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CONSTRUCTIONS OF DESIGNS

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INTRODUCTION .

Our purpose is to give a systematic exposition of combinatorial methods in the construction of designs (BIB-designs) . We add some new constructions to other known ones and show how various kinds of designs are related . In this way we may obtain a more homogeneous insight of designs obtained by different methods and demonstrate how some designs are obtainable by other ones . Some constructions, as applied to particular designs, e.g., finite projective and affine spaces, lead to the construction of new designs . Finally, the methods given may help to identify the combinatorial structure of designs obtained with other mathematical techniques, which is often the most useful characteristic of designs in statistical applications .

We devote section 2 to the exposition of designs and the description of the construction methods used . In section 3 we give properties and relationships between the various designs defined in section 2 . We show in section 4 more properties, applications and examples .

1. PRELIMINARY DEFINITIONS AND NOTATIONS .

For the definitions of incidence structure and non-degenerate incidence structure, see [1] , sec.1.1 . We refer to [1] or [2] for most of the well known properties of designs which we quote without proof .

We recall some definitions which we will use later .

<u>Definition 1.1</u> . Let $\mathfrak{S} = \mathfrak{S}(\mathfrak{P}, \mathfrak{B}, I)$ be an incidence structure.

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Let $p \in \mathcal{P}$, $B \in \mathcal{B}$. The *dual* incidence structure is the incidence structure $\mathcal{S}^* = (\mathcal{P}^*, \mathcal{B}^*, I^*)$ where $\mathcal{P}^* = \mathcal{B}$, $\mathcal{B}^* = \mathcal{P}$ and $I^* = \{(B,p)$ such that $(p,B) \in I\}$.

<u>Definition 1.2</u> . A balanced incomplete block design or, more briefly, a *design* D is a non-degenerate incidence structure such that

i) every block is incident with the same positive number of points ,

ii) every pair of points is incident with the same positive number of blocks .

We use these notations :

$\mathcal{P}$: set of points of D , $|\mathcal{P}| = v$.

$\mathcal{B}$: set of blocks of D , $|\mathcal{B}| = b$.

$[B] = k$: number of points incident with any block B (points "on" B).

$[p] = r$: number of blocks incident with any point p (blocks "through" p) .

$[pq] = \lambda$: number of blocks incident with any pair of points p and q (blocks "through" p and q).

$\bar{B}$: points of D which are not incident with the block B (or the complement in $\mathcal{P}$ of the set of points which are incident with B) .

For every design D :

$$b \geq v \ ,$$

(1.1) $vr = bk \quad , \quad \lambda(v-1) = r(k-1)$ (cf. [1] or [2]) .

<u>Definition 1.3</u> . A *symmetric* design is a design such that $v = b$. For a symmetric design, $r = k$ and $[BC] = \lambda$, $\forall B, C \in \mathcal{B}$ ($[BC]$: number of points incident with both blocks B and C) .

Definition 1.4. Let $t>1$. A *t-design* D_t is a non-degenerate incidence structure such that

ti) every block is incident with the same (positive) number of points,

tii) every t points of D_t are incident with the same (positive) number λ_t of blocks.

One may verify (see [3] or [10]) that any t-design is an s-design for all $2 \leq s \leq t$, with

$$(1.2) \qquad \lambda_s = \lambda_t \frac{\binom{v-s}{t-s}}{\binom{k-s}{t-s}} .$$

Any design is a 2-design.

In the following, the parameter λ referring to a t-design D_t will always mean λ_2, i.e., the number of blocks incident with any two points. We have

$$\lambda = \lambda_2 = \lambda_t \frac{\binom{v-2}{t-2}}{\binom{k-2}{t-2}} \qquad \text{and} \qquad \lambda_3 = \lambda \frac{k-2}{v-2} .$$

Definition 1.5. A *simple* design is a design without repeated blocks (*repeated* blocks are blocks which are incident with the same points).

We call a design D with repeated blocks *simplifiable* if the structure we get by counting only once the repeated blocks is a simple design. I.e., let $\mathbb{R}$ be the following equivalence relation: $B \mathbb{R} B'$ if B, B' are repeated blocks; then D is simplifiable if the incidence structure $D' = D'(\mathcal{P}, \mathcal{B}/\mathbb{R}, I')$ with the incidence I' induced by I, is a simple design.

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Note that for simple designs, any block may be identified with the set of points with which it is incident, and $B \neq C$ as blocks if and only if $B \neq C$ as point sets. We will also use the notation B for the set of all points through B for non-simple designs if this doesn't cause any ambiguity.

Note that all symmetric designs are simple.

2. CONSTRUCTIONS OF DESIGNS.

We give several combinatorial constructions of designs obtained from other designs. Some of them are well known and appear frequently in the literature on designs (cf. [1], [2] and [10]), some are more recent or new. These constructions however are not all independent (see Theorems 3.4, 3.5, 3.7, sec.3).

In order to avoid burdening our exposition too much with statements of propositions, we will use the following conventions. The proofs given with the several designs will show the following two facts: a) that the structure we are intruducing is a design (i.e., axioms i), ii) of Definition 1.2 hold), b) that the parameters (b,v,r,k,λ) are exactly the ones indicated in order. When axiom i) obviously holds we only prove axiom ii). Where not otherwise stated we suppose D is a simple design. We will also mention when the construction introduced gives a design with repeated blocks.

