ON PERMUTATIONAL CENTRAL LIMIT THEOREMS FOR GENERAL MULTIVARIATE LINEAR RANK STATISTICS

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SUMMARY. For multivariate linear rank statistics, permutational central limit theorems have been proved, mostly, either incorrectly or under unnecessarily stringent regularity conditions. These theorems are revisited here with some special emphasis on a novel martingale approach.

1. INTRODUCTION

In nonparametric multivariate analysis, permutational central limit theorems (PCLT) play a vital role. In the context of multvariate multisample rank order tests, Puri and Sen(1966,Theorem 4.1) considered a PCLT and this was later [Puri and Sen(1969,Theorem 3.2)] extended to general linear models. In either case, the proof provided by these authors is not correct (as will be explained in Section 2); nevertheless, the theorems remain valid under the stringent regularity conditions stated there. These theorems are multivariate generalizations of the classical (univariate) PCLT, which in its most general form is due to Hájek(1961). He was able to incorporate a powerful (quadratic mean) equivalence result for linear rank statistics and linear combinations of independent random variables (r.v.) which provide the desired result through the classical CLT. Puri and Sen(1971, Theorem 5.4.1) adapted this technique in the multivariate case; but, their result on the PCLT does not properly follow from the Hájek(1961) result, though the conclusion on the unconditional null distribution remains true. Conclusion on the PCLT in the multivariate case based on the moment-convergence property [cf. Wald and Wolfowitz(1944)] is, of course, valid, but demands comparatively stringent regularity conditions. All these call for a re-examination of multivariate PCLT.

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The object of the present investigation is to provide a systematic account of multivariate PCLT's along with a novel martingale approach. The main results along with the preliminary notions are presented in Section 2 and their proofs are considered in Section 3. In this context, a martingale approach, developed in the context of progressive censoring by Chatterjee and Sen(1973) and extended further by Sen(1979), is incorporated in a convenient proof of a multivariate PCLT under less stringent regularity conditions.

2. THE MAIN THEOREMS

Let $\underset{\sim}{X}_i = (X_{i1},\dots, X_{ip})'$, i=1,...,n, be n independent and identically distributed (i.i.d.) r.v.'s with a continuous distribution function (d.f.) F, defined on the Euclidean space R^p, where p is a positive integer. Let $\underset{\sim}{c}_i = (c_{i1},\dots,c_{iq})'$, i=1,...,n, be a set of known (regression) vectors, where $q \geq 1$. For each j (=1,...,p), let R_{ij} be the rank of X_{ij} among $X_{1j},\dots,X_{nj}$, for i=1, ...,n (ties among the observations are neglected, in probability, as F is assumed to be continuous), and let $a_{nj}(1),\dots,a_{nj}(n)$ be a set of scores. Then, a set of multivariate linear rank statistics (LRS) $\underset{\sim}{L}_n = ((L_{njk}))$ may be defined by

$$L_{njk} = \Sigma_{i=1}^{n}(c_{ij} - \bar{c}_j)a_{nj}(R_{ij}) \;;\; \bar{c}_j = n^{-1}\Sigma_{i=1}^{n}c_{ij} \;, \tag{2.1}$$

for j=1,...,p and k=1,...,q. In general, because of the inter-dependence of the p variates, $\underset{\sim}{L}_n$ is not genuinely distribution-free. However, it is permutationally (conditionally) distribution-free under the following rank permutation model , due to Chatterjee and Sen(1964). Consider the <u>rank-collection matrix</u> $\underset{\sim}{\mathbb{R}}_n$ (of order pxn) specified by

$$\underset{\sim}{\mathbb{R}}_n = (\underset{\sim}{R}_1,\dots,\underset{\sim}{R}_n) = \begin{pmatrix} R_{11},\dots, R_{n1} \\ \cdots\cdots\cdots\cdots \\ R_{1p},\dots, R_{np} \end{pmatrix} , \tag{2.2}$$

