ASUMPTOTIC PROPERTIES OF LIKELIHOOD RATIO TESTS
BASED ON CONDITIONAL SPECIFICATION

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Summary. The problem of testing a general parametric hypothesis following a preliminary test on some other parametric restraints is considered. These tests are based on appropriate likelihood ratio statistics. The effect of the prelimianry test on the size and power of the ultimate test is studied. In this context, some asymptotic distributional properties of some likelihood ratio statistics are studied and incorporated in the study of the main results.

1. Introduction.

In a parametric model, the underlying distributions are of assumed forms and the *parameter* $\underset{\sim}{\theta}$ belongs to a *parameter space* Ω . Let $L_n(\underset{\sim}{\theta})$, $\underset{\sim}{\theta} \in \Omega$, be the *likelihood function* and let ω be a subspace of Ω . For testing the null hypothesis H_0: $\underset{\sim}{\theta} \in \omega$ against H_1: $\underset{\sim}{\theta} \notin \omega$, the usual *likelihood ratio* test (LRT) is based on the (log-) *likelihood ratio statistic* (LRS)

$$\mathcal{L}_n^{(0)} = 2 \log\{\sup_{\underset{\sim}{\theta}\in\Omega} L_n(\underset{\sim}{\theta})/\sup_{\underset{\sim}{\theta}\in\omega} L_n(\underset{\sim}{\theta})\} \tag{1.1}$$

and the null hypothesis H_0 is rejected when $\mathcal{L}_n^{(0)}$ is significantly large. This unrestricted LRT possesses some (asymptotic) optimal properties when $\underset{\sim}{\theta}$ is not restricted to some particular subspace of Ω .

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In certain problems of inference, $\Omega^*(\subseteq \Omega)$, a subset of Ω, may be identified form some extraneous considerations, and one may like to test for H_0: $\underset{\sim}{\theta} \in \omega$, given that $\underset{\sim}{\theta} \in \Omega^*$. This may conveniently be made by setting $\overline{H}_0$: $\underset{\sim}{\theta} \in \omega^*(=\omega \cap \Omega^*)$ and $\overline{H}_1$: $\underset{\sim}{\theta} \in \Omega^*\backslash\omega^*$, so that the corresponding LRT is based on

$$\overline{L}_n = 2 \log\{\sup_{\underset{\sim}{\theta}\in\Omega^*} L_n(\underset{\sim}{\theta})/\sup_{\underset{\sim}{\theta}\in\omega^*} L_n(\underset{\sim}{\theta})\} \tag{1.2}$$

and $\overline{H}_0$ is rejected for significantly large values of L_n. When $\underset{\sim}{\theta} \in \Omega^*$, the restricted LRT based on $\overline{L}_n$ usually performs better than $L_n^{(0)}$ and, under suitable regularity conditions, given $\underset{\sim}{\theta} \in \Omega^*$, L_n possesses some (asymptotic) optimality properties too. On the other hand, if, contrary to the assumption, $\underset{\sim}{\theta} \notin \Omega^*$, then L_n may become inefficient and even inconsistent. Hence, one may not advocate the restricted LRT unless one has high confidence in the assumption that $\underset{\sim}{\theta} \in \Omega^*$.

In a variety of practical problems, though some Ω^* may be framed from certain practical considerations, there may not be sufficient grounds to enforce a restricted LRT. At the same time, considerations of the possible gain in power (when $\underset{\sim}{\theta} \in \Omega^*$) advocate the use of the restricted LRT over the unrestricted one. Often, as a compromise, in such a case, a *preliminary test* is made of H_0^*: $\underset{\sim}{\theta} \in \Omega^*$ (against H_1^*: $\underset{\sim}{\theta} \notin \Omega^*$) and an appropriate LRS is used (depending on the acceptance/rejection of H_0^*). For this preliminary test, consider the LRS

$$L_n^* = 2 \log\{\sup_{\underset{\sim}{\theta}\in\Omega} L_n(\underset{\sim}{\theta})/\sup_{\underset{\sim}{\theta}\in\Omega^*} L_n(\underset{\sim}{\theta})\} , \tag{1.3}$$

where H_0^* is rejected when L_n^* is $\geq L_{n,\alpha^*}^*$, the upper $100\alpha^*\%$ point of the distribution of L_n^* (under H_0^*) and α^* $(0 < \alpha^* < 1)$ is the *level of significance* (size) of the preliminary test. Then the actual test for H_0: $\underset{\sim}{\theta} \in \omega$ is based on the test statistic L_n, where

$$L_n = \begin{cases} \overline{L}_n, \text{ if } L_n^* < L_{n,\alpha^*}^* , \\ L_n^{(0)}, \text{if } L_n^* \geq L_{n,\alpha^*} . \end{cases} \tag{1.4}$$

Since, in general, $L_n^{(0)}$, L_n^* and $\bar{L}_n$ are not independent (even under H_0 and asymptotically), it is clear from (1.4) that the distribution of L_n (even under H_0) generally depends on α^* as well as the joint distribution of $L_n^{(0)}$, L_n^* and $\bar{L}_n$ [actually, through the bivariate distributions of $(\bar{L}_n, L_n^*)$ and $(L_n^{(0)}, L_n^*)$]. We are primarily concerned with a systematic study of the asymptotic properties of the test leased on L_n. In particular, the effect of the preliminary test on the size and power of the ultimate test is the main objective of our study. In some special problems (arising in the classical analysis of variance tests for some linear models with normally distirbuted errors), the effect of preliminary tests on the size and power of some ultimate tests has been studied by Bechhofer (1951) and Bozivich, Bancroft and Hartley (1956), among others. Some nonparametric procedures are due to Tamura (1956) and Saleh and Sen (1980), among others. Sen (1979) has recently studied some asymptotic properties of maximum likelihood estimators (MLE) under conditional specification. These results are incorporated here in the main study. In this context, the (joint) distribution theory of correlated quadratic forms (in normally distributed random vectors), studied by Jensen (1970) and Khatri, Krishnaiah and Sen (1977) palys a vital role.

Along with the preliminary notions, the proposed precedure is outlined in Section 2. Section 3 deals with the asymptotic distribution theory of various statistics involved in the proposed testing procedures. These results are then incorporated in Section 4 in the study of the *asymptotic size* and *asymptotic power function* of the test based on L_n in (1.4). Some general remarks are made in the concluding section.

2. Basic regularity conditions and the proposed tests.

Bearing in mind that, typically, a multisample situation may be involved in a preliminary testing problem, as in Sen (1979), we conceive of the following model. Let there be k (≥ 1) independent samples; for the i^{th} sample, let $\underset{\sim}{X}_{i1},\ldots,\underset{\sim}{X}_{in_i}$ be n_i independent and identically distributed random vectors (i.i.d.r.v.) with a distribution function (df) $F_i(x;\underset{\sim}{\theta})$, for $i=1,\ldots,k$, where $\underset{\sim}{x} \in E^p$, the p (≥ 1)-dimensional Euclidean space and $\underset{\sim}{\theta} = (\theta_1,\ldots,\theta_t)' \in \Omega \subseteq E^t$, for some $t\geq 1$. Note that F_i may not depend on all $\theta_1,\ldots,\theta_t$, for every $i=1,\ldots,k$, but each element of $\underset{\sim}{\theta}$ is associated with at least one df. Further, we assume that for each $\underset{\sim}{\theta} \in \Omega$ and $i(=1,\ldots,k)$, $F_i(x,\underset{\sim}{\theta})$ admits a density function $f_i(\underset{\sim}{x};\underset{\sim}{\theta})$ (with respect to some sigma-finite measure μ). Then, the log-likelihood function is defined by

$$\log L_n(\underset{\sim}{\theta}) = \log L_n(\underset{\sim}{X}_n,\underset{\sim}{\theta}) = \sum_{i=1}^{k} \sum_{j=1}^{n_i} \log f_i(\underset{\sim}{X}_{ij},\underset{\sim}{\theta}) , \quad \underset{\sim}{\theta} \in \Omega , \tag{2.1}$$

where $\underset{\sim}{X}_n = (\underset{\sim}{X}_{11},\ldots,\underset{\sim}{X}_{kn_k})$ and $n = n_1+\cdots+n_k$. Suppose now that a subset ω ($\subset \Omega$) be specified by

$$\omega = \{\underset{\sim}{\theta} \in \Omega: \ \underset{\sim}{h}(\underset{\sim}{\theta}) = (h_1(\underset{\sim}{\theta}),\ldots,h_r(\underset{\sim}{\theta}))' = \underset{\sim}{0}\} , \text{ for some } r<t, \tag{2.2}$$

where $\underset{\sim}{h}(\underset{\sim}{\theta})$ satisfies some regularity conditions, to be specified later on. We are primarily interested in testing H_0: $\underset{\sim}{\theta} \in \omega$ against H_1: $\underset{\sim}{\theta} \notin \omega$.