Let D be a design with parameters (b,v,r,k,λ).

The following four designs occur frequently in the literature (see [1], [2]). One may easily check i) and ii) for each.

1) The *complementary design* $\mathcal{C}D$.

Let $\mathcal{C}D$ be the incidence structure whose points are the points of D and whose blocks are the sets $\bar{B} = \mathcal{P} - B$, $\forall B \in \mathcal{B}$. $\mathcal{C}D$ is a design with parameters

$$(b , v , b-r , v-k , b-2r+\lambda) .$$

Construction 1) may also be given for designs with repeated blocks. We have that $\mathcal{C}D$ is simple iff D is simple and that $\mathcal{C}D$ is simplifiable iff D is simplifiable .

2) The *subtract design* $\mathcal{S}D$.

Let $\mathcal{S}D$ be the incidence structure whose points are the points of D and whose blocks are all the k-subsets of $\mathcal{P}$ ($k = [B]$) which are not blocks of D . D is a design with parameters

$$\left(\binom{v}{k} - b \ , \ v \ , \binom{v-1}{k-1} - r \ , \ k \ , \binom{v-2}{k-2} - \lambda \right) .$$

Note that $\mathcal{S}D$ cannot be defined if D has repeated blocks .

Let now D be a symmetric design (so its parameters are (v,v,k,k,λ)) and assume $\lambda > 1$.

3) The *derived design* $\mathcal{D}_B D$ with respect to a block B .

Fix $B \in \mathcal{B}$. Let $\mathcal{D}_B D$ be the incidence structure whose points are the points incident with B and whose blocks are the sets $B \cap B_i$, $\forall B_i \in \mathcal{B}$, $B_i \neq B$. $\mathcal{D}_B D$ may have repeated blocks . $\mathcal{D}_B D$ is a design with parameters

$$(v-1 , k , k-1 , \lambda , \lambda - 1) .$$

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4) The *residual design* $\mathfrak{R}_B D$ with respect to a block B .

Fix $B \in \mathfrak{B}$. Let $\mathfrak{R}_B D$ be the incidence structure whose points are the points of $\bar{B} = \mathfrak{P} - B$ and whose blocks are the set differences $B - B_i$, $\forall B_i \in \mathfrak{B} - \{B\}$. $\mathfrak{R}_B D$ is a design with parameters

$$(v-1 , v-k, k , k-\lambda , \lambda) .$$

5) The *intersection design* $\mathfrak{J}D$.

Let D be a symmetric design with $\lambda > 1$.

Let $\mathfrak{J}D$ be the incidence structure which has the same points as D and which has as blocks all the intersections of unordered pairs of distinct blocks of D: $B_i \cap B_j$, $i \neq j$. $\mathfrak{J}D$ is a design with parameters :

$$\left(\binom{v}{2} , v , \binom{k}{2} , \lambda , \binom{\lambda}{2} \right) .$$

Proof. Every block of $\mathfrak{J}D$ is composed of the λ points incident with two distinct blocks of the symmetric design D . Any two points of D are incident with λ blocks of D from which $\binom{\lambda}{2}$ pairs of distinct blocks arise thus giving $\binom{\lambda}{2}$ intersections of pairs . As different pairs of blocks may have the same intersection as point sets, $\mathfrak{J}D$ may have repeated blocks (this case occurs frequently) . The remaining parameters of $\mathfrak{J}D$ are obtained by direct enumeration, or by applying (1.1) .

$\mathfrak{J}D$ is obtained in [9] by considering the "dual" structure of blocks and blocks which are incident with a pair of points, for all the pairs of points .

6) The *union design* $\mathfrak{U}D$.

Let D be a symmetric design .

Let $\mathfrak{U}D$ be the incidence structure which has the same points of D and which has as blocks all the unions of unordered pairs of distinct blocks of D : $B_i \cup B_j$, $i \neq j$. $\mathcal{U}D$ is a design with parameters :

$$\left(\binom{v}{2} , v , \frac{(v-1)(2k-\lambda)}{2} , 2k-\lambda , \binom{2k-\lambda}{2} \right).$$

Proof. The number of points of the union of two blocks of D is $2k-\lambda$. In order to enumerate the number of blocks of the type $B_i \cup B_j$ incident with a pair of points, let us consider the dual structure of D . D^* is a simple symmetric design with the same parameters v,k,λ as D . Given any two points p,q of D , the number of unions $B_i \cup B_j$ incident with the points p,q is given by the number of pairs B_i, B_j which have two distinct incidences with p,q , eventually repeated . This is equivalent to enumerating in the dual D^* the pairs of points (duals of B_i, B_j) having distinct incidences, eventually repeated, with two blocks (duals of p,q) . This is the number of pairs of distinct points in the set of $2k-\lambda$ points of a two-blocks union, i.e., $\binom{2k-\lambda}{2}$. The other parameters are obtained with direct enumeration or from (1.1) .

$\mathcal{U}D$ is obtained in [6], with similar enumerative methods.

7) The *symmetric difference design* $\mathcal{W}D$.

Let D be a symmetric design.