where $\underset{\sim}{R}_i' = (R_{i1},\dots,R_{ip})$, for i=1,...,n. Consider now a permutation of the columns of $\underset{\sim}{\mathbb{R}}_n$ so that the top row is in the natural order (viz.,1,...,n),and denote the resulting matrix (termed the reduced <u>rank-collection matrix</u>) by $\underset{\sim}{\mathbb{R}}_n^*$. Note that the totality of $(n!)^p$ rank collection matrices may thus be partitioned into $(n!)^{p-1}$ subsets, where each subset corresponds to a particular reduced rank collection matrix $\underset{\sim}{\mathbb{R}}_n^*$ and the subset $\mathcal{S}(\underset{\sim}{\mathbb{R}}_n^*)$ has cardinality n!. The conditional distribution of $\underset{\sim}{\mathbb{R}}_n$ over the appropriate $\mathcal{S}(\underset{\sim}{\mathbb{R}}_n^*)$ is uniform, irrespective of the (continuous) d.f. F. We denote this conditional (permutational) probability measure by $\mathcal{P}_n$. Then, we have [see Puri and Sen(1969)]

$$E_{\mathcal{P}_n} \underset{\sim}{L}_n = \underset{\sim}{0} \quad \text{and} \quad V_{\mathcal{P}_n} \underset{\sim}{L}_n = \underset{\sim}{V}_n \otimes \underset{\sim}{C}_n , \tag{2.3}$$

where $\underset{\sim}{V}_n = ((v_{njj'}))_{j,j'=1,\ldots,p}$ has the elements

$$v_{njj'} = (n-1)^{-1}\Sigma_{i=1}^n \{a_{nj}(R_{ij}) - \bar{a}_{nj}\}\{a_{nj'}(R_{ij'}) - \bar{a}_{nj'}\} ; \tag{2.4}$$

$$\bar{a}_{nj} = n^{-1}\Sigma_{i=1}^n a_{nj}(i) , \ j=1,\ldots,p \tag{2.5}$$

and

$$\underset{\sim}{C}_n = \Sigma_{i=1}^n (\underset{\sim}{c}_i - \bar{\underset{\sim}{c}}_n)(\underset{\sim}{c}_i - \bar{\underset{\sim}{c}}_n)' ; \ \bar{\underset{\sim}{c}}_n = (\bar{c}_1,\ldots,\bar{c}_q)' \tag{2.6}$$

Our primary concern is to study the asymptotic multi-normality of $\underset{\sim}{L}_n$ under the <u>permutation model</u> $\mathcal{P}_n$; this is termed the multivariate PCLT.

Since the $\underset{\sim}{c}_i$ are specified vectors, without any loss of generality, we may assume that the rank of $\underset{\sim}{C}_n = ((C_{nkk'}))$ is q , for n adequately large. More specifically, we let $\underset{\sim}{C}_n = \underset{\sim}{D}_n \underset{\sim}{Q}_n \underset{\sim}{D}_n$ where $\underset{\sim}{D}_n = \text{Diag}(C_{n11}^{1/2},\ldots,C_{nqq}^{1/2})$ and assume that π_n, the smallest characteristic root of $\underset{\sim}{Q}_n$, satisfies the following:

$$\pi_n \geq Q_o > 0 , \text{ for every } n \geq n_o \ (> q). \tag{2.7}$$

Further, as in Majumdar and Sen(1978), we let

$$\xi_n = \max_{1\leq i\leq n} \{(\underset{\sim}{c}_i - \bar{\underset{\sim}{c}}_n)' \underset{\sim}{C}_n^{-1} (\underset{\sim}{c}_i - \bar{\underset{\sim}{c}}_n)\} \tag{2.8}$$

