A Continuous Time Model of Electricity Consumption Which Incorporates Time of Day Pricing, a Demand Charge, and Other Commodities

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Over an interval of time $[0,T]$, say, a month, a consumer chooses a path of electricity consumption $c(t)$ with units in kilowatts per hour and, under conventional pricing, is billed proportionally to the total kilowatt hours consumed over the period, computed as $\int_0^T c(t)dt$. Time of day pricing uses, instead, the billing formula $\int_0^T r(t)\, c(t)\, dt$ where $r(t)$ is the rate schedule with units in dollars per kilowatt hour.

The method of computing a demand charge varies from utility to utility. The method described here is that used in the North Carolina Time of Day Rate Demonstration Project. At selected times during $[0,T]$ denoted as

$$t_1, t_2, t_3, \ldots, t_K$$

the average kilowatts per hour that a customer consumes over the next fifteen minutes is computed,

$$(1/\Delta) \int_{t_i}^{t_i+\Delta} c(t)dt.$$

The maximum of these as i ranges from 1 to K is, say, D. An amount $d \cdot D$ is added to the customer's bill where d has units in dollars per kilowatt per month. For notational convenience, set

$$I_i(t) = \begin{cases} 1/\Delta & t_i \leq t \leq t_i + \Delta \\ 0 & \text{otherwise} \end{cases}$$

whence

$$\int_0^T I_i(t)c(t)dt = (1/\Delta)\int_{t_i}^{t_i+\Delta} c(t)dt.$$

Let q denote an N-vector of conventional commodities, let p be an N-vector of corresponding prices, and let y denote a consumer's total expenditures over the period $[0,T]$. The consumer is constrained to choices of electricity consumption paths $c(t)$ and quantities of other commodities q which satisfy

$$q_0(c,q,D) = \int_0^T r(t)c(t)dt + p'q + dD - y \leq 0$$

$$q_i(c,D) = \int_0^T I_i(t)c(t)dt - D \leq 0$$

$$i = 1,2,\ldots,K.$$

These equations define the income constraint set.

The consumer is presumed to choose an electricity consumption path (t) and other commodities q by maximizing a functional $u(c,q)$ over the income constraint set. The functional $u(c,q)$ maps a square integrable function $c(t) \in L_2[0,T]$ and a commodity vector $q \in R^N$ into the real line. One has in mind, possibly, an analogy to the Generalized Leontief functional form

$$u(c,q) = \sum_{i=1}^{N} \alpha_i \ln q_i + \int_0^T \alpha(t) \ln [c(t)]dt$$

$$+ \sum_{i=1}^{N} \sum_{i=1}^{N} \beta_{ij}(q_i q_j)^{\frac{1}{2}} + \int_0^T \int_0^T \beta(s,t)[c(t)c(s)]^{\frac{1}{2}} ds\, dt$$

$$+ \sum_{i=1}^{N} \int_0^T \beta_i(t)[q_i c(t)]^{\frac{1}{2}} dt$$

Electricity is an input into household production so that this objective function is to be regarded as a composition of household production functions and a utility function. There is no mathematical impediment to the construction of a household production type of model but since electricity consumption is not measured by end use in the available data there is little point to the exercise.

To solve this optimization problem, a notion of differentiation of a functional such as $u(c,q)$ with respect to a square integrable function $c(t)$ is required. To do this, suppose that $K(t,x)$, $(\partial/\partial x)K(t,x)$, and $(\partial^2/\partial x^2)K(t,x)$ are continuous with $(\partial^2/\partial x^2)K(t,x)$ bounded over $0 \leq t \leq T$, $-\infty < x < \infty$ and that a functional $f(c)$ is defined by

$$f(c) = \int_0^T K[t,c(t)]dt.$$

It follows that, for $c,h \in L_2[0,T]$,

$$f(c+h) - f(c) = \int_0^T (\partial/\partial x)K[t,c(t)]h(t)dt + o(\| h \|)$$

where $\|h\| = \left[\int_0^T h^2(t)dt\right]^{\frac{1}{2}}$. This equation suggests a definition of a derivative $\nabla_c f(c)$ and a definition of a differential $\nabla_c f(c)h$.

