INVARIANCE PRINCIPLES FOR RECURSIVE RESIDUALS

by

Pranab Kumar Sen

Department of Biostatistics
University of North Carolina at Chapel Hill

Institute of Statistics Mimeo Series No. 1304

September 1980

INVARIANCE PRINCIPLES FOR RECURSIVE RESIDUALS*

by

PRANAB KUMAR SEN

University of North Carolina, Chapel Hill

ABSTRACT

A general class of recursive residuals is defined by means of lower-triangular (orthonormal) transformations and, for these residuals, some (weak) invariance principles are established under appropriate regularity conditions. The theory is then incorporated in the study of the robustness of some CUSUM procedures.

AMS 1979 Classification No: 60F17, 62F05.

Key Words & Phrases: Constancy of regression relationships over time; CUSUM tests; invariance principles; orthonormal transformations; recursive residuals; tightness; Wiener process.

---

*Work partially supported by the National Heart, Lung and Blood Institute, Contract No. NIH-NHLBI-71-2243 from the National Institutes of Health.

1. INTRODUCTION

In the context of testing for constancy of regression relationships over time, Brown, Durbin and Evans (1975) have considered some *CUSUM* tests based on suitably defined *recursive residuals*. When the *error components* are assumed to be independent and identically distributed (i.i.d.) according to a normal distribution, these recursive residuals are mutually independent and distributed according to a common normal distribution, so that the *Brownian motion* approximation can be readily incorporated for some CUSUM test procedures based on these residuals. The situation differs considerably when the errors are not normally distributed: the residuals remain uncorrelated but not necessarily independent or normally distributed. Several discussants of the Brown et al. (1975) paper, including P. R. Fisk, G. Phillips and R. E. Quandt, have raised the issues of incorporating more general forms of *orthonormal transformations* for the definition of the recursive residuals and establishing *weak invariance principles* for related CUSUM test statistics without imposing normality on the distribution of the errors. Naturally, these would cast light on the *robustness* of the CUSUM procedures suggested by Brown et al. (1975).

The object of the present investigation is to study some weak invariance principles for recursive residuals generated by a class of orthonormal transformations (when the errors are not necessarily normally distributed). Along with the preliminary notions, the orthonormal transformations are introduced in Section 2. Section 3 deals with the main theorems and their proofs. Section 4 is devoted to the applications of the main theorems to some CUSUM procedures based on recursive residuals and relates to some studies of their asymptotic properties.

2. PRELIMINARY NOTIONS

Consider the *regression model*

$$Y_t = \underset{\sim}{\beta}_t' \underset{\sim}{x}_t + e_t, \quad t = 1,\dots,n, \tag{2.1}$$

where, at time t, Y_t is the *dependent variate*, $\underset{\sim}{x}_t = (x_{t1},\dots,x_{tk})'$ (for some $k \geq 1$) is the vector of *regressors*, $\underset{\sim}{\beta}_t = (\beta_{t1},\dots,\beta_{tk})'$ is the vector of *regression coefficients* and the e_t are the *error components*. For testing the null hypothesis (H_0) that $\underset{\sim}{\beta}_t = \underset{\sim}{\beta}$ (unknown), $\forall$ t along with the i.i.d. character of the e_t, i.e., the constancy of the regression relationships over time, Brown, Durbin and Evans (1975) considered the *regression residuals*

$$w_r = (Y_r - \underset{\sim}{b}_{r-1}' \underset{\sim}{x}_r)/\{1 + \underset{\sim}{x}_r'(\underset{\sim}{X}_{r-1}'\underset{\sim}{X}_{r-1})^{-1}\underset{\sim}{x}_r\}^{-1/2}, \quad k+1 \leq r \leq n, \tag{2.2}$$

where, for every r (= $k,\dots,n$),

$$\underset{\sim}{X}_r' = (\underset{\sim}{x}_1,\dots,\underset{\sim}{x}_r), \quad \underset{\sim}{Y}_r' = (Y_1,\dots,Y_r) \quad \text{and} \quad \underset{\sim}{b}_r = (\underset{\sim}{X}_r'\underset{\sim}{X}_r)^{-1}\underset{\sim}{X}_r'\underset{\sim}{Y}_r . \tag{2.3}$$