An unrestircted MLE ($\hat{\underset{\sim}{\theta}}_n$) of $\underset{\sim}{\theta}$ is an element of Ω, such that

$$\log L_n(\underset{\sim}{X}_n,\hat{\underset{\sim}{\theta}}_n) = \sup_{\underset{\sim}{\theta}\in\Omega} \log L_n(\underset{\sim}{X}_n,\underset{\sim}{\theta}) , \tag{2.3}$$

while, $\check{\underset{\sim}{\theta}}_n$, the restricted MLE is an element of ω, such that

$$\log L_n(\underset{\sim}{X}_n,\check{\underset{\sim}{\theta}}_n) = \sup_{\underset{\sim}{\theta}\in\omega} \log L_n(\underset{\sim}{X}_n,\underset{\sim}{\theta}_n) . \tag{2.4}$$

Suppose now that a subset Ω^* ($\subset \Omega$) be specified by

$$\Omega^* = \{\underset{\sim}{\theta} \in \Omega: \ \underset{\sim}{g}(\underset{\sim}{\theta}) = (g_1(\underset{\sim}{\theta}),\ldots,g_s(\underset{\sim}{\theta}))' = \underset{\sim}{0}\} , \text{ for some } s < t , \tag{2.5}$$

where $\underset{\sim}{g}$ satisfies certain regularity conditions, to be specified later on. Then, by (2.2), (2.5) and the definition of ω^* ($= \omega \cap \Omega^*$), we have

$$\omega^* = \{\underset{\sim}{\theta} \in \Omega: \ \underset{\sim}{h}(\underset{\sim}{\theta}) = \underset{\sim}{0}, \ \underset{\sim}{g}(\underset{\sim}{\theta}) = 0\} . \tag{2.6}$$

Let $\hat{\underset{\sim}{\theta}}{}_n^*$ and $\check{\underset{\sim}{\theta}}{}_n^*$ be respectively MLE of $\underset{\sim}{\theta}$ under $\underset{\sim}{\theta} \in \Omega^*$ and $\underset{\sim}{\theta} \in \omega^*$. Then, parallel to (1,1), (1,2) and (1,3), we have

$$L_n^{(0)} = 2 \log L_n(\underset{\sim}{X}_n, \hat{\underset{\sim}{\theta}}_n) - 2 \log L_n(\underset{\sim}{X}_n, \check{\underset{\sim}{\theta}}_n) , \tag{2.7}$$

$$\bar{L}_n = 2 \log L_n(\underset{\sim}{X}_n, \hat{\theta}_n^*) - 2 \log L_n(\underset{\sim}{X}_n, \check{\theta}_n^*) , \tag{2.8}$$

$$L_n^* = 2 \log L_n(\underset{\sim}{X}_n, \hat{\underset{\sim}{\theta}}_n) - 2 \log L_n(\underset{\sim}{X}_n, \hat{\theta}_n^*) . \tag{2.9}$$

For latter use, we also introduce the following statistic

$$\begin{aligned} \bar{L}_n^* = \bar{L}_n + L_n^* &= 2 \log L_n(\underset{\sim}{X}_n, \hat{\underset{\sim}{\theta}}_n) - 2 \log L_n(\underset{\sim}{X}_n, \check{\underset{\sim}{\theta}}{}_n^*) \\ &= 2 \log\{\sup_{\underset{\sim}{\theta} \in \Omega} L_n(\underset{\sim}{\theta}) / \sup_{\underset{\sim}{\theta} \in \omega^*} L_n(\underset{\sim}{\theta})\} . \end{aligned} \tag{2.10}$$

Finally, we formulate the test function ν_n, corresponding to (1.4), as follows. Let $\bar{\alpha}(0 < \bar{\alpha} < 1)$ and $\alpha^0(0 < \alpha^0 < 1)$ be positive numbers and α^* be defined as in (1.3)-(1.4). Then, we take

$$\nu_n = \begin{cases} 1, \text{ if } L_n^* < L_{n,\alpha^*}^*, \ \bar{L}_n \geq \bar{L}_{n,\bar{\alpha}} , \\ \quad \text{or } L_n^* \geq L_{n,\alpha^*}^*; \ L_n^{(0)} \geq L_{n,\alpha^0}^{(0)} , \\ 0, \text{ otherwise,} \end{cases} \tag{2.11}$$

where $\bar{L}_{n,\bar{\alpha}}$ and $L_{n,\alpha^0}^{(0)}$ are respectively the upper $100\bar{\alpha}\%$ and $100\alpha^0\%$ points of the (null) distributions of L_n and $L_n^{(0)}$. The *size* of the test of (1.4) and (2.11) is therefore

$$\alpha_n = \sup_{\underset{\sim}{\theta} \in \omega} E_\theta\{\nu_n\} \tag{2.12}$$

and its *power* is given by

$$\beta_n(\underset{\sim}{\theta}) = E_\theta\{\nu_n\} , \quad \theta \in \Omega , \tag{2.13}$$

We are primarily concerned with the study of (2.12)-(2.13). For this purpose, we introduce the following regularity conditions [adapted from Aitchison and Silvey (1958) and Sen (1979)]:

[A1] Ω is a convex, compact subspace of E^t, and for every $\underset{\sim}{\theta}_1 \neq \underset{\sim}{\theta}_2$ (both in Ω), for at least one $i(=1,\ldots,k)$,

$$f_i(\underset{\sim}{x};\underset{\sim}{\theta}_1) \neq f_i(\underset{\sim}{x};\underset{\sim}{\theta}_2), \text{ at least on a set of measure nonzero} \tag{2.14}$$

[A2] For every $\underset{\sim}{\theta} \in \Omega$ and every $i(=1,\ldots,k)$, $Z_i(\underset{\sim}{\theta}) = \int_{E^p} \log f_i(\underset{\sim}{x};\underset{\sim}{\theta})\, dF_i(\underset{\sim}{x};\theta_0)$ exists, where $\underset{\sim}{\theta}_0$ is the true parameter point. Note that for the i^{th} density, the Kullback-Leibler information is

$$I_i(\underset{\sim}{\theta},\underset{\sim}{\theta}_0) = \int_{E^p} \log\{f_i(\underset{\sim}{x};\underset{\sim}{\theta}_0)/f_i(\underset{\sim}{x};\underset{\sim}{\theta})\}dF_i(\underset{\sim}{x};\underset{\sim}{\theta}_0) = Z_i(\underset{\sim}{\theta}_0) - Z(\underset{\sim}{\theta}) , \tag{2.15}$$

where $I_i(\underset{\sim}{\theta},\underset{\sim}{\theta}_0) \geq 0$, $\forall\underset{\sim}{\theta} \in \Omega$ and the strict equality sign holds only when $f_i(\underset{\sim}{x};\underset{\sim}{\theta}) = f_i(\underset{\sim}{x};\underset{\sim}{\theta}_0)$ almost everywhere (a.e.)

[A3] For every $\underset{\sim}{\theta} \in \Omega$ and $i(=1,\ldots,k)$, $\log f_i(\underset{\sim}{x};\underset{\sim}{\theta})$ is (a.e.) twice differentiable with respect to $\underset{\sim}{\theta}$ and

$$|(\partial^s/\partial\theta_a^{s_1}\partial\theta_b^{s_2}) \log f_i(\underset{\sim}{x};\underset{\sim}{\theta})| \leq G_s(x),\ \forall\underset{\sim}{x} \in E^p,\ \underset{\sim}{\theta} \in \Omega , \tag{2.16}$$

where $s_1 \geq 0$, $s_2 \geq 0$, $s_1 + s_2 = s = 1,2$ and $1 \leq a, b \leq t$, and where

$$\int_{E^p} G_s(\underset{\sim}{x})dF_i(\underset{\sim}{x};\underset{\sim}{\theta}_0) < \infty,\ \forall i(=1,\ldots,k) \text{ and } s = 1,2 . \tag{2.17}$$

Further,

$$\lim_{\delta\downarrow 0} \max_{i,a,b}\left\{E\left[\sup_{\underset{\sim}{\theta}:\ ||\underset{\sim}{\theta}-\underset{\sim}{\theta}_0||<\delta}\left|\frac{\partial^2}{\partial\theta_a\partial\theta_b}\log f_i(\underset{\sim}{x}_i;\underset{\sim}{\theta})\Big|_{\underset{\sim}{\theta}} - \frac{\partial^2}{\partial\theta_a\partial\theta_b}\log f_i(\underset{\sim}{x}_i;\underset{\sim}{\theta})\Big|_{\underset{\sim}{\theta}_0}\right|\right]\right\} = 0 \tag{2.18}$$

[It is also possible to avoid (2.18) by some alternative conditions, as in Huber (1967) and Inagaki (1973), but, those will require in turn, some other extra conditions on the first derivatives.]