The square integrable function

$$\nabla_c f(c)(t) = (d/dx)K[t,c(t)]$$

may be regarded as the derivative of $f(c)$ evaluated at c and

$$\nabla_c f(c)h = \int_0^T \nabla_c f(c)(t)h(t)dt$$

may be regarded as the differential approximation to the difference

$$f(c+h) - f(c)$$

for small $||h||$. Under similar regularity conditions, if $f(c) = \int_0^T \int_0^T K[t,s,c(t),c(s)]dt\, ds$ then

$$\nabla_c f(c)(t) = \int_0^T (\partial/\partial x_1)K[t,s,c(t),c(s)] + (\partial/\partial x_2)K[s,t,c(s),c(t)]ds \quad .$$

In general, if one can find a square integrable function $F(t)$ which satisfies the equation

$$f(c+h) - f(c) = \int_0^T F(t)h(t)dt + o(||h||)$$

for square integrable c,h then

$$\nabla_c f(c)(t) = F(t).$$

Differentiation with respect to $q \in R^N$ retains its conventional meaning

$$\nabla_q' f(q) = \left(\frac{\partial}{\partial q_1} f(q), \frac{\partial}{\partial q_2} f(q), \ldots, \frac{\partial}{\partial q_N} f(q) \right).$$

Thus

$$f(q+h) - f(q) = \nabla_q' f(q)h + o(||h||)$$

$$= \sum_{i=1}^N (\partial/\partial q_i) f(q)h_i + o(||h||)$$

where $||h|| = \left(\sum_{i=1}^N h_i^2\right)^{1/2}$.

As shown in detail later in this section, the Kuhn-Tucker first order conditions for the consumers optimization problem are

$$\nabla_c u(c,q)(t) - \lambda_0 r(t) - \sum_{i=1}^{K} \lambda_i I_i(t) = 0$$

$$\nabla_q u(c,q) - \lambda_0 p = 0$$

$$- \lambda_0 d - \sum_{i=1}^{K} \lambda_i (-1) = 0$$

$$\int_0^T r(t)c(t)dt + p'q + dD = y$$

$$\lambda_i \left[\int_0^T I_i(t)c(t)dt - D \right] = 0$$

for $i = 0,1,\ldots,K$. Assume that exactly one constraint $\hat{i}$ is binding; this is approximately the same as assuming that the consumers chosen consumption path $c(t)$ has a single maximum. In this case the first order conditions are

$$\nabla_c u(c,q)(t) - \lambda_0 r(t) - \lambda_{\hat{i}} I_{\hat{i}}(t) = 0$$

$$\nabla_q u(c,q) - \lambda_0 p = 0$$

$$\lambda_0 d + \lambda_{\hat{i}} = 0$$

$$\int_0^T r(t)c(t)dt + p'q + dD = y$$

$$\int_0^T I_{\hat{i}}(t)c(t)dt = D$$

Algebraic manipulation yields

$$\lambda_0 = \left[\int_0^T \nabla_c u(c,q)(t)c(t)dt + \nabla'_q u(c,q)q \right] / y$$

$$\lambda_{\hat{i}} = \lambda_0 d$$

whence the first order conditions become

$$\nabla_c u(c,q)(t) = \lambda_0 \hat{r}(t),$$

$$\nabla_q u(c,q) = \lambda_0 p,$$

$$\int_0^T \hat{r}(t)c(t)dt + p'q = y$$

$$\hat{r}(t) = r(t) + dI_{\hat{i}}(t),$$

Marginal utility is proportional to price at each instant of time. The result generalizes; conditional on

$$\int_0^T c(t)I_{\hat{i}}(t)dt = \int_0^T c(t)I_{\hat{\hat{i}}}(t)dt = \max_i \int_0^T c(t)I_i(t)dt$$

