ABEL DISTRIBUTIONS

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1. Definition

Dwass (1979) utilized the identity

$$(\alpha+\beta)^{(n)} = \sum_{x=0}^{n} \binom{n}{x} \alpha^{(x)} \beta^{(n-x)} \tag{1}$$

(where $a^{(b)} = a(a-1)\ldots(a-b+1)$) to define various "generalized binomial" distributions. Note that α and β need not be positive; by reversing signs of both α and β in (1), the equivalent identity

$$(\alpha+\beta)^{[n]} = \sum_{x=0}^{n} \binom{n}{x} \alpha^{[x]} \beta^{[n-x]} \tag{1'}$$

is obtained, where $a^{[b]} = a(a+1)\ldots(a+b-1)$.

In a similar way, Abel's identity (e.g. Riordan (1979))

$$\alpha^{-1}(\alpha+\beta)^{n} = \sum_{x=0}^{n} \binom{n}{x} (\alpha+x)^{x-1} (\beta-x)^{n-x} \tag{2}$$

can be used to define distributions

$$\Pr[X=x] = P_x = \alpha(\alpha+\beta)^{-n} \binom{n}{x} (\alpha+x)^{x-1} (\beta-x)^{n-x} \tag{3}$$

provided α and β are such that $P_x \geq 0$ for $x = 0,1,\ldots,n$, with at least one positive P_x. Such values of α and β are

(a) $\alpha > 0$, $\beta > n$

(b) $\alpha < -n$, $\beta < 0$.

We may call these distributions Abel distributions (with parameters $n;\alpha;\beta$). From here on, we suppose (a) holds.

2. Some Properties

An alternative form for (3) is

$$P_x = \{\binom{n}{x}\theta^x(1-\theta)^{n-x}\}(1+\alpha^{-1}x)^{x-1}(1-\beta^{-1}x)^{n-x} \tag{3'}$$

with $\theta = \alpha(\alpha+\beta)^{-1}$. As $\alpha,\beta \to \infty$ with n and θ fixed, the distribution tends to binomial with parameters n, θ. On the other hand, if $n,\beta \to \infty$ with $n\beta^{-1} = \lambda_0$ and $\alpha = \lambda_1/\lambda_0$ fixed then

$$P_x \to e^{-\lambda_1}\frac{\lambda_1^x}{x!}(1+\frac{\lambda_0}{\lambda_1}x)^{x-1}e^{-\lambda_0 x} = \frac{\lambda_1(\lambda_1+\lambda_0 x)^{x-1}e^{-\lambda_1-\lambda_0 x}}{x!} \tag{4}$$

The distribution tends to a Lagrange double Poisson (generalized Borel-Tanner) distribution. Jain and Consul (1971) have shown that the Lagrange double negative binomial

$$P_x = \frac{N}{N+(r+1)x}\binom{N+(r+1)x}{x}Q^{-N}(\frac{P}{Q})^x \quad (x = 0,1,\ldots;\ p = Q-1)$$

tends to Lagrange double Poisson as $N \to \infty$, $r \to 0$, $P \to 0$ with $NP = \lambda_1$, $rP = \lambda_2$, but (4) is not the same as the Lagrange double binomial of Consul and Shenton (1972).

The expected value of the distribution of X can be obtained from the formula (Riordan (1979), Table 1.2)

$$\alpha^{-1}(\alpha+\beta+F)^n \equiv \sum_{x=0}^{n}\binom{n}{x}(\alpha+x)^x(\beta-x)^{n-x}$$

where F^k is interpreted as k!. This gives

$$E[\alpha+X] = \alpha(\alpha+\beta)^{-n}\sum_{j=0}^{n}\binom{n}{j}j!(\alpha+\beta)^{n-j} = \alpha\sum_{j=0}^{n}\frac{n^{(j)}}{(\alpha+\beta)^j}$$

whence

$$E[X] = \alpha \sum_{j=1}^{n} n^{(j)} (\alpha+\beta)^{-j}$$

$$= \frac{n\alpha}{\alpha+\beta}\{1 + \frac{n-1}{\alpha+\beta} + \frac{(n-1)^{(2)}}{(\alpha+\beta)^2} + \ldots + \frac{(n-1)!}{(\alpha+\beta)^{n-1}}\} . \tag{5}$$