Let $\mathcal{W}D$ be the incidence structure on the same points of D which has as blocks the symmetric differences of unordered pairs of distinct blocks of D : $(B_i \cup B_j)-(B_i \cap B_j) = B_i \Delta B_j$, $i \neq j$. $\mathcal{W}D$ is a design with parameters

$$\left(\binom{v}{2} , v , k(v-k) , 2(k-\lambda) , \binom{2(k-\lambda)}{2} \right).$$

Proof. We proceed as in the proof of 6) . Every symmetric difference

$B_i \triangle B_j$ has $2(k-\lambda)$ points. Given any two points p,q of D, the number of blocks of the form $B_i \triangle B_j$ incident with points p,q is given by the number of pairs B_i, B_j which have precisely two distinct incidences with p,q, i.e., two distinct and not repeated incidences. If we consider the dual design D^* as we did in the proof of 6), we find that the number we are looking for is the number of pairs of points of D^* (duals of blocks of D) which have precisely two unrepeated incidences with the pair of blocks p^*, q^* (duals of p,q) of D^*. This corresponds to enumerating the distinct pairs of points of the symmetric difference $p^* \triangle q^*$. In D^* we have $|p^* \triangle q^*| = 2(k-\lambda)$, giving $\binom{2(k-\lambda)}{2}$ as was to be shown. The remaining parameters are obtained from (1.1).

$\mathcal{W}D$ is obtained by Majindar ([4], Theor.1) by making use of the incidence matrix of D and by considering the dual structure. He obtains in the same way (cf.[4],Theor.2) the complementary design of $\mathcal{U}D$ (i.e. $\mathcal{C}(\mathcal{U}D)$ with our notation, cf. constructions 1), 6)).

7') The *difference design* $\mathcal{V}D$.

Let D be a symmetric design.

Take the incidence structure $\mathcal{V}D$ on points of D, with blocks being the set differences of all ordered pairs of distinct blocks of D, $B_i - B_j$, $i \neq j$. $\mathcal{V}D$ has parameters

$$\big(v(v-1),\ v,\ k(v-k),\ k-\lambda,\ (k-\lambda)(k-\lambda-1) \big).$$

Proof. The blocks of $\mathcal{V}D$ all have $k-\lambda$ points. There are λ blocks of D which are incident with any two points p,q of D, and $v-2k+\lambda$ blocks of D which are not incident with p,q. Then there are $\lambda(v-2k+\lambda)$ differences $B_i - B_j$ of pairs of blocks B_i, B_j, with B_i incident with p,q and B_j not incident with p,q; these are the blocks $B_i - B_j$ incident with p,q. We may derive the other parameters from (1.1) and easily check that $(k-\lambda)(k-\lambda-1) = \lambda(v-2k+\lambda)$.

Note. All designs 3),..,7') may have repeated blocks. This happens frequently for designs 3), 5). We'll say more on this subject in section 3.

Now let D be a t-design, $t \geq 3$. We can obtain the following designs 8), 9).

8) The *restricted* or *internal design* D_p, with respect to a point p.

Let D be a t-design, $t \geq 3$. Fix $p \in \mathcal{P}$. Let D_p be the incidence structure with points set $\mathcal{P} - \{p\}$ and with blocks the blocks of D incident with p. D_p is a (t-1)-design with parameters

$$\left(r \, , \, v-1 \, , \, \lambda \, , \, k-1 \, , \, \lambda \frac{k-2}{v-2} \right) .$$

Here λ is the parameter of D as a BIB-design, i.e. $\lambda = \lambda_2(D)$. The parameters of D_p are also to be meant as BIB-design parameters.

Proof. Every block of D_p has k-1 points. Given t-1 points of D $p_1,..,p_{t-1}$ distinct from p, there are λ_t blocks of D which are incident with $p_1,..,p_{t-1}, p$. Then, through any t-1 points of D_p there are λ_t blocks incident with p (i.e. blocks of D_p). Therefore D_p is a (t-1)-design with $\lambda_{t-1}(D_p) = \lambda_t(D)$. The number of blocks of D_p incident with any two points p_1,p_2 of D_p is the number of blocks of D incident with p_1,p_2,p, i.e., $\lambda_3(D)$. From (1.2) we have $\lambda(D_p) = \lambda_3(D) = \lambda \frac{k-2}{v-2}$.

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9) The *external design* $D_{\bar{p}}$ with respect to a point p .

Let D be a t-design, $t \geq 3$. Let $D_{\bar{p}}$ be the incidence structure with point set $\mathcal{P} - \{p\}$ and blocks the blocks of D not incident with p . $D_{\bar{p}}$ is a (t-1)-design with parameters

$$(b-r \; , \; v-1 \; , \; r-\lambda \; , \; k \; , \; \lambda \frac{v-k}{v-2}) .$$

Proof. Every block of $D_{\bar{p}}$ has k points . Given any t-1 points of D different from p , there are $\lambda_{t-1}(D)$ blocks of D incident with them. Of these blocks, $\lambda_t(D)$ are incident with p and $\lambda_{t-1}(D) - \lambda_t(D)$ are not . Then $D_{\bar{p}}$ is a (t-1)-design with

$\lambda_{t-1}(D_{\bar{p}}) = \lambda_{t-1}(D) - \lambda_t(D) = \lambda_t \dfrac{v-k}{k-t+1}$ (use (1.2) of section 1).

The blocks of $D_{\bar{p}}$ incident with any two points p_1, p_2 are the blocks of D incident with p_1, p_2 and not incident with p , so

$\lambda(D_{\bar{p}}) = \lambda - \lambda(D_p) = \lambda \dfrac{v-k}{v-2}$.