and define the (extended) <u>Noether condition</u> as

$$\xi_n \to 0 \text{ as } n \to \infty ; \tag{2.9}$$

also, the <u>extended Hájek</u>(1968) <u>condition</u> is defined by

$$\sup_{n\geq n_o} \{n\, \xi_n\} \leq \xi^* < \infty . \tag{2.10}$$

In the multi-sample case, treated by Puri and Sen(1966), (2.10) holds, while in the linear model case, Puri and Sen(1969) assumed that (2.10) holds; this may not be really needed. In both these papers, sufficiently stringent conditions on the scores were imposed which insure that $\underset{\sim}{V}_n$ converges in probability to a positive definite (p.d.) matrix $\underset{\sim}{\nu}$. Basically, for the multivariate PCLT, it suffices to show that for every non-null $\underset{\sim}{\lambda}$ (of order pxq), Trace$(\underset{\sim}{\lambda}\underset{\sim}{L}_n')$ is asymptotically normal (under $\mathcal{G}_n$). If we let $\underset{\sim}{L}_n = (\underset{\sim}{L}_n^{(1)},\ldots,\underset{\sim}{L}_n^{(q)})$, then, Puri and Sen(1966,1969) considered arbitrary linear compounds of the form $\lambda_1\underset{\sim}{L}_n^{(1)}+\ldots + \lambda_q\underset{\sim}{L}_n^{(q)}$, expressed the same in the form of $\Sigma_{i=1}^n g_{ni}[a_{n1}(R_{i1}),\ldots,a_{np}(R_{ip})]'$, and then appealed to Theorem 7.1 of Hájek(1961) to show that under $\mathcal{G}_n$, the later (with suitable g_{ni}) is asymptotically multinormal. There appears to be some flaws in these steps. First, one needs to consider an arbitrary linear compound of all the pq elements of $\underset{\sim}{L}_n$ (not simply a linear combination of its q columns) in order that the Cramér-Wold characterization theorem applies. For such a general $\underset{\sim}{\lambda}$, their simplified form for Trace$(\underset{\sim}{\lambda}\underset{\sim}{L}_n')$ may not be obtainable. Second, even otherwise, Theorem 7.1 of Hájek(1961) does not apply here. In this case, we have vectors $\underset{\sim}{R}_1,\ldots,\underset{\sim}{R}_n$ whose joint distribution under $\mathcal{G}_n$ (being different from their unconditional d.f.) does not conform to the model of Hájek (1961), where $(R_1,\ldots,R_n)$ assumes all possible permutations of $(1,\ldots,n)$ with the equal probability $(n!)^{-1}$ and $\underset{\sim}{a}_n(i), i=1,\ldots,n$ were p-vectors. This explains the inadequacy of the proofs of the multivariate PCLT's in Puri and Sen(1966,1969). Puri and Sen(1971, Theorem 5.4.1) have sketched a different approach. Let $F_{[j]}$ be the jth marginal d.f. for F, for $j=1,\ldots,p$, and let $\underset{\sim}{U}_i = (U_{i1},\ldots,U_{ip})'$ with $U_{ij} = F_{[j]}(X_{ij})$, $j=1,\ldots,p$, $i=1,\ldots,n$. Also, for eacj $j(=1,\ldots,p)$, let

$$a_{nj}^o(u) = a_{nj}(i) \text{ for } (i-1)/n<u\le i/n,\ i=1,\ldots,n. \tag{2.11}$$

Finally, let

$$L_n^o = \Sigma_{i=1}^n[\, a_{n1}^o(U_{i1}),\ldots,a_{np}^o(U_{ip})]'\ (\underset{\sim}{c}_i - \bar{\underset{\sim}{c}}_n)'. \tag{2.12}$$