Their *CUSUM* test is then based on the statistics

$$D_n^+ = (n-k)^{-1/2} s_n^{-1}\left\{\max_{k+1\leq r\leq n} W_r\right\} \quad \text{and} \quad D_n = (n-k)^{-1/2} s_n^{-1}\left\{\max_{k+1\leq n\leq n} |W_r|\right\}, \tag{2.4}$$

where

$$W_r = \textstyle\sum_{i=k+1}^{r} w_i, \quad k+1 \leq r \leq n \quad \text{and} \tag{2.5}$$

$$s_n^2 = (n-k)^{-1}(\underset{\sim}{Y}_n - \underset{\sim}{X}_n\underset{\sim}{b}_n)/(\underset{\sim}{Y}_n - \underset{\sim}{X}_n\underset{\sim}{b}_n).$$

When the e_t are i.i.d.r.v. (random variables) with a normal distribution, the w_i are also i.i.d.r.v. with the same normal distribution, so that the Brownian motion approximation for the W_r can readily be incorporated in the study of the distributon of D_n^+ or D_n.

However, when the e_t are normally distributed, the w_i are not necessarily independent, nor normally distributed, and hence, invariance principles for the recursive residuals remain to be explored. Note that by (2.1) and (2.2), for each $r(> k)$, w_r can be expressed as a linear combination of e_i, $i \le r$ (plus a nonstochastic component which vanishes under H_0). Keeping this in mind, we conceive of a sequence $\{U_i, i \ge 1\}$ of i.i.d.r.v.'s, assume that

$$(2.6) \qquad EU_i = 0,\ 0 < \sigma^2 = EU_i^2 < \infty,$$

let $\underset{\sim}{U}_n = (U_1, \ldots, U_n)'$, $n \ge 1$ and define

$$(2.7) \qquad \underset{\sim}{V}_n = (V_{m+1}, \ldots, V_n)' = \underset{\sim}{A}_n \underset{\sim}{U}_n,\ n \ge m+1,$$

where m is a nonnegative integer, $\underset{\sim}{A}_n$ is an $(n-m) \times n$ matrix with row vectors $\underset{\sim}{a}_{nj} = (\underset{\sim}{a}_j, \underset{\sim}{0}_{n-j})$, $m+1 \le j \le n$ and the $\underset{\sim}{a}_{nj}$ satisfy the following (*orthonormality*) condition:

$$(2.8) \qquad \underset{\sim}{A}_n \underset{\sim}{A}_n' = \underset{\sim}{I}_{n-m}\ .$$

Note that by (2.8), $\underset{\sim}{a}_j \underset{\sim}{a}_j' = 1$, $\forall\ j \ge m+1$. The class of residuals $\underset{\sim}{V}_n$, generated by the class of $\underset{\sim}{A}_n$ (of lower triangular or trapizoidal forms) in (2.8), contains the w_r as special cases, for which $m = k$. In view of the normality of the e_t in (2.1), Brown et al. (1975) did not require any further condition on the $\underset{\sim}{x}_t$ (or the $\underset{\sim}{A}_n$). However, in our case, in the absence of this normality assumption, for the study of invariance principles pertaining to the $\underset{\sim}{V}_n$, we may need some additional regularity conditions, which are posed below.

