A FISHERIAN DETOUR OF THE STEP-DOWN PROCEDURE

by

Pranab Kumar Sen

Department of Biostatistics
University of North Carolina at Chapel Hill

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PRANAB KUMAR SEN

University of North Carolina, Chapel Hill

In a Step-down procedure, apart from an hierarchy of the component hypotheses (leading to the steps), one also needs to settle on their individual levels of significnace (constrained on the overall one). The Fisher method of combining independent tests is extended to the step-down procedure and its Bahadur-efficiency and (asymptotic) optimality results are considered.

1. Introduction. For general multivariate analysis of variance (MANOVA) models, the exact distribution theory of the conventional likelihood ratio (or other allied) test statistics, often, gets quite complicated. This is especially the case with incomplete multiresponse designs and hierarchical designs [see, Roy et. al. (1971)], where the conventional methods may encounter considerable difficulties. Further, unlike the univariate case, in a general multivariate model, restriction to invariant tests may not yield an unique best test among them, leaving some room for other ad hoc procedures. One such procedure is the step-down procedure, proposed by J. Roy (1958), which can be viewed as a tributary of the general union-intersection procedure of Roy (1953), and is usually simpler to construct. According to this procedure,

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an overall hypothesis, viewed as a (finite) intersection of m (≥ 1) component ones, is tested in steps: Testing for the j^{th} component is made, if and only if, in all the preceding steps, the component null hypotheses were accepted, $1 \leq j \leq m$, so that the overall null hypothesis is accepted only when all the components are accepted. Step-down testing procedures are generally not invariant ones, and the ordering of the steps (or related component hypotheses) may sometimes be advocated on the ground of their relative importance (and, otherwise, in an arbitrary manner). Further, one needs to choose the levels of significance for the m component tests (say, $\alpha_1,\ldots,\alpha_m$), subject to some specified α $(0 < \alpha < 1)$. Usually, the component hypotheses test statistics (say, $T_1,\ldots,T_m$) are quasi-independent, when the null hypothesis holds, so that

$$1 - \alpha = (1-\alpha_1)\cdots(1-\alpha_m) \ , \tag{1.1}$$

and then, the α_j may be determined suitably either according to some index of preference for the m components or by the equality-criterion: $\alpha_1=\ldots=\alpha_m=1 - (1-\alpha)^{1/m}$.

For combining independent tests, based on the probability integral transformation on the test statistics, various methods are available in the literature [see, van Zwet and Oosterhoff (1967) and the references cited therein]; among these the omnibus test of Fisher (1932) possesses some asymptotic optimality properties [see Littell and Folks (1971), and others]. The object of the present investigation is to focus on this probability integral transformation on the component test statistics $T_1,\ldots,T_m$, in a step-down procedure, and to show that the theory, due to Fisher (1932), available for independent tests works out as well in this case. In particular, the exact distribution theory and the Bahadur-efficiency results hold for a step-down procedure as well. Along with the preliminary notions, the proposed procedure is formulated in Section 2. Section 3 is devoted to the study of the Bahadur (1960) efficiency results. The concluding section deals with the asymptotic optimality of the proposed procedure for regular MANOVA models and some general comments.

2. The proposed procedure. For a general class of hypothesis testing problems, Roy (1953) introduced the union-intersection (UI-) principle where the null hypothesis H_0 (and the alternative H) are expressed as the intersection (and union) of some component hypotheses H_{0i} (and H_i), $i \in J$. In a step-down procedure, the index set J is finite [and we denote its cardinality by m (≥ 1)], so that

$$H_0 = \bigcap_{i=1}^{m} H_{0i} \quad \text{and} \quad H = \bigcup_{i=1}^{m} H_i \quad , \tag{2.1}$$

where there is an hierarchy in the formulation and testing of the component hypotheses (H_{0i} vs. H_i, $i=1,\ldots,m$). First, H_{01} is tested against H_1 (based on the test statistic T_1): If H_1 is accepted, then the process terminates along with the rejection of H_0; otherwise, one proceeds to the second step. At the j^{th} step, H_{0j} is tested against H_j (based on a statistic T_j) only when H_{0i}, $i < j$, are all accepted; if H_j is accepted, the process terminates at that step along with the rejection of H_0 and, otherwise, one proceeds to the next step, for $j=1,\ldots,m$. Note that H_0 is accepted only when $H_{01},\ldots,H_{0m}$ are all accepted, and at the j^{th} step, one usually makes a conditional test, given the preceding steps, for $j=2,\ldots,m$. Let $\mathcal{B}_k$ be the sigma-field generated by the random elements entering the first k steps, for $k=1,\ldots,m$ and let $\mathcal{B}_0$ be the trivial sigma-field. Then $\mathcal{B}_k$ is nondecreasing in k $(0 \leq k \leq m)$. Let then

$$G_j(t) = P\{T_j \leq t | \mathcal{B}_{j-1}, H_{0i}, i < j\}, \; 1 \leq j \leq m \, . \tag{2.2}$$