[A4] For every $i(=1,\ldots,k)$ and $\underset{\sim}{\theta} \in \Omega$,

$$\int_{E^p} (\partial^2/\partial\theta_a\partial\theta_b) f_i(\underset{\sim}{x};\underset{\sim}{\theta}) d\mu(\underset{\sim}{x}) = 0, \forall\ a,b=1,\ldots,t, \tag{2.19}$$

We then define for each $i(=1,\ldots,k)$

$$\underset{\sim}{B}_{\underset{\sim}{\theta}}^{(i)} = E_{\underset{\sim}{\theta}}[(-\underset{\sim}{\partial}^2/\partial\underset{\sim}{\theta}\partial\underset{\sim}{\theta}') \log f_i(\underset{\sim}{x}_i;\underset{\sim}{\theta})] . \tag{2.20}$$

[A5] $\underset{\sim}{B}_{\underset{\sim}{\theta}}^{(1)},\ldots,\underset{\sim}{B}_{\underset{\sim}{\theta}}^{(k)}$ are all continuous in $\underset{\sim}{\theta}$ in some neighborhood of $\underset{\sim}{\theta}_0$ and

$$\underset{\sim}{B}_{\underset{\sim}{\theta}_0}^{*} = \sum_{i=1}^{k} (n_i/n)\underset{\sim}{B}_{\underset{\sim}{\theta}_0}^{(i)} \text{ is positive definite (p.d.)} \tag{2.21}$$

[A6] $\lim_{n\to 0} (n^{-1}n_i) = \rho_i (0 < \rho_i < 1)$ exists, for every $i(=1,\ldots,k)$ and $\sum_{i=1}^{k} \rho_i = 1$.

Note that under [A6], as $n\to\infty$,

$$\underset{\sim}{B}_{\theta_0}^{*} \to \bar{\underset{\sim}{B}}_{\underset{\sim}{\theta}_0} = \sum_{i=1}^{k} \rho_i \underset{\sim}{B}_{\underset{\sim}{\theta}_0}^{(i)} . \tag{2.22}$$

[A7] $\underset{\sim}{h}(\underset{\sim}{\theta})$ possesses continuous first and second order derivatives with respect to $\underset{\sim}{\theta}$, $\forall\ \underset{\sim}{\theta} \in \Omega$. Let then

$$\underset{\sim}{C}_{\underset{\sim}{\theta}} = (((\underset{\sim}{\partial}/\partial\underset{\sim}{\theta})\underset{\sim}{h}(\underset{\sim}{\theta}))) \quad \text{(of order } t\times r) \tag{2.23}$$

[A8] $\underset{\sim}{C}_{\underset{\sim}{\theta}_0}$ is of rank $r(<t)$

[A9] $\underset{\sim}{g}(\underset{\sim}{\theta})$ possesses continuous first and second order derivatives with respect to $\underset{\sim}{\theta}$, $\forall\ \underset{\sim}{\theta} \in \Omega$. Let then

$$\underset{\sim}{D}_{\underset{\sim}{\theta}} = (((\underset{\sim}{\partial}/\partial\underset{\sim}{\theta})\underset{\sim}{g}(\underset{\sim}{\theta}))) \quad \text{(of order } t\times s) \tag{2.24}$$

[A10] $\underset{\sim}{D}_{\underset{\sim}{\theta}_0}$ is of rank $s(<t)$,

[A11] $t > s+r$ and the following matrix (of order $(s+t+r) \times (s+t+r)$)

$$\begin{pmatrix} \bar{\mathbf{B}}_{\theta_0} & -\mathbf{C}_{\theta_0} & -\mathbf{D}_{\theta_0} \\ -\mathbf{C}'_{\theta_0} & & \\ & & \mathbf{0} \\ -\mathbf{D}'_{\theta_0} & & \end{pmatrix} \text{ is of full-rank, } \tag{2.25}$$

We denote by

$$\begin{pmatrix} \bar{\mathbf{B}}_{\theta_0} & -\mathbf{C}_{\theta_0} & -\mathbf{D}_{\theta_0} \\ -\mathbf{C}'_{\theta_0} & & \\ & & \mathbf{0} \\ -\mathbf{D}'_{\theta_0} & & \end{pmatrix}^{-1} = \begin{pmatrix} \bar{\mathbf{P}}_{\theta_0} & \bar{\mathbf{Q}}_{\theta_0} \\ \bar{\mathbf{Q}}'_{\theta_0} & \bar{\mathbf{R}}_{\theta_0} \end{pmatrix} , \tag{2.26}$$

$$\begin{pmatrix} \bar{\mathbf{B}}_{\theta_0} & -\mathbf{C}_{\theta_0} \\ -\mathbf{C}'_{\theta_0} & \mathbf{0} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{P}^*_{\theta_0} & \mathbf{Q}^*_{\theta_0} \\ \mathbf{Q}^{*\prime}_{\theta_0} & \mathbf{R}^*_{\theta_0} \end{pmatrix} , \tag{2.27}$$

and

$$\begin{pmatrix} \bar{\mathbf{B}}_{\theta_0} & -\mathbf{D}_{\theta_0} \\ -\mathbf{D}'_{\theta_0} & \mathbf{0} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{P}^{**}_{\theta_0} & \mathbf{Q}^{**}_{\theta_0} \\ \mathbf{Q}^{**\prime}_{\theta_0} & \mathbf{R}^{**}_{\theta_0} \end{pmatrix} . \tag{2.28}$$

Note that $\bar{\mathbf{B}}$, $\bar{\mathbf{P}}$, $\bar{\mathbf{R}}$, $\mathbf{P}^*$, $\mathbf{R}^*$, $\mathbf{P}^{**}$ and $\mathbf{R}^{**}$ are all symmetric matrices.

Finally, we denote by

$$\boldsymbol{\Lambda}_n = n^{-1/2}(\partial/\partial\boldsymbol{\theta}) \log L_n(\mathbf{x}_n,\boldsymbol{\theta})\big|_{\boldsymbol{\theta}_0} . \tag{2.29}$$

Then, under the assumed regularity conditions, when $\boldsymbol{\theta}_0$ obtains,

$$\boldsymbol{\Lambda}_n \text{ is asymptotically } N_t(\mathbf{0},\bar{\mathbf{B}}_{\theta_0}) . \tag{2.30}$$

3. Asymptotic distribution theory of $L_n^{(0)}$, L_n^*, $\bar{L}_n$, and $\bar{L}_n^*$.

To study the asymptotic nature of (2.12) and (2.13), we need to study first the asymptotic joint distributions of $(L_n^{(0)}, L_n^*)$ and $(L_n^*, \bar{L}_n)$, when the null hypotheses H_0, H_0^* and $\bar{H}_0$, may or may not hold.

For $L_n^{(0)}$ and L_n^*, we may directly use the results of Sections 3 and 4 of Sen (1979) with the allied restraints in (2.2) and (2.5), while for $\bar{L}_n^*$, we need to put the dual restraints in (2.6). Let us denote by

$$\underset{\sim}{A}^{(0)} = (\underset{\sim}{P}^*_{\underset{\sim}{\theta}_0} - \bar{\underset{\sim}{B}}^{-1}_{\underset{\sim}{\theta}_0})\bar{\underset{\sim}{B}}_{\underset{\sim}{\theta}_0}(\underset{\sim}{P}^*_{\underset{\sim}{\theta}_0} - \bar{\underset{\sim}{B}}^{-1}_{\underset{\sim}{\theta}_0}) = \underset{\sim}{Q}^{*\prime}_{\underset{\sim}{\theta}_0}(-\underset{\sim}{R}^*_{\underset{\sim}{\theta}_0})^{-1}\underset{\sim}{Q}^*_{\underset{\sim}{\theta}_0}, \qquad (3.1)$$

$$\underset{\sim}{A}^* = (\underset{\sim}{P}^{**}_{\underset{\sim}{\theta}_0} - \bar{\underset{\sim}{B}}^{-1}_{\underset{\sim}{\theta}_0})\bar{\underset{\sim}{B}}_{\underset{\sim}{\theta}_0}(\underset{\sim}{P}^{**}_{\underset{\sim}{\theta}_0} - \bar{\underset{\sim}{B}}^{-1}_{\underset{\sim}{\theta}_0}) = \underset{\sim}{Q}^{**\prime}_{\underset{\sim}{\theta}_0}(-\underset{\sim}{R}^{**}_{\underset{\sim}{\theta}_0})^{-1}\underset{\sim}{Q}^{**}_{\underset{\sim}{\theta}_0}, \qquad (3.2)$$

$$\bar{\underset{\sim}{A}}^* = (\bar{\underset{\sim}{P}}_{\underset{\sim}{\theta}_0} - \bar{\underset{\sim}{B}}^{-1}_{\underset{\sim}{\theta}_0})\bar{\underset{\sim}{B}}_{\underset{\sim}{\theta}_0}(\bar{\underset{\sim}{P}}_{\underset{\sim}{\theta}_0} - \bar{\underset{\sim}{B}}^{-1}_{\underset{\sim}{\theta}_0}) = \bar{\underset{\sim}{Q}}'_{\underset{\sim}{\theta}_0}(-\bar{\underset{\sim}{R}}_{\underset{\sim}{\theta}_0})^{-1}\bar{\underset{\sim}{Q}}_{\underset{\sim}{\theta}_0}, \qquad (3.3)$$

$$\bar{\underset{\sim}{A}} = \bar{\underset{\sim}{A}}^* - \underset{\sim}{A}^* = (\bar{\underset{\sim}{P}}_{\underset{\sim}{\theta}_0} - \underset{\sim}{P}^{**}_{\underset{\sim}{\theta}_0})\bar{\underset{\sim}{B}}_{\underset{\sim}{\theta}_0}(\bar{\underset{\sim}{P}}_{\underset{\sim}{\theta}_0} - \underset{\sim}{P}^{**}_{\underset{\sim}{\theta}_0}) \quad . \qquad (3.4)$$