Also

$$\mathrm{var}(X) = \mathrm{var}(\alpha+X) = \alpha \sum_{j=0}^{n} n^{(j)} (\alpha+\beta)^{-j} (j+1)(\alpha+\tfrac{1}{2}j)$$

$$- \alpha^2\{\sum_{j=0}^{n} n^{(j)} (\alpha+\beta)^{-j}\}^2 . \tag{6}$$

More compact formulas can be obtained for expected values of other functions of X. Differentiating (2) s times with respect to β, we obtain

$$\alpha^{-1} n^{(s)} (\alpha+\beta)^{n-s} = \sum_{x=0}^{n} \binom{n}{x} (n-x)^{(s)} (\alpha+x)^{x-1} (\beta-x)^{-s} (\beta-x)^{n-x}$$

whence

$$E[(n-X)^{(s)} (\beta-X)^{-s}] = n^{(s)} (\alpha+\beta)^{-s} \qquad (s = 1,2,\ldots,n) . \tag{7}$$

(For $s > n$, both sides of (7) are zero.)

Differentiating with respect to α leads to

$$E[(X-1)^{(s)} (\alpha+X)^{-s}] = \sum_{j=0}^{s} (-1)^j s^{(j)} n^{(s-j)} \alpha^{-j} (\alpha+\beta)^{-s+j}$$

$$(s = 1,2,\ldots,n-1) . \tag{8}$$

More general formulas, such as

$$E[\frac{(n-X)^{(s)}}{(\beta-X)^s} \cdot \frac{(X-1)^{(t)}}{(\alpha+X)^t}] = \frac{n^{(s)}\alpha}{(\alpha+\beta)^n} \frac{d^t}{d\alpha^t}(\frac{(\alpha+\beta)^{n-s}}{\alpha}) \tag{9}$$

are easily derived.

3. Maximum Likelihood Estimation.

For a random sample $X_1,\ldots,X_N$ of size N, the log-likelihood is

$$L = N \log \alpha - Nn \log(\alpha+\beta) + \sum_{i=1}^{N} \log\binom{n}{X_i} + \sum_{i=1}^{N} (X_i-1)\log(\alpha+X_i)$$

$$- \sum_{i=1}^{N} (n-X_i)\log(\beta-X_i) .$$

The equations for the maximum likelihood estimators are

$$\frac{\partial \log L}{\partial \hat{\alpha}} = \frac{N}{\hat{\alpha}} - \frac{Nn}{\hat{\alpha}+\hat{\beta}} + \sum_{i=1}^{N} \frac{X_i - 1}{\hat{\alpha}+X_i} = 0$$

$$\frac{\partial \log L}{\partial \hat{\beta}} = - \frac{Nn}{\hat{\alpha}+\hat{\beta}} + \sum_{i=1}^{N} \frac{n-X_i}{\hat{\beta}-X_i} = 0 .$$

Combining these equations

$$\frac{N}{\hat{\alpha}} + \sum_{i=1}^{N} \frac{X_i - 1}{\hat{\alpha}+X_i} = \frac{Nn}{\hat{\alpha}+\hat{\beta}} = \sum_{i=1}^{N} \frac{n-X_i}{\hat{\beta}-X_i} .$$

The left-hand side can be written

$$\frac{1+\hat{\alpha}}{\hat{\alpha}} \sum_{i=1}^{N} \frac{X_i}{\hat{\alpha}+X_i}$$

and is a decreasing function of $\hat{\alpha}$. Also

$$E[\frac{\partial^2 \log L}{\partial \alpha^2}] = - \frac{N}{\alpha^2} + \frac{Nn}{(\alpha+\beta)^2} - NE[\frac{X-1}{(\alpha+X)^2}] = - \frac{Nn}{(\alpha+\beta)^2(\alpha+2)}(\beta-n+1+\frac{2\beta}{\alpha})$$

$$E[\frac{\partial^2 \log L}{\partial\alpha\, \partial\beta}] = \frac{Nn}{(\alpha+\beta)^2}$$

$$E[\frac{\partial^2 \log L}{\partial \beta^2}] = \frac{Nn}{(\alpha+\beta)^2} - NE[\frac{n-X}{(\beta-X)^2}] = - \frac{Nn\alpha}{(\alpha+\beta)^2(\beta-n+1)} .$$

The asymptotic value of N var$(\hat{\alpha},\hat{\beta})$ is

$$\frac{(\alpha+\beta)^2(\alpha+2)(\beta-n+1)}{2n(n-1)} \begin{pmatrix} \frac{\alpha}{\beta-n+1} & 1 \\ 1 & \frac{\beta-n+1+2\beta\alpha^{-1}}{\alpha+2} \end{pmatrix}$$

$$= \frac{(\alpha+\beta)^2}{2n(n-1)} \begin{pmatrix} \alpha(\alpha+2) & (\alpha+2)(\beta-n+1) \\ (\alpha+2)(\beta-n+1) & (\beta-n+1)(\beta-n+1+2\beta\alpha^{-1}) \end{pmatrix} .$$