If we assume that at each step, right hand side critical regions are used, then corresponding to some preset $\alpha_1,\ldots,\alpha_m$, satisfying (1.1), we may define $t_1^0,\ldots,t_m^0$, by letting

$$G_j(t_j^0) = 1 - \alpha_j \ , \quad 1 \le j \le m \ . \tag{2.3}$$

Thus, at the j^{th} step, H_j is accepted if $T_j \ge t_j^0$, while $T_i < t_i^0$, for every $i < j$ $(1 \le j \le m)$. In regular MANOVA models [see Roy et. al. (1971, pp. 41-43)], the T_j are appropriate variance-ratio statistics, so that the t_j^0 can easily be computed by reference to standard statistical tables.

To introduce the Fisherian detour, we define

$$U_j = -2 \log \{1 - G_j(T_j)\}, \quad 1 \le j \le m \ , \tag{2.4}$$

and let then

$$U_j^* = U_1 + \ldots + U_j \ , \text{ for } 1 \le j \le m \ ; \quad U^* = U_m^* \ . \tag{2.5}$$

First, we consider the following result underlying the test procedure to be proposed.

Theorem 1. If the G_j, $1 \le j \le m$ *are all continuous everywhere, then, under* H_0, *for every real* t: $0 \le t \le \infty$,

$$P\{U^* \ge t | H_0\} = 2^{-m}(\Gamma m)^{-1} \int_t^\infty u^{m-1} \exp(-\tfrac{1}{2}u)\,du \ . \tag{2.6}$$

Proof. By (2.2) and (2.4), for continuous G_j, the conditional distribution of U_j, given $\mathcal{B}_{j-1}$ (and under H_0) is the simple exponential one (which does not depend on U_i, $i < j$), for $j=1,\ldots,m$. As such, writing (under H_0)

$$\begin{aligned} E\{\exp(itU^*)\} &= E[\exp(itU_{m-1}^*)E\{\exp(itU_m)|\mathcal{B}_{m-1}\}] \\ &= (1-2it)^{-1}E[\exp(itU_{m-1}^*)], \end{aligned} \tag{2.7}$$

we obtain by repeated reduction that

$$E\{\exp(itU^*)\} = (1-2it)^{-m} \ , \tag{2.8}$$

and this implies (2.6). Q.E.D.

Let u^*_α be defined by equating the right hand side of (2.6) to α $(0 < \alpha < 1)$. Note that as in the case of combination of independent test statistics [Fisher (1932)], u^*_α is the upper 100% point of the chi square distribution with 2m degrees of freedom (DF). Then, we propose the following testing procedure:

At the first step, compute U^*_1. If $U^*_1 \geq u^*_\alpha$, stop at this stage along with the rejection of H_0, while, if $U^*_1 < u^*_\alpha$, proceed to the second step. At the j^{th} step (only when $U^*_i < u^*_\alpha$, $\forall\, i < j$), if $U^*_j \geq u^*_\alpha$, the process terminates along with the rejection of H_0, while for $U^*_j < u^*_\alpha$, one proceeds to the next step, for $j=1,\dots,m$. Again, H_0 is accepted only when $U^* \,(= U^*_m) < u^*_\alpha$.

Since the U_j are non-negative, for $j \geq 1$, we obtain that

$$\begin{aligned}\text{Type I error} &= P\{U^*_j \geq u^*_\alpha \text{ for at least one } 1 \leq j \leq m \mid H_0\} \\ &= P\{\max_{1\leq j\leq m} U^*_j \geq u^*_\alpha \mid H_0\} \\ &= P\{U^*_m \geq u^*_\alpha \mid H_0\} = \alpha .\end{aligned}$$

Hence, the proposed Fisherian detour eliminates the need of choosing $\alpha_1,\dots,\alpha_m$ [satisfying (1.1)], and uses the same critical level u^*_α, pertaining to the case of independent test statistics. Computations of the G_j (and hence, U_j), $1 \leq j \leq m$, are generally simple and can be made with the aid of standard statistical tables or existing computer programs on standard statistical distributions.