the first order conditions are as above but with

$$\hat{r}(t) = r(t) + d \max [I_{\hat{i}}(t), I_{\hat{\hat{i}}}(t)].$$

Another implication of the first order conditions is that consumption is homogeneous in prices and income. That is, the electricity consumption path is of the form

$$c(t) = c[(\hat{r},p)/y](t)$$

and the consumption of other commodities is given by a function of the form

$$q = q[(\hat{r},p)/y].$$

Typically, the price path $r(t)$ is periodic over $-\infty < t < \infty$; that is

$$r(t+kT) = r(t), \quad k = 0, \pm 1, \pm 2, \ldots$$

It also seems natural to impose periodicity on marginal utility

$$\nabla_c u(c,q)(t+kT) = \nabla_c u(c,q)(t).$$

The optimal consumption path

$$c[(\hat{r},p)/y](t)$$

will, as a consequence, be periodic.

A stocastic specification of an observed demand path $\tilde{c}_i(t)$ is

$$\tilde{c}_i(t) = c[(\hat{r},p)/y](t-\tau_i) + e_i$$

where $\tau_i \sim f(\tau)$, $0 \leq \tau \leq T$, represents small errors in knowledge of the time of day, and $e_i \sim g(e)$ represents an additive error of observation. The expected demand path is, then,

$$\bar{c}[(\hat{r},p)/y](t) = E[\tilde{c}_i(t)]$$

$$= \int_0^T c[(\hat{r},p)/y](t-\tau)\, f(\tau)\, d\tau$$

$$= \int_0^T f(t-s)\, c[(\hat{r},p)/y](s) ds$$

where $f(\tau)$ has been extended to a periodic function. If $f(\tau)$ is bounded and continuous then

$$\lim_{t \to t_0} \bar{c}\,[(\hat{r},p)/y](t)$$

$$= \int_0^T [\lim_{t \to t_0} f(t-s)] c[(\hat{r},p)/y](s)\, ds$$

$$= \int_0^T f(t_0 - s)\; c[(\hat{r},p)/y](s)\, ds$$

$$= \bar{c}[(\hat{r},p)/y](t_0)$$

whence the expected demand path is continuous regardless of whether or not the optimal demand path is continuous.

The path $\bar{c}[(\hat{r},p)/y]$ is a convolution. Thus, if $f(\tau)$ and $c[(\hat{r},p)/y](t)$ have Fourier expansions with absolutely summable coefficients

$$f(\tau) = \sum_{j=-\infty}^{\infty} \alpha_j\; (1/\sqrt{T})\; e^{i(2\pi/T)j\tau}$$

$$\alpha_j = \int_0^T f(\tau)\; (1/\sqrt{T})\; e^{-i(2\pi/T)j\tau} d\tau$$

$$c[(\hat{r},p)/y](t) = \sum_{j=-\infty}^{\infty} \beta_j\; (1/\sqrt{T}) e^{i(2\pi/T)jt}$$

$$\beta_j = \int_0^T c[(\hat{r},p)/y](t)(1/\sqrt{T}) e^{-i(2\pi/T)jt} dt \quad .$$

then

$$\bar{c}[(\hat{r},p)/y](t) = \sum_{j=-\infty}^{\infty} (\alpha_j \beta_j)(1/\sqrt{T}) e^{i(2\pi/T)jt}.$$

Thus, the Fourier coefficients of the expected demand path are attenuated by those of the timing error distribution $f(\tau)$; they decrease to zero faster than those of the optimal demand path.

When fitting this demand system to data, $\hat{i}$ or $(\hat{i},\hat{\hat{i}})$ as the case may be is found by inspecting the consumption path $c(t)$ which obtains in the data. This introduces an errors in variables problem as, due to error, the $\hat{i}$ which obtains may differ from the optimal $\hat{i}$. This problem is addressed in a later section.