Then, proceeding as in Section 3 of Sen (1979) (with direct extensions for the multiple restraints under consideration), we obtain that under the regularity conditions of Section 2,

$$L_n^{(0)} = \underset{\sim}{\Lambda}'_n \underset{\sim}{A}^{(0)} \underset{\sim}{\Lambda}_n + o_p(1) \quad (\text{under } H_0), \qquad (3.5)$$

$$L_n^* = \underset{\sim}{\Lambda}'_n \underset{\sim}{A}^* \underset{\sim}{\Lambda}_n + o_p(1) \quad (\text{under } H_0^*),$$

$$\bar{L}_n^* = \underset{\sim}{\Lambda}'_n \bar{\underset{\sim}{A}}^* \underset{\sim}{\Lambda}_n + o_p(1) \quad (\text{under } \bar{H}_0), \qquad (3.7)$$

$$\bar{L}_n = \underset{\sim}{\Lambda}'_n \bar{\underset{\sim}{A}} \underset{\sim}{\Lambda}_n + o_p(1) \quad (\text{under } \bar{H}_0), \qquad (3.8)$$

where

$$\underset{\sim}{A}^{(0)}\bar{\underset{\sim}{B}}_{\underset{\sim}{\theta}_0}\underset{\sim}{A}^{(0)} = \underset{\sim}{A}^{(0)}, \quad \underset{\sim}{A}^*\bar{\underset{\sim}{B}}_{\underset{\sim}{\theta}_0}\underset{\sim}{A}^* = \underset{\sim}{A}^*, \quad \bar{\underset{\sim}{A}}^*\underset{\sim}{B}_{\underset{\sim}{\theta}_0}\bar{\underset{\sim}{A}}^* = \bar{\underset{\sim}{A}}^*, \qquad (3.9)$$

$$\bar{\underset{\sim}{A}}\,\bar{\underset{\sim}{B}}_{\theta_0}\bar{\underset{\sim}{A}} = \bar{\underset{\sim}{A}} \quad \text{and} \quad \bar{\underset{\sim}{A}}\,\bar{\underset{\sim}{B}}_{\underset{\sim}{\theta}_0}\underset{\sim}{A}^* = \underset{\sim}{0} \; . \qquad (3.10)$$

From (2.30) and (3.5)-(3.10), we conclude that under the regularity conditions of Section 2,

$$L_n^{(0)} \xrightarrow[\mathcal{D}]{} \chi_r^2 \quad (\text{under } H_0), \quad L_n^* \xrightarrow[\mathcal{D}]{} \chi_s^2 \quad (\text{under } H_0^*) \tag{3.11}$$

$$\overline{L}_n^* \xrightarrow[\mathcal{D}]{} \chi_{r+s}^2 \text{ and } \overline{L}_n \xrightarrow[\mathcal{D}]{} \chi_r^2 \quad (\text{under } \overline{H}_0) \tag{3.12}$$

and, further, $\overline{L}_n$ and L_n^* are asymptotically independent under $\overline{H}_0$; here χ_q^2 stands for a r.v. having the chi square d.f. with q degrees of freedom (DF), and we denote its upper 100α% point by $\chi_q^2(\alpha)$. Then, by (2.11) and (3.11)-(3.12), we have, for $n \to \infty$,

$$\overline{L}_{n,\overline{\alpha}} \to \chi_r^2(\overline{\alpha}), \; L_{n\alpha^*}^* \to \chi_s^2(\alpha^*) \text{ and } L_{n,\alpha^0}^{(0)} \to \chi_r^2(\alpha^0) \; . \tag{3.13}$$

On the other hand, in general,

$$\underset{\sim}{A}^{(0)} \overline{\underset{\sim}{B}}_{\underset{\sim}{\theta}_0} \underset{\sim}{A}^* \text{ is not a null matrix,} \tag{3.14}$$

so that $L_n^{(0)}$ and L_n^* are not, generally, asymptotically independent (even under $\overline{H}_0$). However, joint distributions of correlated quadratic forms, developed in Jensen (1970) and Khatri, Krishmaiah and Sen (1977) may be incorporated here in the study of the joint asymptotic distribution of $L_n^{(0)}$ and L_n^*. First consider the case where $\overline{H}_0$ holds. Consider the following probability density function (pdf)

$$\phi(\underset{\sim}{u};\underset{\sim}{b}) = \sum_{m=0}^{\infty} \frac{\Gamma(m+1/2)}{m!\,\Gamma(1/2)} \sum_{\underset{\sim}{0} \le \underset{\sim}{\alpha} \le \underset{\sim}{m}} a_{\underset{\sim}{\alpha}} \frac{\alpha_1!\alpha_2!\,\Gamma(b_1)\Gamma(b_2)}{\Gamma(b_1+\alpha_1)\,\Gamma(b_2+\alpha_2)} \psi(\underset{\sim}{u};\underset{\sim}{b}) L_{\underset{\sim}{\alpha}}(\underset{\sim}{u};\underset{\sim}{b}) \tag{3.15}$$

where

$$\underset{\sim}{u} = (u_1,u_2) \ge 0, \; \underset{\sim}{b} = (b_1,b_2) > \underset{\sim}{0}, \; \underset{\sim}{\alpha} = (\alpha_1,\alpha_2), \; \underset{\sim}{m} = (m\;,m) \;,$$

$$\psi(\underset{\sim}{u};\underset{\sim}{b}) = \prod_{i=1}^{2} \{e^{-u_i} u_i^{b_i-1} / \Gamma(b_i)\}, \; \underset{\sim}{0} \le \underset{\sim}{u} < \underset{\sim}{\infty}, \; \underset{\sim}{b} > \underset{\sim}{0} \;, \tag{3.16}$$

the Laguerre polynomials $L_{\underset{\sim}{\alpha}}$ are defined by

$$\alpha_1!\alpha_2!\psi(\underset{\sim}{u};\underset{\sim}{b})L_{\underset{\sim}{\alpha}}(\underset{\sim}{u};\underset{\sim}{b}) = (-d/d\underset{\sim}{u})^{\underset{\sim}{\alpha}}[u_1^{\alpha_1} u_2^{\alpha_2} \psi(\underset{\sim}{u};\underset{\sim}{b})] , \quad \underset{\sim}{\alpha}\geq\underset{\sim}{0} \quad (3.17)$$

and the $a_{\underset{\sim}{\alpha}}$ are suitable coefficients. Then, by (2.30), (3.5), (3.6), (3.9), (3.14) and Theorem 2 of Jensen (1970), we conclude that under $\overline{H}_0$ and the regularity conditions of Section 2, for every $\underset{\sim}{x}\geq\underset{\sim}{0}$,

$$\lim_{h\to 0} P\{L_n^{(0)}\leq x_1,\ L_n^{*}\leq x_2|\overline{H}_0\} = \int_{\underset{\sim}{0}}^{\underset{\sim}{x}}\phi(\underset{\sim}{u};\ r/2\ ,\ s/2\ )d\underset{\sim}{u}\ , \qquad (3.18)$$

where $\phi(\underset{\sim}{u};\ \tfrac{1}{2}(r,s))$ is defined by (3.15).

In the above development, we have confined ourselves to the case where an appropriate null hypothesis holds. But to study the nature of (2.12) and (2.13), we need to consider the case where H_0 or H_0^* may not hold. For this purpose, we consider some local alternatives and study the asymptotic behavior of the various LRS under such alternatives. We conceive of the following sequence $\{K_n\}$ of alternatives

$$K_n:\ \underset{\sim}{h}(\underset{\sim}{\theta}_0) = n^{-1/2}\underset{\sim}{\gamma}_1,\ \underset{\sim}{g}(\underset{\sim}{\theta}_0) = n^{-1/2}\underset{\sim}{\gamma}_2\ , \qquad (3.19)$$

where $\underset{\sim}{\gamma}_1$ and $\underset{\sim}{\gamma}_2$ are r- and s-vectors of real arguments and $\underset{\sim}{\theta}_0$ is the true parameter point. Then, under H_0, $\underset{\sim}{\gamma}_1 = \underset{\sim}{0}$; H_0^*, $\underset{\sim}{\gamma}_2 = \underset{\sim}{0}$ and $\overline{H}_0$: $\underset{\sim}{\gamma}_1 = \underset{\sim}{0}$, $\underset{\sim}{\gamma}_2 = \underset{\sim}{0}$. Again, we basically follow the steps in Section 4 of Sen (1979) and define $\underset{\sim}{\gamma}_1^*$, $\underset{\sim}{\lambda}_1^*$, $\underset{\sim}{\gamma}_2^*$ and $\underset{\sim}{\lambda}_2^*$ by letting

$$\underset{\sim}{\gamma}_1 = \underset{\sim}{C}'_{\underset{\sim}{\theta}_0}\underset{\sim}{\gamma}_1^*,\ \underset{\sim}{C}_{\underset{\sim}{\theta}_0}\underset{\sim}{\lambda}_1^* = \overline{\underset{\sim}{B}}_{\underset{\sim}{\theta}_0}\underset{\sim}{\gamma}_1^*,\ \underset{\sim}{\gamma}_2 = \underset{\sim}{D}'_{\underset{\sim}{\theta}_0}\underset{\sim}{\gamma}_2^*,\ \underset{\sim}{D}_{\underset{\sim}{\theta}_0}\underset{\sim}{\lambda}_2^* = \overline{\underset{\sim}{B}}_{\underset{\sim}{\theta}_0}\underset{\sim}{\gamma}_2^*\ . \qquad (3.20)$$