3. Asymptotic efficiency of U^*.

The Bahadur (1960) efficiency of the proposed test will be studied and the same will be incorporated in the study of the asymptotic optimality of the proposed method. At this stage, to make the dependence of the statistics $T_1,\dots,T_m$ on the sample size (n) explicit, we denote them by $T_{1,n},\dots,T_{m,n}$, respectively. Similarly, we denote the null distribution in (2.2) by $G_j^{(n)}$, $1 \le j \le m$; generally, these $G_j^{(n)}$ converge to some G_j^* $(1 \le j \le m)$, as $n \to \infty$. Let then

$$U_{j,n} = -2 \log \{1-G_j^{(n)}(T_{j,n})\},\ 1 \le j \le m;\ U_{(n)}^* = \textstyle\sum_{j=1}^m U_{j,n} . \tag{3.1}$$

Note that (2.6) holds for every n (for which the $G_j^{(n)}$ are properly defined), so that for every (such) n,

$$-2t^{-1} \log P\{U_{(n)}^* \ge t | H_0\} \to 1, \text{ as } t \to \infty . \tag{3.2}$$

We assume next that as $n \to \infty$,

$$n^{-1}U_{j,n} \to \mu_j ,\ 1 \le j \le m, \text{ with probability } 1 , \tag{3.3}$$

where the μ_j are real, nonnegative quantities. In passing, we may remark that if, for $n \to \infty$,

$$n^{-1}T_{j,n} \to \nu_j,\ 1 \le j \le m, \text{ with probability } 1 , \tag{3.4}$$

while for the $G_j^{(n)}$, we have

$$-2n^{-1} \log \{1-G_j^{(n)}(nt)\} \to \psi_j(t),\quad \forall\ t \in [0,\infty) , \tag{3.5}$$

for $1 \le j \le m$, where the ψ_j are nonnegative and continuous functions (of t), then (3.3) holds with

$$\mu_j = \psi_j(\nu_j) \text{ , for } j=1,\dots,m . \tag{3.6}$$

In MANOVA models, as we shall see in the next section, (3.3)-(3.6) holds under fairly general conditions. Then, by (3.1) and (3.3), as $n \to \infty$,

$$n^{-1}U_{(n)}^* \to \mu^* = \mu_1 + \dots + \mu_m \text{ , with probability } 1 . \tag{3.7}$$

By (3.2) and (3.7), we conclude that the (exact) *Bahadur slope* for $U_{(n)}^*$ is equal to μ^*. Finally, the Bahadur efficiency of the proposed Fisherian detour is also equal to μ^*.

From consideration of the best average power (over suitable contours in the parameter space) and other criteria, a likelihood ratio test for H_0 vs. H may be advocated; we denote the allied test statistic by L_n. Usually, $-2 \log L_n$ has asymptotically chi square distribution with an appropriate degrees of freedom, and hence, for $-2 \log L_n$, (3.2) holds. Further, from the results of Bahadur (1967), the exact slope for $-2 \log L_n$ can be obtained in terms of the Kulleack-Liebler information and that can further be simplified if H_0 and H are close to each other.

The Bahadur efficiency of the proposed Fisherian detour with respect to the likelihood ratio test can thus be expressed as a ratio of the two slopes. We intend to study this in more detail in the next section.

4. Asymptotic optimality for MANOVA models. We shall establish the asymptotic optimality (in the light of the Bahadur efficiency) of the proposed Fisherian detour for regular MANOVA models. Consider the usual MANOVA model:

$$\underset{\sim}{Y}(n\times p) = \underset{\sim}{A}\underset{\sim}{\beta} + \underset{\sim}{e},\ A(n\times m),\ \underset{\sim}{\beta}(m\times p),\ \underset{\sim}{e}(n\times p) \tag{4.1}$$

where $\underset{\sim}{\beta}$ is the matrix of unknown parameters, $\underset{\sim}{A}$ is a specified matrix (of rank $m \le n$) and the rows of $\underset{\sim}{e}$ are independently distributed according to $N_p(\underset{\sim}{0}, \underset{\sim}{\Sigma})$, with Σ a positive definite (but, unknown) matrix. We intend to test for

$$H_0:\ \underset{\sim}{C}\underset{\sim}{\beta} = \underset{\sim}{0} \text{ vs. } H:\ \underset{\sim}{C}\underset{\sim}{\beta} \ne \underset{\sim}{0}\ , \tag{4.2}$$

where $\underset{\sim}{C}$ $(r\times m)$ is a specified matrix of rank r $(\le m)$. If we define

$$\underset{\sim}{Q}_e = \underset{\sim}{I}_n - \underset{\sim}{A}(\underset{\sim}{A}'\underset{\sim}{A})^{-1}\underset{\sim}{A}'\ , \tag{4.3}$$

$$\underset{\sim}{Q}_H = \underset{\sim}{A}(\underset{\sim}{A}'\underset{\sim}{A})^{-1}\underset{\sim}{C}'[\underset{\sim}{C}(\underset{\sim}{A}'\underset{\sim}{A})^{-1}\underset{\sim}{C}']^{-1}\underset{\sim}{C}(\underset{\sim}{A}'\underset{\sim}{A})^{-1}\underset{\sim}{A}'\ , \tag{4.4}$$