Then, by using the coordinatewise proof of Hájek(1961), Puri and Sen(1971) showed that

$$(\underset{\sim}{V}_n \otimes \underset{\sim}{C}_n)^{-\frac{1}{2}}\ ||\underset{\sim}{L}_n - \underset{\sim}{L}_n^o\ || \rightarrow 0 \text{, in probability, as } n \rightarrow \infty, \tag{2.13}$$

so that the asymptotic normality (under $\mathcal{G}_n$) of $\underset{\sim}{L}_n^o$ would ensure the same for $\underset{\sim}{L}_n$. However, the classical CLT may not apply to the conditional (permutational) distribution of $\underset{\sim}{L}_n^o$ and hence the proof (under $\mathcal{G}_n$) remains incomplete; though, the asymptotic unconditional multinormality of $\underset{\sim}{L}_n^o$, and hence of $\underset{\sim}{L}_n$, would follow.

From the above discussion, it seems desirable to formulate multivariate PCLT's for linear rank statistics in an unambiguous manner and to provide valid proofs for them. Towards this, we define

$$\underset{\sim}{a}_n^{(i)} = (a_{n1}(R_{i1}),\dots,a_{np}(R_{ip}))',\ i=1,\dots,n, \tag{2.14}$$

$$\underset{\sim}{\bar{a}}_n = (\bar{a}_{n1},\dots, \bar{a}_{np})' , \tag{2.15}$$

$$\gamma_n = \max_{1\le i\le n} (\underset{\sim}{a}_n^{(i)} - \underset{\sim}{\bar{a}}_n)' \underset{\sim}{V}_n^{-1} (\underset{\sim}{a}_n^{(i)} - \underset{\sim}{\bar{a}}_n) , \tag{2.16}$$

where, for the time being, we assume that $\underset{\sim}{V}_n$ is of full rank (otherwise, under $\mathcal{P}_n$, $\underset{\sim}{L}_n$ will have a degenerate d.f.). Note that the $\underset{\sim}{a}_n^{(i)}$ and $\underset{\sim}{V}_n$ are stochastic in nature, and hence, unlike the case of p=1, γ_n is, in general, a r.v. Then, we have the following

<u>Theorem 1. If</u> $\xi_n\gamma_n \to 0$ (<u>in probability</u>), <u>as</u> $n \to \infty$, <u>then, under</u> $\mathcal{P}_n$, $\underset{\sim}{L}_n$ <u>is asymptotically (in probability) normal with mean</u> $\underset{\sim}{0}$ <u>and dispersion matrix</u> $\underset{\sim}{V}_n \otimes \underset{\sim}{C}_n$.

It may be noted that under (2.10), all we need for the above theorem to hold is that $n^{-1}\gamma_n \to 0$ (in probability), as $n \to \infty$, and this can be established under conditions much weaker than the ones in Puri and Sen(1966,1969,1971). In the next theorem, we do not want to impose (2.10) and desire to incorporate (2.9). For this, we suppose that there exist score functions $\phi_j(u), 0<u<1$, $j=1,\dots,p$, such that for the $a_{nj}^{o}(u)$, defined by (2.11),

$$\max_{1\le j\le p}\{ \int_0^1 \{a_{nj}^{o}(u) - \phi_j(u)\}^2 \, du\} \to 0 , \text{ as } n \to \infty , \tag{2.17}$$

where, for each $j(=1,\dots,p)$,

$\phi_j(u) = \phi_{j,1}(u) - \phi_{j,2}(u)$, $0<u<1$, where $\phi_{j,k}(u)$ is nondecreasing, absolutely continuous and square integrable inside (0,1), for k=1,2. (2.18)

These two conditions are less stringent than the ones in Puri and Sen(1966,1969).

<u>Theorem 2. If</u> (2.9), (2.17) <u>and</u> (2.18) <u>hold, then, under</u> $\mathcal{P}_n$, $\underset{\sim}{L}_n$ <u>is asymptotically (in probability) normal with mean</u> $\underset{\sim}{0}$ <u>and dispersion matrix</u> $\underset{\sim}{V}_n \otimes \underset{\sim}{C}_n$.