We assume that

$$(2.9) \qquad \lim_{n\to\infty} \left\{ \max_{m<k\le n} \left(n^{-1} (\textstyle\sum_{j=k}^{n} |a_{jk}|)^2\right) \right\} = 0.$$

Generally, for recursive residuals, the a_{kk} are close to 1, a_{kj}, $j > k$ are all equal to 0 and a_{kj}, $j < k$ are all close to 0. In this respect, we may need the following conditions (where δ_{rs} stands for the Kronecker delta):

$$\sup_{n>m}\left\{\max_{m<k\le n}\ \max_{1\le j\le k}(k|a_{kj} - \delta_{kj}|)\right\} \le c_1 < \infty, \tag{2.10}$$

$$\sup_{n>m}\left\{\max_{1\le i<n-1} i\left|\sum_{k=(m\vee i)+2}^{n}(a_{ki} - a_{ki+1})\right|\right\} \le c_2 < \infty, \tag{2.11}$$

where c_1 and c_2 are finite positive numbers. It may be remarked that (2.10) implies (2.9), but, not conversely. The main results are then presented in the next section.

3. THE MAIN THEOREMS

For every $n(> m)$, we introduce a stochastic process $W_n = \{W_n(t),\ 0 \le t \le 1\}$ by letting $V_i = 0,\ i \le m$ and

$$W_n(t) = \left(\sum_{i\le k_n(t)} V_i\right)/(\sigma\sqrt{n - m}),\ k_n(t) = m + [(n - m)t],\ 0 \le t \le 1. \tag{3.1}$$

Then, W_n belongs to the $D[0,1]$ space endowed with the Skorokhod J_1-topology. Also, let $W = \{W(t),\ 0 \le t \le 1\}$ be a standard Wiener process on $[0,1]$. We are primarily interested in the weak convergence of $\{W_n\}$ to W. Towards this, we consider the following two theorems under different sets of regularity conditions.

Theorem 1. Under (2.6), (2.7), (2.8), (2.9) *and* $\nu_4 = EU_1^4 < \infty$,

$$W_n \underset{\mathcal{D}}{\to} W,\ \textit{in the } J_1\textit{-topology on } D[0,1]. \tag{3.2}$$

Theorem 2. Under (2.7), (2.8), (2.10) *and* (2.11), (2.6) *insures* (3.2).

[Note that in Theorem 1, under the more stringent condition that $\nu_4 < \infty$, we are able to eliminate (2.10) and (2.11), while, in Theorem 2, (2.10) and (2.11) eliminate the need for $\nu_4 < \infty$ and the finiteness of σ^2 in (2.6) suffices.]

Proof of Theorem 1. To establish (3.2), we need to show that (i) the finite-dimensional distributions (f.d.d.) of $\{W_n\}$ converge to those of W and (ii) $\{W_n\}$ is *tight*. For (i), for arbitrary $r(\geq 1)$, $0 \leq t_1 < \ldots < t_r \leq 1$ and $\underset{\sim}{\lambda} = (\lambda_1, \ldots, \lambda_r)'(\neq \underset{\sim}{0})$, on letting $k_n(t_j) = k_j$, $1 \leq j \leq r$, we have

$$\text{(3.3)} \qquad \sum_{j=1}^{r} \lambda_j W_n(t_j) = (\sigma\sqrt{n-m})^{-1} \sum_{j=1}^{r} \lambda_j \sum_{i \leq k_j} V_i$$

$$= (\sigma\sqrt{n-m})^{-1} \sum_{i=1}^{n} \left\{ \sum_{j=1}^{r} \lambda_j \sum_{s=i}^{k_j} a_{si} \right\} U_i$$

$$= \sum_{i=1}^{n} c_{ni} (U_i/\sigma), \text{ say,}$$

(where, conventionally, $\sum_k^q a_{sk} = 0$, for $k > q$). Now, by (2.6) and (2.8), (3.3) has mean 0 and variance $\sum_{i=1}^{n} c_{ni}^2 \to \sum_{j=1}^{r} \sum_{\ell=1}^{r} \lambda_j \lambda_\ell (t_j \wedge t_\ell)$ $= E(\sum_{j=1}^{r} \lambda_j W(t_j))^2 (> 0)$. Hence, it suffices to establish the asymptotic normality of (3.3). For this, we appeal to a special central limit theorem in Hajék and Šidák (1967, p. 153), and, we require to show only that

$$\text{(3.4)} \qquad \lim_{n\to\infty} \left\{ \max_{1 \leq i \leq n} c_{ni}^2 \right\} = 0.$$