Another problem in empirical work with the demand system is that consumption is defined as an implicit function of price. Consumption expressed as an explicit function of price is preferable. A form of Roy's identity is available to address this problem. An indirect utility functional $v(\hat{x},x)$ has income normalized prices as its arguments

$$\hat{x}(t) = \hat{r}(t)/y$$

$$x = p/y.$$

One has in mind as a choice of $v(\hat{x},x)$ a generalization of a flexible functional form along the lines described earlier. Roy's identity is

$$c(t) = \left[\int_0^T \hat{x}(t)\nabla_{\hat{x}} v(\hat{x},x)(t)dt + x'\nabla_x(\hat{x},x)\right]^{-1} \nabla_{\hat{x}} v(\hat{x},x)(t)$$

$$q = \left[\int_0^T \hat{x}(t)\nabla_{\hat{x}} v(\hat{x},x)(t)dt + x'\nabla_x(\hat{x},x)\right]^{-1} \nabla_x v(\hat{x},x)$$

Next, a detailed verification of the first order conditions is presented. The reference for notation, definitions, manipulative results such as the chain rule, and the Kuhn-Tucker first order conditions is Wouk (1979, Ch. 12). The fact that both $L_2[0,T]$ and R^N are self dual is exploited repeatedly in the development. Thus, a bounded linear operator $<\cdot,r'>$ on $L_2[0,T]$ must be of the form

$$<c,r'> = \int_0^T c(t)r'(t)dt \quad c \in L_2[0,T]$$

where r' itself is in $L_2[0,T]$. This being the case, the notation r' to indicate membership in the dual space is dropped in favor of the notation r since r' is, in fact, in $L_2[0,T]$. This gives rise to another fact which is exploited repeatedly

$$\langle c,r\rangle = \int_0^T c(t)r(t)dt = \langle r,c\rangle,$$

$$\langle q,p\rangle = \sum_{i=1}^N q_i p_i \qquad = \langle p,q\rangle.$$

The Frechet partial derivative with respect to c evaluated at (c,q) of a mapping $f(c,q)$ from $L_2[0,T]\otimes R^N$ to, say, R^N is a bounded linear operator denoted as $\nabla_c f(c,q)$ which maps $h \varepsilon L_2[0,T]$, its domain, into R^N, its range, and satisfies

$$f(c+h,q) - f(c,q) = \nabla_c f(c,q)h + o(||h||).$$

The notation $\nabla_c f$ is used frequently to mean $\nabla_c f(c,q)$; Table 1 is provided to avoid any confusion caused by this abbreviation.

1. Frechet Partial Derivatives

Frechet Partial Derivative	Evaluated at	Domain	Range
$\nabla_c u = \langle \cdot, \nabla_c u \rangle$	(c,q)	L_2	R
$\nabla_q u = \langle \cdot, \nabla_q u \rangle$	(c,q)	R^N	R
$\nabla_D u = \langle \cdot, 0 \rangle$	(c,q)	R	R
$\nabla_c g_0 = \langle \cdot, r \rangle$	(c,q,D)	L_2	R
$\nabla_q g_0 = \langle \cdot, p \rangle$	(c,q,D)	R^N	R
$\nabla_D g_0 = \langle \cdot, d \rangle$	(c,q,D)	R	R
$\nabla_c g_i = \langle \cdot, I_i \rangle$	(c,D)	L_2	R
$\nabla_q g_i = \langle \cdot, 0 \rangle$	(c,D)	R^N	R
$\nabla_D g_i = \langle \cdot, -1 \rangle$	(c,D)	R	R
$\nabla_{\hat{r}} v = \langle \cdot, \nabla_{\hat{r}} v \rangle$	$(\hat{r},p)$	L_2	R
$\nabla_p v = \langle \cdot, \nabla_p v \rangle$	$(\hat{r},p)$	R^N	R
$\nabla_{\hat{r}} c$	$(\hat{r},p)$	L_2	L_2
$\nabla_p c$	$(\hat{r},p)$	R^N	L_2
$\nabla_{\hat{r}} q$	$(\hat{r},p)$	L_2	R^N
$\nabla_p q$	$(\hat{r},p)$	R^N	R^N