Notations. Some authors use notation D_B and D^B for $\mathcal{I}_B D$ and $\mathcal{R}_B D$ respectively, and D_p , D^p for D_p , $D_{\bar{p}}$.

Observations on designs 8), 9) .

a) Let D be a Steiner system S(t,k,v) , that is a t-design with $\lambda_t = 1$. Then
   D_p is a Steiner system S(t-1,v-1,k-1) ,
   $D_{\bar{p}}$ is not a Steiner system but it is a (t-1)-design with
   $\lambda_{t-1}(D_p) = \dfrac{v-k}{k-t+1}$.

b) A t-design may have repeated blocks, while a Steiner system is always simple as is any external design obtained from a Steiner system.

c) If D is a t-design with $t \geq 3$, the constructions 8), 9) may be iterated or composed. One gets in this way s-designs with $2 \leq s \leq t$. Young and Edmonds in [10] studied the iterated internal designs of Steiner system $S(5,8,24)$.

d) Conversely, given two designs with parameters as in 8), 9) respectively, we may obtain a design with the same parameters as D. This is the approach used in [7]. This point of view for D_p and $D_{\bar{p}}$ ia also considered in [1].

Other constructions.

We introduce other constructions obtainable from designs whose parameters satisfy some restrictions.

10) The *composition design* $D \times D'$.

Let D be a design with parameters (b,v,r,k,λ) and D' be a design with $|\mathcal{P}| = k$ and parameters $(b', k, r', k', \lambda')$. Let us consider for every block B of D the design on points of B which is isomorphic to D'. We obtain for every block B of D b' blocks each of them incident with k' points. The composition design $D \times D'$ has the same points as D and as blocks the set of bb' blocks obtained in that way. $D \times D'$ has parameters

$$(bb', v, rr', k', \lambda\lambda').$$

$D \times D'$ may have repeated blocks.

Proof. Direct check (cf. [3]).

Example.

An easy example of composition is obtained by composing any design D with parameters (b,v,r,k,λ) and the design of all the

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h-tuples ($h<k$) of a set of k elements. Denote this design $U_{h,k}$. We get the composition $D\times U_{h,k}$ with parameters $\left(b\binom{k}{h}, v, r\binom{k-1}{h-1}, h, \binom{k-2}{h-2} \right)$. We will give in section 4 some applications of this kind of construction.

11) The *addition design* $D+D'$.

Let D, D' be designs with the same number of points v and blocks of the same size k: $D\ (b,v,r,k,\lambda)$, $D'\ (b',v,r',k,\lambda')$. Note that one may check that $\frac{b}{b'}=\frac{r}{r'}=\frac{\lambda}{\lambda'}$. Suppose D, D' are given on the same point set $\mathcal{P}$, $|\mathcal{P}|=v$. Take the incidence structure $D+D'$ with the same points as D, D' and with blocks the blocks of D and the blocks of D'. $D+D'$ is a design with parameters

$$(b+b', v, r+r', k, \lambda+\lambda').$$

The proof is a trivial check. More generally, one may obtain from D, D' linear combinations (with integer non-negative coefficients) with repeated blocks (cf. [7]). We confine ourselves to the addition $D+D'$ as we are interested chiefly in simple designs. Note that $D+D'$ is a simple design iff D, D' are simple designs with no common blocks.

Using the idea of the addition design, the subtract design 2) is obtained as the unique design δD such that $D+\delta D=U_{k,v}$.

Finding non-trivial examples of designs D, D' that satisfy conditions of 11) (for example, simple and non-isomorphic D, D') is not easy. We will see in section 4 how to enlarge the class of addable designs D, D' in a systematic way using complement and subtract designs.

A similar construction to 11) which is more difficult to exploit is the following.

12) The *subtraction design* $D - D'$.

Let D be a design with parameters (b,v,r,k,λ) . Let D' be an incidence structure with the same points as D and whose block set is a subset $\mathcal{B}' \subset \mathcal{B}$ of the block set of D . D' may or may not be a design. If D' is a design, then we may define the subtraction design $D - D'$ to be the incidence structure with the same points as D (and D') and whose blocks are the blocks of D not of D' : $\mathcal{B} - \mathcal{B}'$.

The proof given the above conditions imposed on D , D' is trivial. If D' has parameters (b', v, r', k, λ') with $\frac{b'}{b} = \frac{r'}{r} = \frac{\lambda'}{\lambda}$ then $D - D'$ has parameters

$$(b-b' , v , r-r' , k , \lambda-\lambda') .$$

Note. The hypotheses for constructions 11) and 12) are not equivalent. The ones in 12) are more restrictive. Indeed, for $D + D'$ to be a design, D , D' must have the same v and k , while for $D - D'$ to be a subtraction design, D , D' must have the same v and k and also D' must have as blocks a subset of the block set of D . We will give in section 4 an application of this construction.

3. COMBINATORIAL PROPERTIES AND RELATIONSHIPS BETWEEN DESIGNS .

We first note that since complementary design $\mathcal{C}D$ preserves symmetry of D , we may perform any of the constructions obtained in section 2 from a symmetric design D on $\mathcal{C}D$. This is also true for t-designs, as the following proposition shows.

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Proposition 3.1. (cf. [10]). Let D be a t-design with $v \geq k+t$. Then the complementary design of D, $\mathfrak{C}D$, is a t-design with parameters as in section 2, 1).