Proofs of these theorems along with other comments are presented in the next section.

3. PROOFS OF THE THEOREMS

Let us first consider the proof of Theorem 1. For an arbitrary non-null matrix $\underset{\sim}{\lambda}$ (of order pxq), we like to show that under $\mathcal{G}_n$, $\mathrm{Tr}(\underset{\sim}{\lambda}\underset{\sim}{L}_n')$ is asymptotically normal. If we define the $\underset{\sim}{a}_n^{(i)}$ and $\underset{\sim}{\bar{a}}_n$ as in (2.14)-(2.15), we have then

$$\mathrm{Tr}(\underset{\sim}{\lambda}\underset{\sim}{L}_n') = \Sigma_{i=1}^n \Sigma_{j=1}^p \Sigma_{k=1}^q (c_{ik} - \bar{c}_k)\, \lambda_{jk} [\, a_{nj}(R_{ij}) - \bar{a}_n \,] \tag{3.1}$$

$$= \Sigma_{i=1}^n \underset{\sim}{d}_i' [\underset{\sim}{a}_n^{(i)} - \underset{\sim}{\bar{a}}_n \,] \; ; \quad d_{ij} = \Sigma_{k=1}^q \lambda_{jk}(c_{ik} - \bar{c}_k),\ 1 \le j \le p,\ 1 \le i \le n,$$

and $\underset{\sim}{d}_i' = (d_{i1},\ldots,d_{ip})$, for $i=1,\ldots,n$. Note that by (2.3), $E_{\mathcal{G}_n}\mathrm{Tr}(\underset{\sim}{\lambda}\underset{\sim}{L}_n') = 0$ and

$$E_{\mathcal{G}_n}(\mathrm{Tr}(\underset{\sim}{\lambda}\underset{\sim}{L}_n'))^2 = \Sigma_{j=1}^p \Sigma_{j'=1}^p \Sigma_{k=1}^q \Sigma_{k'=1}^q \lambda_{jk}\lambda_{j'k'} V_{njj'} C_{nkk'} = \tau_n^2, \text{ say.} \tag{3.2}$$

Express the reduced rank collection matrix $\underset{\sim}{\mathbb{R}}_n^*$ as $(\underset{\sim}{R}_1^*,\ldots,\underset{\sim}{R}_n^*)$, so that the first element of $\underset{\sim}{R}_i^*$ is equal to i, for $i=1,\ldots,n$. Define then $S_1,\ldots,S_n$ by letting

$$R_{i1}^* = R_{S_i 1} = i, \text{ for } i=1,\ldots,n. \tag{3.3}$$

Further, let

$$\underset{\sim}{b}_n(i; \underset{\sim}{\mathbb{R}}_n^*) = \underset{\sim}{a}_n^{(S_i)} - \underset{\sim}{\bar{a}}_n \,, \text{ for } i=1,\ldots,n \,, \tag{3.4}$$

and let $\underset{\sim}{Y}_n = (Y_{n1},\ldots,Y_{nn})$ be a random vector which takes on each permutation of $(1,\ldots,n)$ with the equal probability $(n!)^{-1}$. Then, the permutation distribution of $\mathrm{Tr}(\underset{\sim}{\lambda}\underset{\sim}{L}_n')$ agrees with the distribution of

$$Z_n = \Sigma_{i=1}^n \underset{\sim}{d}_i' \underset{\sim}{b}_n(Y_{ni}; \underset{\sim}{\mathbb{R}}_n^*) \qquad [\text{ given } \underset{\sim}{\mathbb{R}}_n^* \,]. \tag{3.5}$$