The limiting correlation between $\hat{\alpha}$ and $\hat{\beta}$ is

$$-\sqrt{\frac{\alpha+2}{\alpha+2\beta(\beta-n+1)^{-1}}} .$$

4. Multivariate Abel Distributions.

From the identity (Riordan (1979), p. 25)

$$(\prod_{j=1}^{m}\alpha_j)^{-1}(\sum_{j=1}^{m}\alpha_j+\beta)^n = \sum_{\Sigma x_j \le n}\cdots\sum \binom{n}{x_1,\ldots,x_m,n-\Sigma_1^m x_j}$$

$$\times \{\prod_{j=1}^{m}(\alpha_j+x_j)^{x_j-1}\}(\beta-\sum_{j=1}^{m}x_j)^{n-\Sigma_1^m x_j} \tag{10}$$

we define the m-variate Abel distribution, with parameters $n;\underset{\sim}{\alpha};\beta$

$$\Pr[\bigcap_{j=1}^{m}(X_j=x_j)] = P_{\underset{\sim}{x}} = (\sum_{j=1}^{m}\alpha_j+\beta)^{-n}\binom{n}{x_1,\ldots,x_m,n-\Sigma_1^m x_j}$$

$$\times \{\prod_{j=1}^{m}\alpha_j(\alpha_j+x_j)^{x_j-1}\}(\beta-\sum_{j=1}^{m}x_j)^{n-\Sigma_1^m x_j}$$

$$(0 \le x_j;\ \sum_{j=1}^{m}x_j \le n). \tag{11}$$

To ensure $P_{\underset{\sim}{x}} \ge 0$ we take $\alpha_j > 0$ $(j = 1,\ldots,m)$; $\beta > n$. Formula (11) can be written

$$P_{\underset{\sim}{x}} = \{n!\prod_{j=1}^{m+1}\frac{\theta_j^{x_j}}{x_j!}\}\{\prod_{j=1}^{m}(1+\alpha_j^{-1}x_j)^{x_j-1}\}(1-\beta^{-1}\sum_{j=1}^{m}x_j)^{n-\Sigma_1^m x_j} \tag{11'}$$

where $\theta_i = \alpha_i(\sum_{j=1}^{m}\alpha_j+\beta)^{-1}$ $(i = 1,\ldots,m)$ and $\theta_{m+1} = \beta(\sum_{j=1}^{m}\alpha_j+\beta)^{-1}$, so that $\sum_{i=1}^{m+1}\theta_i = 1$; and $x_{m+1} = n - \sum_{j=1}^{m}x_j$.

Similarly, as in the univariate case, differentiating (10) s times with respect to β leads to

$$E[\frac{(n-\sum_{j=1}^{m}X_j)^{(s)}}{(\beta-\sum_{j=1}^{m}X_j)^s}] = \frac{n^{(s)}}{(\sum_{j=1}^{m}\alpha_j+\beta)^s} \quad (s = 1,2,\ldots,n) . \tag{12}$$

Comparing this with (7) it follows that $\sum_{j=1}^{m} X_j$ has a univariate Abel distribution with parameters n; $\sum_{j=1}^{m}\alpha_j$; β. A similar argument leads to the conclusion that the marginal distribution of X_1 is Abel with parameters n; α_1; $\sum_{j=2}^{m}\alpha_j+\beta$.

For the conditional distribution of X_1, given $X_j = x_j$ (j = 2,...,m) we have

$$\Pr[X_1=x_1 \mid \bigcap_{j=2}^{m}(X_j=x_j)] \propto \binom{n-\Sigma_2^m x_j}{x_1}\alpha_1(\alpha_1+x_1)^{x_1-1}\left(\beta - \sum_{j=2}^{m}x_j-x_1\right)^{n-\Sigma_2^m x_j - x_1} . \tag{13}$$

This is an Abel distribution with parameters n - $\sum_{j=2}^{m}x_j$; α_1; β - $\sum_{j=2}^{m}x_j$.