Since $\underset{\sim}{\lambda}'\underset{\sim}{\lambda} < \infty$, we obtain from (3.3) that for every i: $1 \leq i \leq n$,

$$(3.5)\qquad c_{ni}^2 = \left\{\sum_{j=1}^r \lambda_j \sum_{s=i}^{k_j} a_{si}\right\}^2/(n - m)$$

$$\leq (\tilde{\lambda}'\tilde{\lambda})\sum_{j=1}^r (\sum_{s=i}^{k_j} a_{si})^2/(n - m)$$

$$\leq r(\tilde{\lambda}'\tilde{\lambda})(\sum_{s=i}^n |a_{si}|)^2/(n - m)$$

$$= \left\{rn/(n - m)\right\}(\tilde{\lambda}'\tilde{\lambda})\left\{n^{-1}(\sum_{s=i}^n |a_{si}|)^2\right\},$$

so that (2.9) insures (3.5). Hence (i) holds.

To establish the tightness of $\{W_n\}$, note that for every $(m \leq)k_1 < k_2 < k_3(\leq n)$, by virtue of (2.7)-(2.8), we may write

$$(3.6)\qquad \sum_{i=k_j+1}^{k_{j+1}} V_i = \sum_{i=k_j+1}^{k_{j+1}} \tilde{a}_{ni}\tilde{U}_n = \tilde{b}_{nj}\tilde{U}_n, \text{ say, } j = 1, 2,$$

where [by (2.8)]

$$(3.7)\qquad \tilde{b}_{nj}\tilde{b}_{nj}' = (k_{j+1} - k_j),\ j = 1,2 \text{ and } \tilde{b}_{n1}\tilde{b}_{n2}' = 0.$$

Thus, by (3.6) and (3.7),

$$(3.8)\qquad E\left\{(\sum_{i=k_1+1}^{k_2} V_i)^2(\sum_{j=k_2+1}^{k_3} V_j)^2\right\}/(\sigma\sqrt{n - m})^4$$

$$= \left\{\sum_{i_1=1}^n\sum_{i_2=1}^n\sum_{i_3=1}^n\sum_{i_4=1}^n b_{1i_1}b_{1i_2}b_{2i_3}b_{2i_4}E(U_{i_1}U_{i_2}U_{i_3}U_{i_4})\right\}/(\sigma\sqrt{n - m})^4.$$

Note that $E(U_{i_1}U_{i_2}U_{i_3}U_{i_4}) = 0$ whenever at least one of the indices i_1, i_2, i_3, i_4 occurs with multiplicity 1. Hence, (3.8) is equal to

$$(3.9)\quad (n-m)^{-2}(\nu_4/\sigma^4)\textstyle\sum_{i=1}^{n} b_{1i}^2 b_{2i}^2 + 3(n-m)^{-2}\Big\{\sum_{1\le i\ne j\le n}(b_{1i}^2 b_{2j}^2 + b_{1i}b_{2i}b_{1j}b_{2j})\Big\}$$

$$\le (\nu_4/\sigma^4)(n-m)^{-2}(\textstyle\sum_{i=1}^{n} b_{1i}^2)(\sum_{j=1}^{n} b_{2j}^2) + 3(n-m)^{-2}\Big\{(\sum_{i=1}^{n} b_{1i}^2)(\sum_{j=1}^{n} b_{2j}^2)\Big\} \qquad (\text{as } \underset{\sim}{b}_{n1}\underset{\sim}{b}'_{n2} = 0)$$

$$= \Big\{(\nu_4/\sigma^4) + 3\Big\}(k_3 - k_2)(k_2 - k_1)/(n-m)^{-2}.$$

Thus, (3.1), (3.8), (3.9) and Theorem 15.6 of Billingsley (1968) insure the tightness of $\{W_n\}$ (where note that $W_n(0) = 0$, with probability 1). Q.E.D.