Then, $\underset{\sim}{\gamma}_1^*$ and $\underset{\sim}{\gamma}_2^*$ are both t-vectors, while $\underset{\sim}{\lambda}_1^*$ and $\underset{\sim}{\lambda}_2^*$ are r- and s-vectors, respectively. Under $\{K_n\}$ in (3.19) and the regularity conditions of Section 2, we have as in Section 4 of Sen (1979),

$$L_n^{(0)} = (\underset{\sim}{\Lambda}_n'\underset{\sim}{Q}_{\underset{\sim}{\theta}_0}^{*\prime}+\underset{\sim}{\lambda}_1^{*\prime})(-\underset{\sim}{R}_{\underset{\sim}{\theta}_0}^{*})^{-1}(\underset{\sim}{Q}_{\underset{\sim}{\theta}_0}^{*}\underset{\sim}{\Lambda}_n+\underset{\sim}{\lambda}_1^{*}) + o_p(1), \qquad (3.21)$$

$$L_n^{*} = (\underset{\sim}{\Lambda}_n'\underset{\sim}{Q}_{\underset{\sim}{\theta}_0}^{**\prime}+\underset{\sim}{\lambda}_2^{*\prime})(-\underset{\sim}{R}_{\underset{\sim}{\theta}_0}^{**})^{-1}(\underset{\sim}{Q}_{\underset{\sim}{\theta}_0}^{**}\underset{\sim}{\Lambda}_n+\underset{\sim}{\lambda}_2^{*}) + o_p(1). \qquad (3.22)$$

Thus, by (2.30), (3.1) (3.2), (3.9) and (3.21)-(3.22), we conclude that under $\{K_n\}$ and the regularity conditions of Section 2, marginally,

$$\mathcal{L}_n^{(0)} \xrightarrow{\mathcal{D}} \chi^2_{r,\Delta^0} \quad \text{and} \quad \mathcal{L}_n^* \xrightarrow{\mathcal{D}} \chi^2_{s,\Delta^*} \quad , \tag{3.23}$$

where $\chi^2_{q,\Delta}$ stands for a r.v. having the noncentral chi-square d.f. with q DF and noncentrality parameter Δ, and

$$\Delta^0 = \underset{\sim}{\gamma}_1^{*\prime} \bar{\underset{\sim}{B}}_{\underset{\sim}{\theta}_0} \underset{\sim}{\gamma}_1^* = \underset{\sim}{\gamma}_1^{*\prime} \underset{\sim}{C}_{\underset{\sim}{\theta}_0} \underset{\sim}{\lambda}_1^* = \underset{\sim}{\gamma}_1^{\prime} \underset{\sim}{\lambda}_1^* = -\underset{\sim}{\gamma}_1^{\prime} \underset{\sim}{B}^*_{\underset{\sim}{\theta}_0} \underset{\sim}{\gamma}_1 \ , \tag{3.24}$$

$$\Delta^* = \underset{\sim}{\gamma}_2^{*\prime} \bar{\underset{\sim}{B}}_{\underset{\sim}{\theta}_0} \underset{\sim}{\gamma}_2^* = -\underset{\sim}{\gamma}_2^{\prime} \underset{\sim}{B}^{**}_{\underset{\sim}{\theta}_0} \underset{\sim}{\gamma}_2 \ . \tag{3.25}$$

In a similar manner, it follows that under $\{K_n\}$ and the regularity conditions of Section 2,

$$\bar{\mathcal{L}}_n^* = (\underset{\sim}{\Lambda}_n^{\prime} \bar{\underset{\sim}{Q}}_{\underset{\sim}{\theta}_0}^{\prime} + \bar{\underset{\sim}{\lambda}}^{*\prime})(-\bar{\underset{\sim}{B}}_{\underset{\sim}{\theta}_0})^{-1}(\bar{\underset{\sim}{Q}}_{\underset{\sim}{\theta}_0} \underset{\sim}{\Lambda}_n + \bar{\underset{\sim}{\lambda}}^*) + o_p(1) \ , \tag{3.26}$$

where letting $\underset{\sim}{\gamma}^{\prime} = (\underset{\sim}{\gamma}_1^{\prime}, \underset{\sim}{\gamma}_2^{\prime})$, we define

$$\underset{\sim}{\gamma} = \begin{pmatrix} \underset{\sim}{C}^{\prime}_{\underset{\sim}{\theta}_0} \\ \underset{\sim}{D}^{\prime}_{\underset{\sim}{\theta}_0} \end{pmatrix} \bar{\underset{\sim}{\gamma}}^* \quad \text{and} \quad (\underset{\sim}{C}_{\underset{\sim}{\theta}_0} \ , \ \underset{\sim}{D}_{\underset{\sim}{\theta}_0})\bar{\underset{\sim}{\lambda}}^* = \bar{\underset{\sim}{B}}_{\underset{\sim}{\theta}_0} \bar{\underset{\sim}{\gamma}}^* \tag{3.27}$$

so that

$$\bar{\mathcal{L}}_n^* \xrightarrow{\mathcal{D}} \chi^2_{r+s,\Delta^*} \ ; \ \bar{\Delta}^* = -\underset{\sim}{\gamma}^{\prime} \bar{\underset{\sim}{B}}_{\underset{\sim}{\theta}_0} \underset{\sim}{\gamma} \ . \tag{3.28}$$

Since $\bar{\mathcal{L}}_n^* = \bar{\mathcal{L}}_n + \mathcal{L}_n^*$, $\bar{\mathcal{L}}_n \geq 0$, $\mathcal{L}_n^* \geq 0$, by (3.22), (3.23), (3.26), (3.28) and the Cochran theorem, we conclude that under $\{K_n\}$,

$$\bar{\mathcal{L}}_n \xrightarrow{\mathcal{D}} \chi^2_{r,\bar{\Delta}} ; \ \bar{\Delta} = \bar{\Delta}^* - \Delta^* = -\underset{\sim}{\gamma}^{\prime} \bar{\underset{\sim}{B}}_{\underset{\sim}{\theta}_0} \underset{\sim}{\gamma} + \underset{\sim}{\gamma}_2^{\prime} \underset{\sim}{B}^{**}_{\underset{\sim}{\theta}_0} \underset{\sim}{\gamma}_2 \tag{3.29}$$

and furhter that

$$\bar{\mathcal{L}}_n \text{ and } \mathcal{L}_n^* \text{ are asymptotically independent under } \{K_n\}. \tag{3.30}$$

The situation is somewhat different with the joint distribution of $(\mathcal{L}_n^{(0)}, \mathcal{L}_n^*)$

under $\{K_n\}$. Firstly, by (3.14), (3.21) (3.22) and (2.30), they are not generally (asymptotically) independent. Secondly, we have the non-central case where Theorem 2 of Jensen (1970) may not apply. However, we are able to use the results in Khatri, Krishnaiah and Sen (1977) and these provide us with some exact as well as asymptotic expressions.

Note that by [A11], (2.30), (3.21), and (3.22), we can use a suitable (not necessarily unique) non-singular transformation on $\underset{\sim}{\Lambda}_n$ and write

$$L_n^{(0)} = \underset{\sim}{Z}_n^{(1)\prime}\underset{\sim}{Z}_n^{(1)} + o_p(1), \quad L_n^* = \underset{\sim}{Z}_n^{(2)\prime}\underset{\sim}{Z}_n^{(2)} + o_p(1) \qquad (3.31)$$

where $\underset{\sim}{Z}_n^{(1)}$ and $\underset{\sim}{Z}_n^{(2)}$ are r- and s-vectors and, under $\{K_n\}$,

$$\underset{\sim}{Z}_n = (\underset{\sim}{Z}_n^{(1)\prime}, \underset{\sim}{Z}_n^{(2)\prime})' \text{ is asymptotically } N_{r+s}(\underset{\sim}{\zeta}, \Sigma^*) \qquad (3.32)$$

where $\underset{\sim}{\zeta}$ and $\underset{\sim}{\Sigma}^*$ depend on $\overline{\underset{\sim}{B}}_{\underset{\sim}{\theta}_0}$, $\underset{\sim}{Q}^*_{\underset{\sim}{\theta}_0}$, $\underset{\sim}{Q}^{**}_{\underset{\sim}{\theta}_0}$, λ_1^*, λ_2^*, $\underset{\sim}{R}^*_{\underset{\sim}{\theta}_0}$ and $\underset{\sim}{R}^{**}_{\underset{\sim}{\theta}_0}$.

