek-Šidák approach to the asymptotic distribution of multivariate rank order statistics. *Jour. Multivar. Anal.* 3, 57-70.
9. PURI, M.L. and SEN, P.K. (1969a). Analysis of covariance based on general rank scores. *Ann. Math. Statist.* 40, 610-618.
10. PURI, M.L. and SEN, P.K. (1969b). A class of rank order tests for a general linear hypothesis. *Ann. Math. Statist.* 40, 1325-1343.
11. QUADE, D. (1967). Rank analysis of covariance. *Jour. Amer. Statist. Assoc.* 62, 1187-1200.
12. SEN, P.K. (1976). A two-dimensional functional permutational central limit theorem for linear rank statistics. *Ann. Probability* 4, 13-26
13. SEN, P.K. (1979a). Weak convergence of some quantile processes arising in progressively censored tests. *Ann. Statist.* 7, 414-431.
14. SEN, P.K. (1979b). Rank analysis of covariance under progressive censoring. *Sankhyā, Ser. A.* 41.
15. SEN, P.K. (1980). Asymptotic theory of some time-sequential tests based on progressively censored quantile processes. *Contributions to Statistics: 60-th Birthday Volume for Professor C.R. Rao.* (in Press).
16. SEN, P.K. (1981). The Cox regression model, invariance principles for some induced quantile processes and some repeated significance tests. *Ann. Statist.* 9,
17. SEN, P.K. and PURI, M.L. (1970). Asymptotic theory of likelihood ratio and rank order tests in some multivariate linear models. *Ann. Math. Statist.* 41, 87-100.