Proof. See [10].

The same kind of property holds for subtract designs.

Proposition 3.2. (cf. [10]). Let D be a simple t-design. The subtract ("opposite" in [10]) design of D, $\mathfrak{S}D$, is a t-design with parameters as in section 2, 2).

We pointed out in section 2, 5) that the intersection design $\mathfrak{J}D$ may have repeated blocks. Proposition 3.3 gives a sufficient condition on a symmetric design D for $\mathfrak{J}D$ be simplifiable (see Def. 1.5). Theorems 3.4 and 3.5 are meant to prove in addition that derived and intersection designs $\mathfrak{D}_B D$ and $\mathfrak{J}D$ with repeated blocks don't occur more frequently than residual and union designs $\mathfrak{R}_B D$ and $\mathfrak{U}D$ with repeated blocks, in spite of what one might suppose from an examination of most common cases (cf. observations in [1], page 3, footnote 2)). On the contrary, there is a bijective correspondence between the former and the latter types.

Proposition 3.3. Let D be a symmetric design. If in the dual design D^* every line has the same number of points, then the intersection design $\mathfrak{J}D$ is simplifiable.

Proof. A "line" in a design is the set of points incident with all blocks through two distinct points. If every line of D^* has the same cardinality h, then it follows by dualizing this property that in D we have that the number of blocks incident with all

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Proof. A "line" in a design is the set of points incident with all blocks through two distinct points. If every line of D^* has the same cardinality h, then it follows by dualizing this property that in D we have that the number of blocks incident with all points common to any two blocks B_i, B_j (that is, the points of $B_i \cap B_j$) is constant and equal to h. Then, any intersection of pairs of blocks is repeated $\binom{h}{2}$ times, as many times as are the pairs of blocks both containing $B_i \cap B_j$. This means that every block of $\mathcal{J}D$ is repeated $\binom{h}{2}$ times. It follows that $\binom{h}{2}$ divides the parameters b, r, λ of D, so if we count only once the $\binom{h}{2}$ repetitions we get the simple design $\mathcal{J}D$ with parameters

$$\left(\frac{v(v-1)}{h(h-1)}, v, \frac{k(k-1)}{h(h-1)}, \lambda, \frac{\lambda(\lambda-1)}{h(h-1)} \right).$$

Corollary 3.3. If in the dual design of D every line has the same cardinality h, then the derived design $\mathcal{D}_B D$ is simplifiable, for each $B \in \mathcal{B}$, thus giving a design with parameters

$$\left(\frac{v-1}{h-1}, v, \frac{k-1}{h-1}, \lambda, \frac{\lambda-1}{h-1} \right).$$

The following theorem proves that constructions 1), 3) and 4) of section 2 are not independent.

Theorem 3.4. Let D be a symmetric design. Every residual design with respect to a block B of D is the complementary of the derived design of $\mathcal{C}D$ with respect to its block $\bar{B}$. That is

$$\mathcal{R}_B D = \mathcal{C}_{\bar{B}} \left[\mathcal{D}_{\bar{B}}(\mathcal{C}D) \right]. \tag{3.1}$$

We also have obviously

$$\mathcal{D}_B D = \mathcal{C}_B \left[\mathcal{R}_{\bar{B}}(\mathcal{C}D) \right]. \tag{3.2}$$

The subscript of the symbol $\mathcal{C}$ recalls the point set on which the complementary is defined.

Proof. Let B_i be the blocks of D, $B_i \neq B$. The blocks of $\mathcal{C}D$ are of the form $\bar{B}, \bar{B}_i$. Then the blocks of $\mathcal{D}_{\bar{B}}(\mathcal{C}D)$ are of the form $\bar{B} \cap \bar{B}_i$

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and the ones of $\mathcal{E}_{\bar{B}}\left[\mathcal{D}_{\bar{B}}(\mathcal{E}D)\right]$ are of the form $\bar{B}-\bar{B}_i = B_i - B$ and these are the blocks of $\mathcal{R}_B D$.

Example. The most notable example of a residual design is the residual design of a finite projective space $D = PG(d,q)$, $d \geq 3$, with respect to one of its hyperplanes $B = PG(d-1,q)$. $\mathcal{R}_B D$ is the affine space $AG(d,q)$.

The derived design of D with respect to B , $\mathcal{D}_B D$, is the design whose blocks are the (d-2)-dimensional subspaces PG(d-2,q) which are contained in $B = PG(d-1,q)$. These are the intersections with B of all the other hyperplanes PG(d-1,q) which are contained in $D = PG(d,q)$. The design $\mathcal{D}_B D$ with repeated blocks obtained in this way has parameters $(v-1, k, k-1, \lambda, \lambda-1)$ where $v = \frac{q^{d+1}-1}{q-1}$, $k = \frac{q^d-1}{q-1}$, $\lambda = \frac{q^{d-1}-1}{q-1}$. That is

$$\left(q\,\frac{q^d-1}{q-1},\ \frac{q^d-1}{q-1},\ q\,\frac{q^{d-1}-1}{q-1},\ \frac{q^{d-1}-1}{q-1},\ q\,\frac{q^{d-2}-1}{q-1} \right).$$

The dual of D is also a projective space PG(d,q) where every line has q+1 points . Then, by Corollary 3.3 , $\mathcal{D}_B D$ is a simplifiable design which simplifies to a symmetric design with parameters $(v',k',\lambda') = \left(\frac{q^d-1}{q-1},\ \frac{q^{d-1}-1}{q-1},\ \frac{q^{d-2}-1}{q-1} \right)$ (after division by $q = h-1$ of b',r',λ'). This is the projective space PG(d-1,q).