Furhter, Σ^* is non-singular. Let then

$$\underset{\sim}{\Delta} = \text{Diag}(\delta_1 \underset{\sim}{I}_r, \delta_2 \underset{\sim}{I}_s), \quad \underset{\sim}{\delta} = (\delta_1, \delta_2) > \underset{\sim}{0}, \qquad (3.33)$$

$$\underset{\sim}{R} = \underset{\sim}{I}_{r+s} - \underset{\sim}{\Delta}^{-1}\underset{\sim}{\Sigma}^* \text{ and } B = \underset{\sim}{\Delta}^{-1}\underset{\sim}{\zeta}\underset{\sim}{\zeta}'. \qquad (3.34)$$

The choice of $\underset{\sim}{\delta}$ is arbitrary and the convergence rate of the infinite expansion (to follow) depends on $\underset{\sim}{\delta}$ and $\underset{\sim}{\Sigma}^*$. Let then

$$\ell_0 = |\underset{\sim}{I}-\underset{\sim}{R}|^{-1/2}\exp\{-1/2 \text{ trace}[(\underset{\sim}{I}-\underset{\sim}{R})^{-1}B]\}. \qquad (3.35)$$

and for every k>0,

$$\phi_i(\omega;k) = \{(2\delta_i)^k \Gamma(k)\}^{-1}\omega^{k-1}e^{-\omega/2\delta_i}, \quad i=1,2. \qquad (3.36)$$

Finally, let

$$\phi^*(\omega_1,\omega_2) = \ell_0 \Sigma_{\underset{\sim}{j}\geq\underset{\sim}{0}} \ell_{\underset{\sim}{j}} \phi_1(\omega_1;\ r/2+j_1)\phi_2(\omega_2;\ s/2+j_2) \quad (3.37)$$

where $\underset{\sim}{j} = (j_1, j_2)$ and the coefficients $\ell_{\underset{\sim}{j}}$, $\underset{\sim}{j} \geq \underset{\sim}{0}$ are suitable constants. Some formulae for the computation of the $\ell_{\underset{\sim}{j}}$ are also given in Khatri, Krishnaiah and Sen (1977). Then, by (3.31), (3.32) and by (2.11)-(2.12) of Khatri, Krishnaiah and Sen (1977), we obtain that under $\{K_n\}$ and the regularity conditions of Section 2, for every $\underset{\sim}{x} = (x_1, x_2) \geq 0$,

$$\lim_{n\to 0} P\{L_n^{(0)} \leq x_1,\ L_n^* \leq x_2 | K_n\} \tag{3.38}$$
$$= \int_{\underset{\sim}{0}}^{\underset{\sim}{x}} \phi^*(\underset{\sim}{\omega})\, d\underset{\sim}{\omega} = \ell_0 \sum_{\underset{\sim}{j}\geq\underset{\sim}{0}} \ell_{\underset{\sim}{j}} \Phi_1(x_1; r/2 + j_1) \Phi_2(x_2; s/2 + j_2),$$

where Φ_j is the d.f. corresponding to the pdf ϕ_j, $j=1,2$.
Note that by (3.34), $|\underset{\sim}{I}-\underset{\sim}{R}| = |\underset{\sim}{\Sigma}^*| / \delta_1^r \delta_2^s$ and we need to choose δ_1, δ_2, such that $\underset{\sim}{I}-\underset{\sim}{R}$ is p.d. In the central case (where $\zeta=0$), we may take $\ell_o=1$, by letting $\delta_1^r \delta_2^s = |\Sigma^*|$.

4. Asymptotic performance of the three LRT.

We shall study now the comparative performance of the three LRT's based on $L_n^{(0)}$, $\overline{L}_n$ and L_n. In addition to (2.11), we let

$$\nu_{n\alpha}^{(0)} = \begin{cases} 1, & L_n^{(0)} \geq L_{n,\alpha}^{(0)}, \\ 0, & \text{otherwise} \end{cases}; \tag{4.1}$$

$$\overline{\nu}_{n\alpha} = \begin{cases} 1, & \overline{L}_n \geq \overline{L}_{n,\alpha}, \\ 0, & \text{otherwise} \end{cases}; \tag{4.2}$$

and

$$\nu_{n\alpha^*}^* = \begin{cases} 1, & L_n^* \geq L_{n,\alpha^*}^*, \\ 0, & \text{otherwise} \end{cases} \tag{4.3}$$

where, by virtue of (3.13), in the asymptotic case, we may replace the exact critical values $L_{n,\alpha}^{(0)}$ and $\overline{L}_{n,\alpha}$ by $\chi_r^2(\alpha)$ and L_{n,α^*}^* by $\chi_s^2(\alpha^*)$.

Note that under the regularity conditions of Section 2, for any (fixed) $\underset{\sim}{\theta} \notin \Omega^*$ and α^*: $0 < \alpha^* < 1$, $E_{\underset{\sim}{\theta}}\{\nu_{n\alpha^*}^*\} \to 1$ as $n\to\infty$, so that by (2.11), (4.1) and (4.3), for every (fixed) $\alpha^* \in (0,1)$ and

$$[\underset{\sim}{\theta} \notin \Omega^*] \Longrightarrow \nu_n \text{ is asymptotically equivalent to } \nu^0_{n\alpha^0}. \qquad (4.4)$$

The picture is different when $\underset{\sim}{\theta} \in \Omega^*$ (or on a shrinking boundary of Ω^*). In fact, this is domain where we would like to study the behavior of $\nu^{(0)}_{n\alpha}$, $\bar{\nu}_{n\alpha}$ and ν_n. Towards this end, we consider the set of alternatives $\{K_n\}$ in (3.19), so that H_0, H^*_0 and $\bar{H}_0$ are respectively characterized by $\underset{\sim}{\chi}_1=\underset{\sim}{0}$, $\underset{\sim}{\chi}_2=\underset{\sim}{0}$ and $\underset{\sim}{\chi}_1=\underset{\sim}{0}$, $\underset{\sim}{\chi}_2=\underset{\sim}{0}$.

Note that by (4.1) and the results of Section 3,

$$E_{\underset{\sim}{\theta}_0}\{\nu^{(0)}_{n\alpha}|H_0\} = \alpha, \text{ whatever } \underset{\sim}{\chi}_2 \text{ may be }. \qquad (4.5)$$

Also, note that by (3.29), for $\underset{\sim}{\chi}_1=\underset{\sim}{0}$,

$$\bar{\Delta} = -(0', \underset{\sim}{\chi}_2')\bar{\underset{\sim}{R}}_{\underset{\sim}{\theta}_0}\begin{pmatrix}\underset{\sim}{0}\\ \underset{\sim}{\chi}_2\end{pmatrix} + \underset{\sim}{\chi}_2'\underset{\sim}{R}^{**}_{\underset{\sim}{\theta}_0}\underset{\sim}{\chi}_2 \geq 0, \qquad (4.6)$$

where the equality sign holds when $\underset{\sim}{\chi}_2=\underset{\sim}{0}$. Thus, by (3.29) and (4.2),

$$E_{\underset{\sim}{\theta}_0}\{\bar{\nu}_{n\alpha}|\underset{\sim}{\chi}_1=\underset{\sim}{0}\} \to P\{\chi^2_{r,\bar{\Delta}} \geq \chi^2_r(\alpha)\}(\geq \alpha), \qquad (4.7)$$

where the equality sign holds when $\bar{\Delta}=0$ (i.e., $\underset{\sim}{\chi}_2=\underset{\sim}{0}$). This explains the lack of robustness of the restricted LRT based on $\bar{L}_n$. Under $\bar{H}_0$ (i.e., $\underset{\sim}{\chi}_1=\underset{\sim}{0}$, $\underset{\sim}{\chi}_2=\underset{\sim}{0}$), of course, $\bar{\Delta}=0$ and the size of the test based on $\bar{L}_n$ is α. But, under H_0: $\underset{\sim}{\chi}_1=\underset{\sim}{0}$, when nothing is specified of $\underset{\sim}{\chi}_2$, this may be generally $\geq \alpha$. Thus, unless we feel that $\bar{\Delta}$, defined by (4.6) is very close to 0, the use of the restricted LRT may result in a significance level greater than the specified level α. The discouraging fact is that for the set of $\underset{\sim}{\chi}_2$ leading to large values of $\bar{\Delta}$, $E_{\underset{\sim}{\theta}_0}(\bar{\nu}_{n\alpha}|\underset{\sim}{\chi}_1=\underset{\sim}{0})$ tends to 1, so that the restricted LRT may even be inconsistent against such alternatives.