Proof of Theorem 2. Since (2.10) $\Longrightarrow$(2.9), the proof of the convergence of f.d.d.'s of $\{W_n\}$ to those of W, as has been sketched in (3.3)-(3.5), holds for the hypothesis of Theorem 2. Thus, it suffices to show that under (2.6), (2.7), (2.8), (2.10) and (2.11), $\{W_n\}$ is tight. For this, note that by (3.1),

$$(3.10)\quad W_n(t) = (\sigma\sqrt{n-m})^{-1}\textstyle\sum_{i\le k_n(t)} a_{ii}U_i + (\sigma\sqrt{n-m})^{-1}\sum_{i\le k_n(t)}\sum_{j<i} a_{jj}U_j$$

$$= W_{n1}(t) + W_{n2}(t), \quad 0 \le t \le 1 \quad (\text{say}).$$

Since the $a_{ii}U_i$ are independent r.v.'s and (2.10) insures that $n^{-1}\sum_{i=1}^{n} a_{ii}^2 \to 1$ and $n^{-1}\Big\{\max_{1\le i\le n} a_{ii}^2\Big\} \to 0$, as $n \to \infty$, under (2.6) and (2.10), $\{W_{n1}\}$ converges weakly to W; see Problem 1 (p. 67) of Billingsley (1968) in this respect. This, in turn, insures that $\{W_{n1}\}$ is tight. Thus, it remains to establish the tightness of $\{W_{n2}\}$.

Let $S_k = U_1 + \cdots + U_k$, $k \ge 0$ and $S_0 = 0$. Then, by the Hájek-Rényi inequality, for every $c > 0$, $0 \le a < 1/2$ and $q \ge 1$,

$$(3.11)\qquad P\left\{\max_{1\le k\le q}(k/n)^{-a}n^{-1/2}|S_k| > c\sigma\right\}$$

$$\le c^{-2}\left\{n^{-1}\sum_{k=1}^{q}(k/n)^{-2a}\right\} \le c^{-2}\int_0^{q/n}t^{-2a}dt$$

$$= (1 - 2a)^{-1}c^{-2}(q/n)^{1-2a}.$$

Also, $U_i = S_i - S_{i-1}$, $i \ge 1$, so that for every $m + 1 < q \le n$,

$$(3.12)\qquad (\sigma\sqrt{n - m})^{-1}(\sum_{i=m+1}^{q}\sum_{j=1}^{i-1}a_{ij}U_j)$$

$$= \left\{(n - m)/n\right\}^{-1/2}\left\{\sum_{j=1}^{m}(a_{m+1j} + \sum_{i=m+2}^{q}(a_{ij} - a_{ij+1}))(S_j/\sigma\sqrt{n})\right.$$

$$\left. + \sum_{j=m+1}^{q-1}(a_{j+1j} + \sum_{i=j+2}^{q}(a_{ij} - a_{ij+1}))(S_j/\sigma\sqrt{n})\right\},$$

where by (2.10), $|ia_{ij}| < c_1$, $\forall\ 1 \le j \le i - 1$, $i \ge m + 1$, while by (2.11), $|j\sum_{i=(j\vee m)+2}^{q}(a_{ij} - a_{ij+1})| < c_2$, $\forall\ i \ge 1$, $q \ge m + 1$. Hence, by (3.11) and (3.12), we have for $q = m + [(n - m)\delta] + 1$, $0 < \delta < 1$,

$$(3.13)\qquad \max_{m<k\le q}(\sigma\sqrt{n - m})^{-1}|\sum_{i=m+1}^{k}\sum_{j=1}^{i-1}a_{ji}U_i|$$