Starting with $D = PG(d,q)$ one may obtain an example of a residual design with repeated blocks by considering $\mathcal{E}_B(\mathcal{D}_B D) = \mathcal{R}_{\bar{B}}(\mathcal{E}D)$ and an example of a simple derived design by taking $\mathcal{E}_{\bar{B}}(\mathcal{R}_B D) = \mathcal{D}_{\bar{B}}(\mathcal{E}D)$ (see Theorem 3.4) .

The relationship observed between derived, residual and complementary designs holds for $\mathfrak{I}D$, $\mathfrak{U}D$ and complementary designs also. This is shown by the following theorem.

Theorem 3.5. Let D be a symmetric design. The union design $\mathfrak{U}D$ is the complementary design of the intersection design obtained from $\mathscr{C}D$. That is:

(3.3) $$\mathfrak{U}D = \mathscr{C}\left[\mathfrak{J}(\mathscr{C}D)\right].$$

Also the following obviously holds (though it is not as obvious as it would be if $\mathscr{C}D$ was the set-theoretic complement: $\mathscr{C}D$ is the operator defined in section 2, 1)):

(3.4) $$\mathfrak{J}D = \mathscr{C}\left[\mathfrak{U}(\mathscr{C}D)\right].$$

Proof. We use the same type of argument as in Theorem 3.4. Let B_i be the blocks of D. Then the blocks of $\mathscr{C}D$ are of the form $\bar{B}_i$, the blocks of $\mathfrak{J}(\mathscr{C}D)$ are all the intersections $\bar{B}_i \cap \bar{B}_j = \overline{B_i \cup B_j}$, $i \neq j$. Then the blocks of $\mathscr{C}\left[\mathfrak{J}(\mathscr{C}D)\right]$ are all the unions $B_i \cup B_j$.

We give now a new proof of a theorem by Mendelsohn (cf. [5]) which shows (and make use of) the incidence duality relating the construction of derived and internal designs, as well as residual and external designs.

Theorem 3.6. There is no symmetric t-design with $t \geq 3$.

Observation. We recall that given a t-design D, $t \geq 3$, the internal design of D with respect to a point p has by its definition as points all the points of D different from p and as blocks all blocks incident with p. Given a symmetric design D, the derived design with respect to a block B, $\mathfrak{D}_B D$, is obtained by the "dual" definition of the former: $\mathfrak{D}_B D$ has as blocks all blocks of D different from B and as points, all points incident with B.

Proof of Theorem 3.6. Let D be a t-design, $t \geq 3$. Suppose D is symmetric. Then its dual D^* is symmetric. Let $\mathfrak{D}_p D^*$ be the

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derived design of D^* with respect to a block p (p is a point of D). It follows from the above observation that the dual structure of $\mathcal{D}_p D^*$ is the internal structure of D with respect to p . As D is a t-design, $t \geq 3$, its internal structure is the internal design D_p . Then D_p and $\mathcal{D}_p D^*$ are dual designs. This implies they are both symmetric since the Fisher inequality $b \geq v$ holds for both of them. But D_p being symmetric forces D to be a degenerate structure. Indeed, the parameters of D_p (see sec.2, 8)) lead to $k = v-1$ for the parameters of D . This means D should be the degenerate structure with v points and with all (v-1)-point subsets as blocks.

Theorem 3.4 has the following "dual" statement which relates internal, external and complementary designs.

<u>Theorem 3.7</u> . Let D be a t-design , $t \geq 3$. Then the following relation holds :

(3.5) $$D_{\bar{p}} = \mathcal{C}\left[(\mathcal{C}D)_p\right] .$$

We also have obviously :

(3.6) $$D_p = \mathcal{C}\left[(\mathcal{C}D)_{\bar{p}}\right] .$$

Proof. $(\mathcal{C}D)_p$ is a design with point set $\mathcal{P} - \{p\}$ and with blocks of the form $\bar{B} - \{p\}$, $\forall B \in \mathcal{B}$ such that $p \in \bar{B}$, i.e. $p \notin B$. $\mathcal{C}\left[(\mathcal{C}D)_p\right]$ is a design with point set $\mathcal{P} - \{p\}$ whose blocks are the complements of $\bar{B} - \{p\}$ in $\mathcal{P} - \{p\}$, $\forall B \in \mathcal{B}$ such that $p \notin B$. But these are precisely all blocks B of D such that $p \notin B$, i.e., the block set of $D_{\bar{p}}$.

It is possible to exploit Theorems 3.4 , 3.5 , 3.7 to obtain constructions 4), 6), 9) from 3), 5), 8) (sec.2) respectively without need of any proof by using the following corollaries.

Corollary 3.4 (of Theorem 3.4) . Given any symmetric design D , if $\mathcal{D}_B D$ is the derived design with respect to a block B of D , then the structure $\mathcal{R}_B D$ is a design since $\mathcal{R}_B D = \mathcal{C}_{\bar{B}}\left[\mathcal{D}_{\bar{B}}(\mathcal{C}D)\right]$.