Now, by (2.11), (3.11). (3.12) and (3.18), we obtain that

$$E\{\nu_n|H_0\} \to \bar{\alpha}(1-\alpha^*) + \int_{\chi_r^2(\alpha^0)}^{\infty} \int_{\chi_s^2(\alpha^*)}^{\infty} \phi(\underset{\sim}{u};\ \tfrac{1}{2}(r,s))d\underset{\sim}{u}, \tag{4.8}$$

where $\phi(\cdot,\cdot)$ is defined by (3.15)-(3.17) and the coefficients therein depend on $\underset{\sim}{A}^*$, $\underset{\sim}{A}^{(0)}$ and $\bar{\underset{\sim}{B}}_{\underset{\sim}{\theta}_0}$ (and these may be evaluated by a procedure suggested in Khatri, Krishnaiah and Sen (1977)). Alternately, the right hand side of (4.8) is bounded from above by $\bar{\alpha}(1-\alpha^*) + \alpha^* \wedge \alpha^0$, which may be equated to the desired α. Actually, if both $\bar{\alpha}$ and α^0 are chosen very close to (but less than) α and α^* is small, then this upper bound provides a close approximation, or even, in (4.8), the series in (3.15) may be approximated by a few terms. Parallel to (4.7), we now consider the case, where $\underset{\sim}{\gamma}_1=\underset{\sim}{0}$ but $\underset{\sim}{\gamma}_2$ may not be $\underset{\sim}{0}$. Then, by (2.11), (3.19) and the results of Section 3,

$$\begin{aligned} E_{\underset{\sim}{\theta}_0}\{\nu_n|\underset{\sim}{\gamma}_1=0\} &\to P\{\chi^2_{s,\Delta^*} < \chi^2_s(\alpha^*)\}P\{\chi^2_{r,\bar{\Delta}} \geq \chi^2_r(\bar{\alpha})\} \\ &+ \ell_0\Sigma_{\underset{\sim}{j}\geq 0}\ell_{\underset{\sim}{j}}[1-\Phi_1(\chi^2_r(\alpha^0);r/2+j_1)][1-\Phi_2(\chi^2_s(\alpha^*);s/2+j_2)] , \end{aligned} \tag{4.9}$$

where $\bar{\Delta}$, defined by (4.6) is ≥ 0 (with the equality sign for $\underset{\sim}{\gamma}_2=\underset{\sim}{0}$) and $\ell_0, \ell_{\underset{\sim}{j}}$, $\underset{\sim}{j}\geq\underset{\sim}{0}$ as well as Φ_1, Φ_2 are defined as in (3.38) and these depend on $\underset{\sim}{A}^{(0)}$, $\underset{\sim}{A}^*$, $\bar{\underset{\sim}{B}}_{\underset{\sim}{\theta}_0}$ and $\underset{\sim}{\gamma}_2$. In this context, note that

$$(1-\alpha^*\geq)P\{\chi^2_{s,\Delta^*} < \chi^2_s(\alpha^*)\} \text{ is } \searrow \text{ in } \Delta^*(\geq 0), \tag{4.10}$$

$$(\bar{\alpha} \leq)P\{\chi^2_{r,\bar{\Delta}} \geq \chi^2_r(\bar{\alpha})\} \text{ is } \nearrow \text{ in } \bar{\Delta}(\geq 0), \tag{4.11}$$

and the second term on the right hand side of (4.9) is bounded by

$$\alpha^0 \wedge P\{\chi^2_{s,\Delta^*} \geq \chi^2_s(\alpha^*)\} . \tag{4.12}$$

Thus, unlike (4.7), though (4.9) is affected by $\underset{\sim}{\gamma}_2 \neq \underset{\sim}{0}$ it may not converge to 1 as $\bar{\Delta}$ or Δ^* blows up. Or, in the other words, it is more robust against $\underset{\sim}{\gamma}_2 \neq \underset{\sim}{0}$, than the restricted LRT. Hence, from considerations of validity-robustness, ν_n may be preferred to $\bar{\nu}_{n\alpha}$.

If, in particular, $\underset{\sim}{A}^{(0)}\bar{\underset{\sim}{B}}_{\underset{\sim}{\theta}_0}\underset{\sim}{A}^*$ is a null matrix, then (4.9) reduces to

$$\begin{aligned} &P\{\chi^2_{s,\Delta^*} < \chi^2_s(\alpha^*)\}P\{\chi^2_{r,\bar{\Delta}} \geq \chi^2_r(\bar{\alpha}) + \\ &\qquad P\{\chi^2_{s,\Delta^*} \geq \chi^2_s(\alpha^*)\}P\{\chi^2_{r,0} \geq \chi^2_r(\alpha^0)\} \\ &= \alpha_0 + P\{\chi^2_{s,\Delta^*} < \chi^2_s(\alpha^*)\}[P\{\chi^2_{r,\bar{\Delta}} \geq \chi^2_r(\bar{\alpha})\} - \alpha^0] , \end{aligned} \tag{4.13}$$

and the validity-robustness picture becomes more clear. In this case, we have

$$\alpha = \alpha^0 + (1-\alpha^*)(\bar{\alpha}-\alpha^0) , \tag{4.14}$$

so that on letting $\alpha^0 = \bar{\alpha} = \alpha$, one may choose α^* arbitrarily.

Let us now proceed to the study of the asymptotic power functions of the three LRT's. As in (4.4), for any fixed alternative, there is not much interest in studying these (as the limits dagenerate at α or 1), and hence, we confine ourselves to local alternatives, as in (3.19), for which the limits are different from 1.

First, consider the case of the unrestricted LRT $\nu^{(0)}_{n\alpha}$. From (3.21), (3.23) and (3.24), we obtain that

$$\lim_{n\to\infty} E\{\nu^{(0)}_{n\alpha}|K_n\} = P\{\chi^2_{r,\Delta^0} \geq \chi^2_r(\alpha)\} , \tag{4.15}$$

where Δ^0 is defined by (3.24). Similarly, by (3.28) and (3.29),

$$\lim_{n\to\infty} E\{\bar{\nu}_{n\alpha}|K_n\} = P\{\chi^2_{r,\bar{\Delta}} \geq \chi^2_r(\alpha)\} , \tag{4.16}$$

where $\bar{\Delta}$ is defined by (3.29). For a comparison of (4.15) and (4.16), we may note that by (2,25)-(2,28),

$$-\bar{\underset{\sim}{R}}_{\underset{\sim}{\theta}_0} = [\begin{pmatrix} -\underset{\sim}{C}'_{\underset{\sim}{\theta}_0} \\ -\underset{\sim}{D}'_{\underset{\sim}{\theta}_0} \end{pmatrix} \bar{\underset{\sim}{B}}^{-1}_{\underset{\sim}{\theta}_0}(-\underset{\sim}{C}_{\underset{\sim}{\theta}_0} \ -\underset{\sim}{D}_{\underset{\sim}{\theta}_0})]^{-1} \tag{4.17}$$

$$-\underset{\sim}{R}^*_{\underset{\sim}{\theta}_0} = (\underset{\sim}{C}'_{\underset{\sim}{\theta}_0}\overline{\underset{\sim}{B}}^{-1}_{\underset{\sim}{\theta}_0}\underset{\sim}{C}_{\underset{\sim}{\theta}_0})^{-1} \text{ and } -\underset{\sim}{R}^{**}_{\underset{\sim}{\theta}_0} = (\underset{\sim}{D}'_{\underset{\sim}{\theta}_0}\overline{\underset{\sim}{B}}^{-1}_{\underset{\sim}{\theta}_0}\underset{\sim}{D}_{\underset{\sim}{\theta}_0})^{-1} , \qquad (4.18)$$

Thus,

$$\Delta^0 = \underset{\sim}{\gamma}_1'(\underset{\sim}{C}'_{\underset{\sim}{\theta}_0}\overline{\underset{\sim}{B}}^{-1}_{\underset{\sim}{\theta}_0}\underset{\sim}{C}_{\underset{\sim}{\theta}_0})^{-1}\underset{\sim}{\gamma}_1 = \underset{\sim}{\gamma}_1^{*\prime}\underset{\sim}{C}'_{\underset{\sim}{\theta}_0}(\underset{\sim}{C}'_{\underset{\sim}{\theta}_0}\overline{\underset{\sim}{B}}^{-1}_{\theta_0}\underset{\sim}{C}_{\underset{\sim}{\theta}_0})^{-1}\underset{\sim}{C}_{\underset{\sim}{\theta}_0}\underset{\sim}{\gamma}_1^* , \qquad (4.19)$$

$$\begin{aligned}\overline{\Delta} &= \underset{\sim}{\gamma}'\{(\underset{\sim}{C}_{\underset{\sim}{\theta}_0}\underset{\sim}{D}_{\underset{\sim}{\theta}_0})'\overline{\underset{\sim}{B}}^{-1}_{\underset{\sim}{\theta}_0}(\underset{\sim}{C}_{\underset{\sim}{\theta}_0}\underset{\sim}{D}_{\underset{\sim}{\theta}_0})\}^{-1}\underset{\sim}{\gamma} - \underset{\sim}{\gamma}_2'(\underset{\sim}{D}'_{\underset{\sim}{\theta}_0}\overline{\underset{\sim}{B}}^{-1}_{\underset{\sim}{\theta}_0}\underset{\sim}{D}_{\theta_0})^{-1}\underset{\sim}{\gamma}_2 \\ &= (\underset{\sim}{\gamma}_1-\underset{\sim}{\Gamma}\underset{\sim}{\gamma}_2)'\underset{\sim}{\Sigma}^*(\underset{\sim}{\gamma}_1-\underset{\sim}{\Gamma}\underset{\sim}{\gamma}_2) \quad , \qquad (4.20)\end{aligned}$$

where writing $(\underset{\sim}{C}_{\underset{\sim}{\theta}_0}\underset{\sim}{D}_{\underset{\sim}{\theta}_0})'\overline{\underset{\sim}{B}}^{-1}_{\theta_0}(\underset{\sim}{C}_{\underset{\sim}{\theta}_0}\underset{\sim}{D}_{\underset{\sim}{\theta}_0}) = \begin{pmatrix}\underset{\sim}{\Sigma}_{11} & \underset{\sim}{\Sigma}_{12} \\ \underset{\sim}{\Sigma}_{21} & \underset{\sim}{\Sigma}_{22}\end{pmatrix}$, we have

$$\underset{\sim}{\Gamma} = -\underset{\sim}{\Sigma}_{12}\underset{\sim}{\Sigma}_{22}^{-1} \text{ and } \underset{\sim}{\Sigma}^* = (\underset{\sim}{\Sigma}_{11}-\underset{\sim}{\Sigma}_{12}\underset{\sim}{\Sigma}_{22}^{-1}\underset{\sim}{\Sigma}_{21})^{-1} . \qquad (4.21)$$