Since under $\mathcal{G}_n$, $\underset{\sim}{\mathbb{R}}_n^*$ is held fixed, while the vector $\underset{\sim}{Y}_n$ has the discrete uniform distribution [over the set of permutations of $(1,\ldots,n)$], we may now virtually repeat the proof of Theorem 3 of Hoeffding(1951) and obtain the asymptotic normality of Z_n (given $\underset{\sim}{\mathbb{R}}_n^*$) under the sole condition that $\gamma_n \xi_n \to 0$ as $n \to \infty$. Since γ_n is a r.v., whenever $\gamma_n \xi_n \to 0$, in probability, as $n \to \infty$, the method of moment proof of Hoeffding holds for all $\underset{\sim}{\mathbb{R}}_n^*$, excepting a subset with probability tending to 0 as $n \to \infty$, and hence, the aforesaid normality holds, in probability. This completes the proof of Theorem 1.

To prove Theorem 2, let $F_{[jj']}$ be the bivariate marginal d.f. for the (j,j')th variates, for the d.f. F, for $j \ne j'=1,\ldots,p$. Let then $\underset{\sim}{\nu} = ((\nu_{jj'}))$ be defined by

$$\nu_{jj'} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_j(F_{[j]}(x))\phi_{j'}(F_{[j']}(y))dF_{[jj']}(x,y) - \bar{\phi}_j\bar{\phi}_{j'} \,, \tag{3.6}$$

for $j,j'=1,\ldots,p$, where

$$\bar{\phi}_j = \int_0^1 \phi_j(u)du \text{ , for } j= 1,\ldots,p. \quad (3.7)$$

Note that expressing (2.4) and (2.5) in the integral form involving the empirical d.f.'s and then using (2.17)-(2.18) along with the usual Glivenko-Cantelli lemma type result, we obtain that under (2.17) and (2.18),

$$\underset{\sim}{V}_n \rightarrow \underset{\sim}{\nu} \text{ , in probability, as } n \rightarrow \infty \text{ .} \quad (3.8)$$

The proof of (3.8) is essentially similar to that of Theorem 3.1 of Puri and Sen(1969), and hence, the details are omitted. Now, without any essential loss of generality, we assume that $\underset{\sim}{\nu}$ is positive definite (otherwise, the limiting permutation distribution of $\underset{\sim}{L}_n$ will be singular, in probability). Let us also define the S_i as in (3.3) and for every $k: 1\leq k\leq n$, let $\underset{\sim}{S}_{n,k} = (S_1,\ldots,S_k)'$ and let $S_{n,0} = 0$. Let now $\mathcal{B}_{n,k}$ be the sigma-field generated by $\underset{\sim}{S}_{n,k}$ (under $\mathcal{P}_n$), for $k=1,\ldots,n$ and let $\mathcal{B}_{n,0}$ be the trivial sigma field. Let then

$$\underset{\sim}{L}_{n,k} = E_{\mathcal{P}_n}(\underset{\sim}{L}_n \mid \mathcal{B}_{n,k}) \text{ , for } k = 0,1,\ldots,n. \quad (3.9)$$

At this stage, we appeal to Chatterjee and Sen(1973, Section 4), for the case of $p = 1$, and Sen(1979, Section 2), for general $p \geq 1$, and obtain that

$$\underset{\sim}{L}_{n,k} = \Sigma_{i=1}^{k}[a_{n1}(R_{S_i 1})-a^*_{n1}(k),\ldots, a_{np}(R_{S_i p})-a^*_{np}(k)]'(\underset{\sim}{c}_{S_i} - \bar{\underset{\sim}{c}}_n) \quad (3.10)$$

where

$$a^*_{nj}(k) = (n-k)^{-1}[n\cdot\bar{a}_{nj} - \Sigma_{i=1}^{k} a_{nj}(R^*_{ij})], \; j=1,\ldots,p; k=0,\ldots,n, \quad (3.11)$$

and, conventionally, we let $a^*_{nj}(n) = 0$, for $j=1,\ldots,p$. By (3.9), we conclude that under $\mathcal{P}_n$, for every n,

$$\{ \underset{\sim}{L}_{n,k}, \mathcal{B}_{n,k}: 0 \leq k \leq n \} \text{ is a zero mean martingale.} \quad (3.12)$$