$$\le c(c_1 + c_2)\{(n - m)/n\}^{-1/2}\sum_{j=1}^{q}j^{-1}(j/n)^{a}$$

$$\le c(c_1 + c_2)\{(n - m)/n\}^{-1/2}(\int_0^{q/n}t^{-1+a}dt)$$

$$= c(c_1 + c_2)\{(n - m)/n\}^{-1/2}a^{-1}(q/n)^{a},$$

with probability greater than

$$(3.14)\qquad 1 - (q/n)^{1-2a}c^{-2}(1 - 2a)^{-1}.$$

If we let $a = 1/3$, then for every $\varepsilon > 0$ and $\eta > 0$, there exist a

δ: $0 < \delta < 1$ and a sample size n_0 $(= n_0(\varepsilon,\eta))$, such that for $q = m + [(n - m)\delta] + 1$, the right hand side of (3.13) is bounded from above by ε, while (3.14) is bounded from below by $1 - \eta$, for every $n \geq n_0$, i.e.,

(3.15) $$P\left\{\sup_{0<t\leq\delta} |W_{n2}(t)| > \varepsilon\right\} < \eta, \ \forall\ n \geq n_0.$$

On the other hand, defining $k_n(s) = k$ and $k_n(t) = q$, for $\delta \leq s < t \leq 1$, we have by (2.10) and (3.10),

(3.16) $$E\{[W_{n2}(t) - W_{n2}(s)]^2\}$$

$$= (\sigma\sqrt{n - m})^{-2} E\left\{(\textstyle\sum_{j=k+1}^{q}\sum_{i=1}^{j-1} a_{ji}U_i)^2\right\}$$

$$= (\sigma\sqrt{n - m})^{-2} E\left\{(\textstyle\sum_{i=1}^{q-1}(\sum_{j=k\vee i+1}^{q} a_{ji})U_i)^2\right\}$$

$$= (n - m)^{-1}\textstyle\sum_{i=1}^{q-1}(\sum_{j=k\vee i+1}^{q} a_{ji})^2$$

$$\leq (n - m)^{-1}\textstyle\sum_{i=1}^{q-1}(q - k\vee i)\sum_{j=k\vee i+1}^{q} a_{ji}^2$$

$$\leq c_1^2(n - m)^{-1}\left\{k(q - k)\textstyle\sum_{j=k+1}^{q} j^{-2} + \sum_{i=k+1}^{q}(q - i)\sum_{j=i+1}^{q} j^{-2}\right\}$$

$$\leq c_1^2\{(q - k)^2/(n - m)^2\}\{(n - m)(q + 2k + 2)/3q(k + 1)\}$$

$$\leq c_1^2\delta^{-1}\{(q - k)/(n - m)\}^2$$

Hence, using Theorem 12.3 of Billingsley (1968, p. 95) along with (3.16), we obtain that for every $\varepsilon > 0$, $\eta > 0$ (and $\delta > 0$, defined by (3.15)), there exist a ρ: $0 < \rho < 1$ and a sample size n_0', such that for $n \geq n_0'$, (3.17) $$P\left\{\sup_{\delta\le s<t\le s+\rho\le 1} |W_{n2}(t) - W_{n2}(s)| > \varepsilon\right\} < \eta.$$

Then, (3.15) and (3.17) insure the tightness of $\{W_{n2}\}$. Q.E.D.