Hence, from (4.19), (4.20) and (4.21), we have

$$\overline{\Delta} - \Delta^0 = (\underset{\sim}{\gamma}_1-\underset{\sim}{\Gamma}\underset{\sim}{\gamma}_2)'\underset{\sim}{\Sigma}^*(\underset{\sim}{\gamma}_1-\underset{\sim}{\Gamma}\underset{\sim}{\gamma}_2) - \underset{\sim}{\gamma}_1'\Sigma_{11}^{-1}\underset{\sim}{\gamma}_1 . \qquad (4.22)$$

From (4.22), we immediately claim that

$$\underset{\sim}{\gamma}_2 = \underset{\sim}{0} \Rightarrow \overline{\Delta} \geq \Delta^0, \text{ with = holding for } \underset{\sim}{\Sigma}_{12} = \underset{\sim}{0}. \qquad (4.23)$$

Hence, if H_0^*: $\underset{\sim}{\gamma}_2 = \underset{\sim}{0}$ holds, then the restricted LRT $\overline{\nu}_{n\alpha}$ has an asymptotic power (against: $\underset{\sim}{h}(\underset{\sim}{\theta}) = n^{-1/2}\underset{\sim}{\gamma}_1$) greater than or equal to that of the unrestricted LRT $\nu_{n\alpha}^{(0)}$. The picture may be different when H_0^* may not hold. For example, if $\underset{\sim}{\gamma}_2 \neq \underset{\sim}{0}$ but $\underset{\sim}{\gamma}_1 = \underset{\sim}{\Gamma}\underset{\sim}{\gamma}_2$, then by (4.22), $\overline{\Delta} = 0$, $\Delta^0 > 0$, so that the unrestricted LRT performs better than the restricted one. In general,

$$\overline{\Delta} \geq \Delta^0 \text{ when } ch_1(\underset{\sim}{\Sigma}^*(\underset{\sim}{\gamma}_1-\underset{\sim}{\Gamma}\underset{\sim}{\gamma}_2)(\underset{\sim}{\gamma}_1-\underset{\sim}{\Gamma}\underset{\sim}{\gamma}_2)') \geq ch_1(\underset{\sim}{\Sigma}_{11}^{-1}\underset{\sim}{\gamma}_1\underset{\sim}{\gamma}_1'), \qquad (4.24)$$

where ch_1 stands for the largest characteristic root. Clearly, in a neighborhood of $\underset{\sim}{\Gamma}\underset{\sim}{\gamma}_2$, this may not hold. This explains the lack of efficiency-robustness of the restricted LRT, when H_0^* may not hold.

For the preliminary test LRT ν_n in (2.11), we obtain that

$$\lim_{n\to\infty} E\{\nu_n|K_n\} = P\{\chi^2_{s,\Delta*} < \chi^2_s(\alpha*)\}P\{\chi^2_{r,\bar{\Delta}} \geq \chi^2_r(\bar{\alpha})\}$$

$$+ P\{\chi^2_{s,\Delta*} \geq \chi^2_s(\alpha*),\ \chi^2_{r,\Delta^0} \geq \chi^2_r(\alpha^0)\}\ , \qquad (4.25)$$

where $(\chi^2_{s,\Delta*}, \chi^2_{r,\Delta^0})$ has (jointly) a bivariate chi-square distribution (non-central case), given by (3.37), with the coefficients depending on $\Delta*$, Δ^0 and $-\bar{R}_{\theta_0}$. If, in particular, $A^{(0)}\bar{B}_{\theta_0}A*$ is 0, then (4.25) reduces to

$$P\{\chi^2_{r,\Delta^0} \geq \chi^2_r(\alpha^0)\} + P\{\chi^2_{s,\Delta*} < \chi^2_s(\alpha*)\}[P\{\chi^2_{r,\bar{\Delta}} \geq \chi^2_r(\bar{\alpha})\} - P\{\chi^2_{r,\Delta^0} \geq \chi^2_r(\alpha^0)\}], \quad (4.26)$$

so that by arguments similar to in (4.22)-(4.24), we conclude that (4.26), lies in between (4.15) and (4.16). In particular, if $\alpha^0 = \bar{\alpha} = \alpha$, then (4.26) reduces further to

$$P\{\chi^2_{r,\Delta^0} \geq \chi^2_r(\alpha)\}P\{\chi^2_{s,\Delta*} \geq \chi^2_s(\alpha*)\} +$$

$$[1 - P\{\chi^2_{s,\Delta*} \geq \chi^2_s(\alpha*)\}]P\{\chi^2_{r,\bar{\Delta}} \geq \chi^2_r(\alpha)\}, \qquad (4.27)$$

which is an weighted average of (4.15) and (4.16). In general, (for $A^{(0)}\bar{B}_{\theta_0}A^*$ not necessarily 0), the second term on the right hand side of (4.25) can be evaluated by using (3.38) and it may be concluded that the asymptotic power of ν_α lies in between that of $\nu^{(0)}_{n\alpha}$ and $\bar{\nu}_{n\alpha}$, and further, ν_n is more (less) efficiency-robust than $\bar{\nu}_n(\nu^{(0)}_n)$ when H^*_0 may not hold.

5. Some general remarks.

From the results of Section 4, it follows that unlike the case of the unrestricted LRT, for the preliminary test LRT, the computation of the size needs elaborate expansion as in (3.38). The situation becomes simpler when $A^{(0)}\bar{B}_{\theta_0}A^*$ is 0; the later case arises in many linear models, where the design matrix permits this condition. Also, both $\bar{\nu}_{n\alpha}$ and ν_n have size and power affected by the validity of H_0^*. But, ν_n is more robust than $\bar{\nu}_{n\alpha}$ against departures from $\underset{\sim}{g}(\underset{\sim}{\theta}) = \underset{\sim}{0}$. Thus, from validity-robustness point of view, ν_n may be preferred to $\bar{\nu}_{n\alpha}$. On the other hand, for H_0^* being true, but $\underset{\sim}{h}(\underset{\sim}{\theta}) \neq \underset{\sim}{0}$, the asymptotic power of $\bar{\nu}_{n\alpha}$ is better than that of ν_n and $\nu_{n\alpha}^{(0)}$, although a different picture may emerge when H_0^* may not hold and χ_1 is close to $\underset{\sim}{\Gamma}\chi_2$, in which case, ν_n performs better than $\bar{\nu}_{n\alpha}$. Thus, from the efficiency-robustness point of view, ν_n may be preferred to $\nu_{n\alpha}^{(0)}$ or $\bar{\nu}_{n\alpha}$.

The actual computations of the asymptotic size and power function of the three LRT depend on $\bar{\underset{\sim}{R}}_{\underset{\sim}{\theta}_0}$ as well as χ_1, χ_2 . In some simple case, this may however be done. For example, for testing for the intercept parameter (when the regression parameter may or may not be equal to 0) in a simple regression model (which includes the two-sample location model as a special case), this comparative picture is very similar to the nonparametric case dealt with in Saleh and Sen (1980). A definite advantage of ν_n over $\bar{\nu}_{n\alpha}$ (or $\nu_{n\alpha}^{(0)}$) may be seen from the numerical values presented there.

References

1. Aitchison, J. and Silvey, S.D.: Maximum likelihood estimation of parameters subject to restraints. *Ann. Math. Statist. 29*, 813-828, 1958

2. Bechhofer, R.E.: The effect of preliminary test of significance on the size and power of certain tests of univariate linear hypotheses. *Unpublished Ph. D. Thesis*, Columbia University, 1951

3. Bozivich, H., Bancroft, T.A. and Hartley, H.: Power of analysis of variance tests procedures for certain incompletely specified models, I. *Ann. Math. Statist. 27*, 1017-1043, 1956

4. Huber, P.J.: The behavior of maximum likelihood estimators under nonstandard conditions. *Proc. 5th Berkeley Symp. Math. Statist. Prob. 1*, 221-234, 1967

5. Inagaki, N.: Asymptotic relations between the likelihood estimating function and the maximum likelihood estimator. *Ann. Inst. Statist. Math.* *25*, 1-16, 1973

6. Jensen, D.R.: The joint distribution of traces of Wishart matrices and some applications. *Ann. Math. Statist.* *41*, 133-145, 1970

7. Khatri, C.G., Krishnaiah, P.R. and Sen, P.K.: A note on the joint distribution of correlated quadratic forms. *Jour. Statist. Plann. Inf.* *1*, 299-307, 1977

8. Saleh, A.K. Md.E. and Sen, P.K.: Nonparametric tests for location after a preliminary test on regression. (to be published), 1980

9. Sen, P.K.: Asymptotic properties of maximum likelihood estimators based on conditional specification. *Ann. Statist.* *7*, 1019-1033, 1979

10. Tamura, R.: Nonparametric inference with a preliminary test. *Bull. Math. Statist.* *14*, 39-61, 1965