Thus, to prove Theorem 2, it suffices to consider for an arbitrary non-null $\underset{\sim}{\lambda}$ (of order pxq), the partial sequence $\{ Tr(\underset{\sim}{\lambda}\underset{\sim}{L}'_{n,k}); 0\leq k\leq n\}$ and verify the conditions for the martingale central limit theorem [viz., Dvoretzky (1972)].Note that by (3.12), under $\mathcal{P}_n$, $\{Tr(\underset{\sim}{\lambda}\underset{\sim}{L}'_{n,k}), 0\leq k\leq n\}$ also forms a martingale sequence. Hence, if we define

$$Y_{n,k} = Tr(\underset{\sim}{\lambda}\underset{\sim}{L}'_{n,k}) - Tr(\underset{\sim}{\lambda}\underset{\sim}{L}'_{n,k-1}), \; k=1,\ldots,n, \quad (3.13)$$

then, it suffices to show that for τ^2_n, defined by (3.2), as $n \rightarrow \infty$,

$$\{ \Sigma_{i=1}^{n} E_{\mathcal{G}_n}[Y_{n,i}^2 \mid \mathcal{B}_{n,i-1}]\}/\tau_n^2 \overset{p}{\to} 1 , \tag{3.14}$$

and

$$\{ \Sigma_{i=1}^{n} E_{\mathcal{G}_n}[Y_{n,i}^2 I(|Y_{n,i}| > \varepsilon\tau_n)]\}/\tau_n^2 \overset{p}{\to} 0, \quad \forall\ \varepsilon > 0. \tag{3.15}$$

Now, (3.14) and (3.15) follow (as a direct vector-extension) precisely on the same line as in (4.32) through (4.40) of Sen(1979), and hence, the proof is complete.

We conclude this section with the remark that this martingale approach adapted from Chatterjee and Sen(1973) and Sen(1979), besides providing a valid proof of the PCLT in the multivariate case, avoids the computational complications and the extra regularity condition (2.10) of the moment-convergence approach in Theorem 1.

REFERENCES

CHATTERJEE, S.K. and SEN, P.K.(1964). Nonparametric tests for the bivariate two-sample location problem. Calcutta Statist.Asso.Bull. 13, 18-58.

--- and --- (1973). Nonparametric testing under progressive censoring. Calcutta Statist.Asso.Bull. 22, 13-50.

DVORETZKY, A.(1972). Central limit theorems for dependent random variables. Proc. 6th Berkeley Symp. Math.Statist.Prob.,Univ.California Press, 2, 515-535.

HÁJEK, J.(1961). Some extensions of the Wald-Wolfowitz-Noether theorem. Ann. Math.Statist. 32, 506-523.

---- (1968). Asymptotic normality of simple linear rank statistics under alternatives. Ann. Math. Statist. 39, 325-346.

HOEFFDING, W.(1951). A combinatorial central limit theorem. Ann. Math. Statist. 22 , 558-566.

MAJUMDAR, H. and SEN, P.K.(1978). Nonparametric tests for multiple regression under progressive censoring. Jour. Multivar. Anal. 8, 73-95.

PURI, M.L. and SEN, P.K.(1966). On a class of multivariate multisample rank order tests. Sankhya, Ser.A 28, 353-376.

--- and --- (1969). A class of rank order tests for a general linear hypothesis. Ann. Math. Statist. 40, 1325-1343.

--- and --- (1971). Nonparametric Methods in Multivariate Analysis. John Wiley, New York.

SEN, P.K.(1979). Rank analysis of covariance under progressive censoring. Sankhya, Ser.A 41, 147-169.

WALD, A. and WOLFOWITZ,J.(1944). Statistical tests based on permutations of the observations. Ann. Math. Statist. 15, 358-372.