If $\underset{\sim}{\alpha},\beta \to \underset{\sim}{\infty}, \infty$ with n, $\underset{\sim}{\theta}$ constant, the distribution (11)' tends to multinomial with parameters n, $\underset{\sim}{\theta}$. If n,β → ∞ with $n\beta^{-1} = \lambda_0$ and $\alpha_j = \lambda_j/\lambda_0$ (j = 1,...,m) fixed, the distribution tends to that of independent Lagrange double Poisson variables with parameters λ_0,λ_j (j = 1,...,m).

The marginal joint distribution of X_1 and X_2 is bivariate Abel with parameters n; α_1, α_2; $\sum_{j=3}^{m}\alpha_j+\beta$. Also

$$E[(\alpha_1+X_1)(\alpha_2+X_2)] = \alpha_1\alpha_2 \sum_{j=0}^{n}(j+1)n^{(j)}\gamma^{-j}$$

where $\gamma = \sum_{j=1}^{n}\alpha_j+\beta$, and hence

$$\begin{aligned}\operatorname{cov}(X_1,X_2) &= \operatorname{cov}(\alpha_1+X_1,\ \alpha_2+X_2)\\ &= \alpha_1\alpha_2\left[\sum_{j=0}^{n}(j+1)n^{(j)}\gamma^{-j} - \left\{\sum_{j=0}^{n}n^{(j)}\gamma^{-j}\right\}^2\right] .\end{aligned} \tag{14}$$

The parallelism between the structures of multivariate Abel and multinomial distributions - in regard to marginal and conditional distributions, for example - is noteworthy. However, the regression of one variable on the others is not linear for the multivariate Abel.

5. A Related Distribution

If X has distribution (3), then n-X has distribution

$$P_x = \frac{\alpha}{(\alpha+\beta)^n} \binom{n}{x} (\beta-n+x)^x (\alpha+n-x)^{n-x-1} \quad (x = 0,1,\ldots,n). \tag{15}$$

This is not an Abel distribution as defined in (3), although it is somewhat similar to an Abel distribution with parameters n, β-n, α+n. It might be called a reverse Abel distribution.

Replacing (β-n) by α, and (α+n) by β we have

$$P_x = \frac{\beta-n}{(\alpha+\beta)^n} \binom{n}{x} (\alpha+x)^x (\beta-x)^{n-x-1}$$

whence

$$\sum_{x=0}^{n} \binom{n}{x} (\alpha+x)^x (\beta-x)^{n-x-1} = \frac{(\alpha+\beta)^n}{\beta-n} \quad (x = 0,1,\ldots,n;\ \beta > n,\ \alpha > 0)\ . \tag{16}$$

The identity (16) can be used to extend further the expected value formulas obtained in Section 2. For example, immediately from (16), if X has distribution (3)

$$E\left[\frac{\alpha+X}{\beta-X}\right] = \frac{\alpha}{\beta-n}\ ; \tag{17}$$

differentiating s times with respect to α

$$E\left[\frac{X^{(s)}}{(\alpha+X)^{s-1}(\beta-X)}\right] = \frac{n^{(s)}}{(\alpha+\beta)^s(\beta-n)}\ . \tag{18}$$

These formulas could, of course, be derived from (2) with patience.

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Appendix

For purposes of calculation, the following formulae are useful

$$E[\alpha+X] = f_{0,n}(\alpha+\beta) \qquad \text{(cf. (5))}$$

$$\mathrm{Var}(X) = \alpha^2 g_n(\alpha+\beta) + \alpha f_{2,n}(\alpha+\beta) \qquad \text{(cf. (6))}$$

$$\mathrm{Cov}(X_1,X_2) = \alpha_1\alpha_2 g_n(\alpha_1+\alpha_2+\beta) \qquad \text{(cf. (14))}$$

where

$$f_{r,n}(y) = \sum_{j=0}^{n} \frac{j^{[r]}{}_n(j)}{r!y^j} ,$$

$$g_n(y) = f_{1,n}(y) - \{f_{0,n}(y)\}^2 ;$$

$$f_{0,0}(y) = 1$$

$$f_{0,n}(y) = 1 + ny^{-1}f_{0,n-1}(y)$$

$$f_{r,n}(y) = f_{r-1,n}(y) + ny^{-1}f_{r,n-1}(y) \qquad (r = 1,2) .$$

NOTE: The Abel distribution (3) was introduced by P.C. Consul (Sankhyā, 36B, 351-357 (1974)) as the quasi-binomial distribution. (See also K.G. Janardan (pp. 359-364 in Statistical Distributions in Scientific Work (Ed. Patel, Kotz and Ord) Reidel, Boston (1974)).)