4. SOME APPLICATIONS TO CUSUM TESTS

We shall discuss the role of Theorems 1 and 2 in the context of some CUSUM tests considered in Brown et al. (1975). Consider first the simple location model, where, in (2.1), $k = 1$ and $x_t = 1$, $\forall$ t. In this case, $b_r = \bar{Y}_r = r^{-1}\sum_{i=1}^{r} Y_i$, $r \ge 1$ and

(4.1) $$w_r = (Y_r - \bar{Y}_{r-1})/(1 + (r-1)^{-1})^{1/2} = \left\{(r-1)Y_r - \sum_{i=1}^{r-1} Y_i\right\}/\sqrt{r(r-1)},$$

$$r \ge 2.$$

Therefore, we have here $m = 1$ and for every $j \ge 2$,

(4.2) $$a_{jj} = (1 - j^{-1})^{1/2},\ a_{ji} = -\{j(j-1)\}^{-1/2},\ 1 \le i < j.$$

Now, (4.2) insures (2.10), while $a_{j1} = \cdots = a_{ji-1}$, $\forall\ j \ge 2$ insure (2.11) with $c_2 = 0$. Thus, by Theorem 2, (3.2) holds here under (2.6). Also, $s_n \to \sigma$, in probability under (2.6) [see Sen and Puri (1970)], so that for the CUSUM test for a shift in location, under H_0 and (2.6),

(4.3) $$D_n^+ \underset{\mathcal{D}}{\to} \sup_{0\le t\le 1} W(t) \quad \text{and} \quad D_n \underset{\mathcal{D}}{\to} \sup_{0\le t\le 1} |W(t)|.$$

This explains the robustness of the CUSUM test for nonnormality; finiteness of the second moment of the e_t suffices.

Consider next the general regression model in (2.1). First, we assume that for some $\lambda > 1/2$,

(4.4) $$\max_{1\le k\le n} x_k'(X_{n-1}'X_{n-1})^{-1}x_k = O(n^{-\lambda}),\quad \forall\ n \ge m+1.$$

Then, by (2.2) and (2.3), we have for every $j \geq m + 1$,

$$(4.5) \quad a_{ji} = (\underset{\sim}{x}_j'(\underset{\sim}{X}_{j-1}'\underset{\sim}{X}_{j-1})^{-1}\underset{\sim}{x}_i)/(1 + \underset{\sim}{x}_j'(\underset{\sim}{X}_{j-1}'\underset{\sim}{X}_{j-1})^{-1}\underset{\sim}{x}_j)^{1/2}, \; 1 \leq i \leq j - 1,$$

$$(4.6) \qquad a_{jj} = \left\{1 + \underset{\sim}{x}_j'(\underset{\sim}{X}_{j-1}'\underset{\sim}{X}_{j-1})^{-1}\underset{\sim}{x}_j\right\}^{-1/2},$$

so that by (4.4), (4.5) and (4.6), for every $1 \leq i \leq j - 1$, $j \geq m + 1$,

$$(4.7) \qquad a_{ji}^2 \leq (\underset{\sim}{x}_j'(\underset{\sim}{X}_{j-1}'\underset{\sim}{X}_{j-1})^{-1}\underset{\sim}{x}_i)^2$$

$$\leq (\underset{\sim}{x}_j'(\underset{\sim}{X}_{j-1}'\underset{\sim}{X}_{j-1})^{-1}\underset{\sim}{x}_j)(\underset{\sim}{x}_i'(\underset{\sim}{X}_{j-1}'\underset{\sim}{X}_{j-1})^{-1}\underset{\sim}{x}_i) = 0(j^{-2\lambda}).$$

Further, by (2.9) and (4.7),

$$(4.8) \qquad \max_{m<k\leq n}\left\{n^{-1}(\textstyle\sum_{j=k}^n|a_{jk}|)^2\right\}$$

$$\leq \max_{m<k\leq n}\left\{n^{-1}(n - k)\textstyle\sum_{j=k}^n a_{jk}^2\right\}$$

$$\leq \max_{m<k\leq n}\left\{\frac{n - k}{n}[a_{jj}^2 + \textstyle\sum_{j=k+1}^n a_{jk}^2]\right\}$$

$$= \max_{m<k\leq n}\left\{1 + 0(k^{-2\lambda+1})\right\} = 0(1), \quad \text{as } \lambda > 1/2$$

Hence, (2.9) holds, so that Theorem 1 holds whenever $\nu_4 < \infty$ and (4.4) holds.