The consumers optimization problem is to

$$\text{maximize:} \quad u(c,q)$$

$$\text{subject to:} \quad g_0(c,q,D) = \langle c,r\rangle + \langle q,p\rangle + \langle D,d\rangle - y \leq 0$$

$$g_i(c,D) = \langle c,I_i\rangle + \langle D,-1\rangle \leq 0 \qquad i = 1,2,\ldots,K.$$

A unique solution (c,q) of the consumers optimization problem which satisfies $g_0(c,q,D) = 0$, $g_i(c,D) = 0$ for some i, $c(t) > 0$ all $t \in [0,T]$, and $q_i > 0$ for $i = 1,2,\ldots N$ is presumed to exist for each (r,p,d) with $r(t) > 0$ all $t \in [0,T]$, $p_i > 0$ for $i = 1,2,\ldots N$, and $d > 0$. The Frechet derivative of $u(c,q)$ is presumed to exist at (c,q).

The Kuhn-Tucher first order conditions for this problem are

$$\nabla_c u(c,q) - \lambda_0\langle\cdot,r\rangle - \textstyle\sum_{i=1}^{K} \lambda_i\langle\cdot,I_i\rangle = 0$$

$$\nabla_q u(c,q) - \lambda_0\langle\cdot,p\rangle = 0$$

$$\lambda_0\langle\cdot,d\rangle - \textstyle\sum_{i-1}^{K} \lambda_i\langle\cdot,-1\rangle = 0$$

$$\langle c,r\rangle + \langle q,p\rangle + \langle D,d\rangle = y$$

$$\lambda_i[\langle c,I_i\rangle + \langle D,-1\rangle] = 0\,.$$

If exactly one constraint $\hat{i}$ is active, then $\lambda_i = 0$ for $i \neq \hat{i},0$ and the first order conditions may be written as

$$\nabla_c u(c,q) - \lambda_0\langle\cdot,r\rangle - \lambda_{\hat{i}}\langle\cdot,I_{\hat{i}}\rangle = 0$$

$$\nabla_q u(c,q) - \lambda_0\langle\cdot,p\rangle = 0$$

$$-\lambda_0\langle\cdot,d\rangle - \lambda_{\hat{i}}\langle\cdot,-1\rangle = 0$$

$$\langle c,r\rangle + \langle q,p\rangle + \langle D,d\rangle = y$$

$$\langle c,I_{\hat{i}}\rangle + \langle D,-1\rangle = 0$$

then

$$[\langle c,\nabla_c u\rangle + \langle q,\nabla_q u\rangle + 0] - \lambda_0[\langle c,r\rangle + \langle q,p\rangle + \langle D,d\rangle]$$

$$- \lambda_{\hat{i}}[\langle c,I_{\hat{i}}\rangle + \langle D,-1\rangle] = 0$$

which simplifies to

$$[\langle c,\nabla_c u\rangle + \langle q,\nabla_q u\rangle] - \lambda_0 y = 0$$

using the income constraint and $\langle c,I_{\hat{i}}\rangle + \langle D,-1\rangle = 0$.

Thus,

$$\lambda_0 = [\langle c,\nabla_c u\rangle + \langle q,\nabla_q u\rangle]/y.$$

Now the equation

$$-\lambda_0\langle \cdot,d\rangle - \lambda_{\hat{i}}\langle \cdot,-1\rangle = 0$$

may be rewritten as

$$\lambda_0 d\langle \cdot,-1\rangle - \lambda_{\hat{i}}\langle \cdot,-1\rangle = 0$$

whence

$$\lambda_{\hat{i}} = \lambda_0 d \quad .$$

The first order conditions become

$$\nabla_c u(c,q) - \lambda_0 \langle \cdot,\hat{r}\rangle = 0$$

$$\nabla_q u(c,q) - \lambda_0 <\cdot,p> = 0$$

$$<c,\hat{r}> + <q,p> \qquad = y$$

with

$$\hat{r} = r + dI_{\hat{i}}$$

$$\lambda_0 = [<c,\nabla_c \vec{u}> + <q,\nabla_q u>]/y \quad .$$

Suppose that there is a differentiable mapping $(c,q) = \{c[(\hat{r},p)/y],$ $q[(\hat{r},p)/y]\}$ of $(\hat{r},p)/y$ into the point (c,q) which solves the consumers optimization problem. The indirect utility function is, then,