Corollary 3.5 (of Theorem 3.5). If $\mathfrak{J}D$ is the intersection design obtained from D , then the structure $\mathcal{U}D$ is a design since $\mathcal{U}D = \mathcal{C}\left[\mathfrak{J}(\mathcal{C}D)\right]$.

Corollary 3.7 (of Theorem 3.7). If D_p is the internal design of D with respect to a point p , then the structure $D_{\bar{p}}$ is a design since $D_{\bar{p}} = \mathcal{C}\left[(\mathcal{C}D)_p\right]$.

4. OTHER PROPERTIES , EXAMPLES AND APPLICATIONS .

Property 4.1 . The addition design D + D' can be combined with the complementary design, and if D + D' is a simple design it can be combined with the subtract design, thus giving new addition designs with the following properties :

$$\mathcal{C}(D + D') = \mathcal{C}D + \mathcal{C}D'$$

$$\mathcal{S}(D) = \mathcal{S}(D + D') + D' .$$

We omit the easy proof.

A systematic method of obtaining addition designs is the following.

Property 4.2 . Let D be a t-design , $t \geq 3$, with parameters (b,v,r,k,λ) . Consider the internal and external designs D_p , $D_{\bar{p}}$

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(see 2. 8),9)) and $U_{k-1,k}$ (degenerate design with k points and with blocks being all the (k-1)-points subsets). Then D_p and the composition $D_{\bar{p}} \times U_{k-1,k}$ (see 2. 10)) satisfy the conditions of 2. 11) with $|\mathcal{P}| = v-1$ and $[B] = k-1$ and so

$$D(p) = D_p + (D_{\bar{p}} \times U_{k-1,k})$$

is a design (indeed it is a (t-1)-design). Furthermore, if D is a Steiner system then D(p) is simple.

Proof. The first assertion is already proven. Let D be a Steiner system. We prove that D(p) is simple. By construction a block B of $D(p) = D_p + (D_{\bar{p}} \times U_{k-1,k})$ is repeated if and only if it is contained in at least two distinct blocks of D . But B has k-1 points, so since D is a Steiner system there is exactly one block of D incident with t of them. Then B cannot be repeated so D(p) is simple.

We get the following parameters for D(p) :

$$(\; r+k(b-r) \; , \; v-1 \; , \; \lambda+(r-\lambda)(k-1) \; , \; k-1 \; , \; \lambda \frac{k-2}{v-2}(v-k+1) \;)$$

i.e. $(\; r(v-k+1) \; , \; v-1 \; , \; \lambda(v-k+1) \; , \; k-1 \; , \; \lambda \frac{k-2}{v-2}(v-k+1) \;)$.

The proportionality of the 1st, 3rd and 5th parameters with the ones of $D_{\bar{p}}$ is no surprise, as D_p and D(p) have the same points and have blocks of the same size (cf. 2. 11)) . As a (t-1)-design, D(p) has parameters $\lambda_{t-1}(D(p)) = \lambda_t(v-k+1)$.

We give now some examples and applications of the constructions given in section 2 .

<u>Example 4.1</u> . (Application of composition, addition and subtract designs) . We prove that in a projective plane PG(2,q) the structure with points being the points of PG(2,q) and with blocks being all

the 4-tuples of points such that exactly three of them are collinear is a design.

In PG(2,q), the 4-tuples of points with no three on a line (4-arcs) give the blocks of a design D_1 on the points of PG(2,q) with parameters (cf. Example 4.2, 2)):

$$b=\frac{(q^2+q+1)(q^2+q)q^2(q-1)^2}{24},\ v=q^2+q+1,\ r=\frac{(q^2+q)q^2(q-1)^2}{6},\ k=4,\ \lambda=\frac{q^2(q-1)^2}{2}.$$

The 4-tuples of points all on a line give a design D_2: it is the composition of design D with blocks being the lines of PG(2,q) with $U_{4,q+1}$. $D_2 = D\times U_{4,q+1}$ has parameters:

$$\left((q^2+q+1)\binom{q+1}{4},\ q^2+q+1,\ (q+1)\binom{q}{3},\ 4,\ \binom{q-1}{2} \right).$$

Then we may form the addition design D_1+D_2 with parameters:

$$v=q^2+q+1,\quad k=4,\quad \lambda=\frac{(q-1)\ q^2(q-1)+q-2}{2}.$$

Therefore the structure we are looking for is clearly the subtract design $\mathfrak{z}(D_1+D_2)$. The parameters of $\mathfrak{z}(D_1+D_2)$ are obtained with easy (but some tedious) calculations:

$$b=\frac{(q^2+q+1)(q^2+q)q^2(q-1)^2}{24},\ v=q^2+q+1,\ r=(q^2+q)q^2(q-1)\frac{2}{3},\ k=4,\ \lambda=2q^2(q-1).$$

For $q=3$ we get:

D_1 design of 4-arcs with parameters (234, 13, 72, 4, 18),

D_2 design of 4-tuples on a line (lines of PG(2,3)) with parameters (13, 13, 4, 4, 1),

D_1+D_2 with parameters (247, 13, 76, 4, 19),

$\mathfrak{z}(D_1+D_2)$ with parameters (468, 13, 144, 4, 36).

The trivial design of all the 4-tuples of PG(2,3) has parameters (715, 13, 220, 4, 55).