Next, we proceed to relax the condition that $\nu_4 < \infty$. Note that for every $j \geq m + 1$ and $1 \leq i \leq j - 2$,

$$(4.9) \quad |a_{ji} - a_{ji+1}| = \left|\left\{\underset{\sim}{x}_j'(\underset{\sim}{X}_{j-1}'\underset{\sim}{X}_{j-1})^{-1}(\underset{\sim}{x}_i - \underset{\sim}{x}_{i+1})\right\}/\left\{1 + \underset{\sim}{x}_j'(\underset{\sim}{X}_{j-1}'\underset{\sim}{X}_{j-1})^{-1}\underset{\sim}{x}_j\right\}^{1/2}\right|$$

$$\leq |\underset{\sim}{x}_j'(\underset{\sim}{X}_{j-1}'\underset{\sim}{X}_{j-1})^{-1}(\underset{\sim}{x}_i - \underset{\sim}{x}_{i+1})|.$$

Thus, if for every $j \geq m + 1$ and $1 \leq i \leq j - 2$,

$$(4.10) \qquad |(\underset{\sim}{x}_i - \underset{\sim}{x}_{i+1})'(\underset{\sim}{X}'_{j-1}\underset{\sim}{X}_{j-1})^{-1}\underset{\sim}{x}_j| = O(j^{-2}),$$

then (2.11) holds, while (2.10) holds when (4.4) holds with $\lambda = 1$. As a result, Theorem 2 holds when (4.4) holds with $\lambda = 1$ and (4.10) holds. Under these extra conditions, we do not need that $\nu_4 < \infty$.

As a simple example, consider the classical polynomial regression model where

$$(4.11) \qquad \underset{\sim}{x}'_t = (1,t,\ldots,t^p), \text{ for some } p \geq 1,\ 1 \leq t \leq n.$$

In this case, (4.4) holds with $\lambda = 1$ and (4.10) holds. Thus, for the related CUSUM test, (4.3) holds whenever $0 < \sigma < \infty$.

So far, we have considered the case where H_0 holds i.e., the U_t have all expectation 0. If $U_i = \xi_i + e_i$, where the e_i are i.i.d.r.v. with mean 0 and if we define $\mu_i = \sum_{j\leq i}\xi_j a_{ij}$, $i \geq m + 1$, then for the $V_i - \mu_i$, $i \geq m + 1$, the same invariance principles hold. Hence, whenever $\{(\sigma\sqrt{n - m})^{-1}\mu_{k_n(t)},\ 0 \leq t \leq 1\}$ converges to a smooth function $\gamma = \{\gamma(t),\ 0 \leq t \leq 1\}$, then the asymptotic power of the CUSUM test can be expressed in terms of the boundary crossing probability of the drifted Wiener process $W + \gamma$. In general, γ is not so simple as to allow an algebraic expression for this probability. However, the recent results obtained by De Long (1980) (in a different context) presents excellent scope for adequate simulation studies.

REFERENCES

[1] BILLINGSLEY, P. (1968). *Convergence of Probability Measures.* John Wiley: New York.

[2] BROWN, R. L., DURBIN, J. and EVANS, J. M. (1975). Techniques for testing constancy of regression relationships over time (with discussions). *Jour. Roy. Statist. Soc. Ser. B,* **37**, 149-192.

[3] DE LONG, D. (1980). Some asymptotic properties of a progressively censored nonparametric test for multiple regression. *Jour. Multivar. Anal.*, **10**, in press.

[4] HÁJEK, J. and ŠIDÁK, Z. (1967). *Theory of Rank Tests.* Academic Press: New York.

[5] SEN, P. K. and PURI, M. L. (1970). Asymptotic theory of likelihood ratio and rank order tests in some multivariate linear models. *Ann. Math. Statist.*, **41**, 87-100.