$$v[(\hat{r},p)/y] = u\{c[(\hat{r},p)/y] \; q[(\hat{r},p)/y]\} .$$

By the chain rule

$$\nabla_{\hat{r}} v = \nabla_c u \nabla_{\hat{r}} c + \nabla_q u \nabla_{\hat{r}} q \; ,$$

$$\nabla_p v = \nabla_c u \nabla_p c + \nabla_q u \nabla_p q \quad .$$

Substitution of the first order conditions derived earlier yields

$$\nabla_{\hat{r}} v = \lambda_0 <\cdot,\hat{r}> \nabla_{\hat{r}} c + \lambda_0 <\cdot,p> \nabla_{\hat{r}} q \; ,$$

$$\nabla_p v = \lambda_0 <\cdot,\hat{r}> \nabla_p c + \lambda_0 <\cdot,p> \nabla_p q \quad .$$

Differentiation of the income constraint

$$<c[(\hat{r},p)/y],\hat{r}> + <q[(\hat{r},p)/y],p> = y$$

yields the equations

$$\langle \cdot, c[(\hat{r},p)/y]\rangle + \langle \cdot, \hat{r}\rangle \nabla_{\hat{r}} c + \langle \cdot, p\rangle \nabla_{\hat{r}} q = 0$$

$$\langle \cdot, q[(\hat{r},p)/y]\rangle + \langle \cdot, \hat{r}\rangle \nabla_p c + \langle \cdot, p\rangle \nabla_p q = 0 \quad .$$

Substitution into the equations for $\nabla_{\hat{r}} v$, $\nabla_p v$ yields

$$\nabla_{\hat{r}} v = -\lambda_0 \langle \cdot, c[(\hat{r},p)/y]\rangle \ ,$$

$$\nabla_p v = -\lambda_0 \langle \cdot, q[(\hat{r},p)/y]\rangle \ .$$

Then

$$\langle \hat{r}, \nabla_{\hat{r}} v\rangle + \langle p, \nabla_p v\rangle = -\lambda_0 [\langle \hat{r}, c[(\hat{r},p)/y]\rangle + \langle p, q[(\hat{r},p)/y]\rangle]$$

$$= -\lambda_0 [\langle c[(\hat{r},p)/y], \hat{r}\rangle + \langle q[(\hat{r},p)/y], p\rangle]$$

$$= -\lambda_0 y \quad .$$

Substituting for λ_0 yields Roy's identity

$$\langle \cdot, c\rangle = y \langle \cdot, \nabla_{\hat{r}} v\rangle / [\langle \hat{r}, \nabla_{\hat{r}} v\rangle + \langle p, \nabla_p v\rangle] \quad ,$$

$$\langle \cdot, q\rangle = y \langle \cdot, \nabla_p v\rangle / [\langle \hat{r}, \nabla_{\hat{r}} v\rangle + \langle p, \nabla_p v\rangle] \quad .$$

If one writes $v(\hat{x},x)$ for the indirect utility function where

$$\hat{x} = \hat{r}/y$$

$$x = p/y$$

then Roy's identity becomes

$$\langle \cdot, c\rangle = \langle \cdot, \nabla_{\hat{x}} v\rangle / [\langle \hat{x}, \nabla_{\hat{x}} v\rangle + \langle x, \nabla_x v\rangle] \quad ,$$

$$\langle \cdot, q\rangle = \langle \cdot, \nabla_x v\rangle / [\langle \hat{x}, \nabla_{\hat{x}} v\rangle + \langle x, \nabla_x v\rangle] \quad .$$

Reference

Wouk, Arthur (1979) A Course of Applied Functional Analysis. New York: John Wiley and Sons.