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Example 4.2. (Application of composition and subtraction designs) Designs from k-arcs of PG(2,q).

Using the same enumerative technique used in [8],n.173 to compute the number of irreducible conics of PG(2,q), it may be shown that

1) The points and irreducible conics of PG(2,q) form a design with parameters ($q^2(q^3-1)$, q^2+q+1 , $q^2(q^2-1)$, $q+1$, $q^2(q-1)$).
2) Fix $k \leq 6$. The set of all k-arcs (sets of k points no three on a line) of PG(2,q) is the block set of a design on the points of PG(2,q) with the following parameters: $v = q^2+q+1$ and:

$$k = 3: \quad \lambda = q^2 ,$$

$$k = 4: \quad \lambda = \frac{q^2(q-1)^2}{2} ,$$

$$k = 5: \quad \lambda = \frac{q^2(q-1)^2(q-2)(q-3)}{6} ,$$

$$k = 6: \quad \lambda = \frac{q^2(q-1)^2(q-2)(q-3)\ (q-4)(q-5)+1}{24} .$$

Let D be the design obtained in 1). For a fixed k, $3 \leq k \leq q+1$, the configuration of points of PG(2,q) and of all k-arcs of PG(2,q) which are contained in irreducible conics is the composition design $D \times U_{k,q+1}$.

Then whenever we may affirm that all k-arcs of PG(2,q) are contained in irreducible conics, we can conclude that the set of all k-arcs of PG(2,q) is the block set of a design on points of PG(2,q). For instance, in PG(2,q), q odd, since all q-arcs are contained in irredicible conics (cf. [8],n.175), we have that the q-arcs of PG(2,q) are the blocks of a design on points of PG(2,q) with parameters

$$(q^2(q^2-1)(q^2+q+1) , q^2+q+1 , q^3(q^2-1), q , q^2(q-1)^2),$$

with repeated blocks for $q = 3$, simple for $q \geq 5$.

Let D_6 be the design obtained in 2) for $k=6$, i.e. the design on points of PG(2,q) with blocks being all the 6-arcs of PG(2,q). Given any q, $q>6$, there are 6-arcs of PG(2,q) which are contained in irreducible conics and 6-arcs which are not. The whole set of 6-arcs forms the design D_6, while the set of all 6-arcs which are contained in irreducible conics forms the design $D \times U_{6,q+1}$ (recall D is the design of conics defined in 1)).

Then we may obtain the subtraction design (see sec.2 12)) $D_6 - (D \times U_{6,q+1})$. We conclude that the set of 6-arcs not contained in conics is the set of blocks of a design on points of PG(2,q), for any $q>6$.

Example 4.3. (External and composition designs obtained from Steiner systems).

Let us consider the Steiner system S(5,8,24) (here S(t,k,v) means a t-design with $\lambda_t = 1$, $|\mathcal{P}| = v$, $|B| = k$). We obtain a Steiner system D = S(3,6,22) by applying the internal "operator" (see 2.8)) repeatedly to two points of S(5,8,24). It is known (cf. [10]) that by applying again the internal operator to a third point p we get a structure D_p on 21 points which is isomorphic to PG(2,4).

The external structure $D_{\bar{p}}$ has interesting geometric properties also:

1) $D_{\bar{p}}$ is a design on points of PG(2,4) with parameters (56,21,16,6,4). These values have a certain interest as there are no designs whose parameters b,r,λ divide the ones of $D_{\bar{p}}$.
2) The blocks of $D_{\bar{p}}$ are (q+2)-arcs (ovals) of PG(2,4). In fact the blocks of $D_{\bar{p}}$ cannot have 3 points on a line of PG(2,4).

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This is because any such three points of $D = S(3,6,22)$ which are collinear in PG(2,4) are already incident with the block of D made by the 5 points of this line of PG(2,4) and the point p. (Recall that there is only one block incident with three points of D). The design of all 6-arcs of PG(2,4) (see Example 4.2, 2)) has parameters (168,21,48,6,12). Then $D_{\bar{p}}$ is a design whose blocks are given by 56 of the total of 168 6-arcs of PG(2,4). To it the subtraction design $D_6 - D_{\bar{p}}$ is associated, and its blocks are the remaining 6-arcs of PG(2,4).

3) We may obtain from $D_{\bar{p}}$ the design $D_{\bar{p}} \times U_{5,6}$ with parameters (336,21,80,5,16), whose blocks are the 336 irreducible conics of PG(2,4) which are contained in 6-arcs of $D_{\bar{p}}$ (out of the total of 1008 irreducible conics of PG(2,4).

5. CONCLUDING REMARKS.

Nowadays most of the current research on combinatorial properties of incidence structures is primarily concerned with some more "complex" designs which have less regularity properties than BIB-designs, e.g. partially balanced block designs, or that have additional structural properties (e.g. resolvable designs). Nevertheless, the combinatorial structural properties of designs, as well as of other more basic and elementary incidence structure, like tactical configurations (cf. [1]), seems not yet to have been fully explored. The basic properties of constructions like internal and external incidence structure and of duality, and the way they are related, are not yet fully understood.

Another inexhausted direction of research is the designs obtainable from projective or affine geometries. Geometric spaces enjoy simple and full regularity in their structure. This suggests that one might try to obtain from them some other regular incidence structures by the use of purely combinatorial and geometric methods similar to those presented here.

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