$$S_e = \underset{\sim}{Y}'\underset{\sim}{Q}_e\underset{\sim}{Y} \text{ and } S_H = \underset{\sim}{Y}'\underset{\sim}{Q}_H\underset{\sim}{Y}\ , \tag{4.5}$$

then the likelihood ratio test for (4.2) rests on the critical region

$$W_1: \ \lambda = |\underset{\sim}{S}_e| / |\underset{\sim}{S}_e + \underset{\sim}{S}_H| \leq \lambda_1 , \tag{4.6}$$

where $P\{W_1 | H_0\} = \alpha$, the desired level of significance, and Theorem 8.6.1 of Anderson (1958) provides the desired chi-square approximation for $-2 \log \lambda$.

It is well-known that $S_e \Sigma^{-1}$ has the Wishart distribution (with n-m DF), so that for every fixed m, as $n \to \infty$,

$$n^{-1} \underset{\sim}{S}_e \to \underset{\sim}{\Sigma} , \text{ with probability } 1 , \tag{4.7}$$

irrespective of H_0 being true or not. On the other hand, if we assume that as $n \to \infty$,

$$n^{-1} \underset{\sim}{A}' \underset{\sim}{A} \to \underset{\sim}{\Lambda} , \ \underset{\sim}{\Lambda} \text{ p.d.} , \tag{4.8}$$

then, it follows by some routine steps that for any (fixed) $\underset{\sim}{\beta}$,

$$n^{-1}(-2 \log \lambda) \to \text{Trace}(\underset{\sim}{\Sigma}^{-1} \underset{\sim}{\beta}' \underset{\sim}{C}' [\underset{\sim}{C} \underset{\sim}{\Lambda}^{-1} \underset{\sim}{C}']^{-1} \underset{\sim}{C} \underset{\sim}{\beta}) , \tag{4.9}$$

with probability 1, as $n \to \infty$. Thus, the (exact) Bahadur slope for the likelihood ratio test is equal to

$$\text{Trace}(\underset{\sim}{\Sigma}^{-1} \underset{\sim}{\beta}' \underset{\sim}{C}' [\underset{\sim}{C} \underset{\sim}{\Lambda}^{-1} \underset{\sim}{C}]^{-1} \underset{\sim}{C} \underset{\sim}{\beta}) . \tag{4.10}$$

To formulate the step-down procedure with the intent of the Fisherian detour, we let $\underset{\sim}{Y} = (\underset{\sim}{Y}_1, \ldots, \underset{\sim}{Y}_p)$, so that each $\underset{\sim}{Y}_j$ is an n-vector. Similarly, let $\underset{\sim}{\beta} = (\underset{\sim}{\beta}_1, \ldots, \underset{\sim}{\beta}_p)$ and let

$$\underset{\sim}{Y}_{(j)} = (\underset{\sim}{Y}_1, \ldots, \underset{\sim}{Y}_j) \ \text{ and } \ \underset{\sim}{\beta}_{(j)} = (\beta_1, \ldots, \beta_j), \ 1 \leq j \leq p ; \tag{4.11}$$

conventionally, we let $\underset{\sim}{Y}_{(0)} = 0$, $\underset{\sim}{\beta}_{(0)} = 0$. Further, let $\underset{\sim}{\Sigma}_{(j)}$ be the upper $j \times j$ minor of $\underset{\sim}{\Sigma} = ((\sigma_{jj'}))$, for $j=1,\ldots,p$. Then, conditional on $\underset{\sim}{Y}_{(j=1)}$, $\underset{\sim}{Y}_j$ follows the model

$$\underset{\sim}{Y}_j = \underset{\sim}{A} \underset{\sim}{\beta}_j + (\underset{\sim}{Y}_{(j-1)} - \underset{\sim}{A} \underset{\sim}{\beta}_{(j-1)}) \gamma_j + \underset{\sim}{e}_j^* , \tag{4.12}$$

where $\underset{\sim}{e}_j^* \sim N(\underset{\sim}{0}, \nu_j^2 \underset{\sim}{I}_n)$, $\nu_j^2 = |\underset{\sim}{\Sigma}_{(j)}| / |\Sigma_{(j-1)}|$ and

(4.13) $$\underset{\sim}{\gamma}_j' = (\sigma_{j1}, \ldots, \sigma_{jj-1}) \underset{\sim}{\Sigma}_{(j-1)}^{-1}, \quad 1 \le j \le p .$$

[For j=1, the second term on the right hand side of (4.12) drops out and $\nu_1^2 = \sigma_{11}$.] Thus, on letting [for each j (=1,...,p)]

(4.14) $$\underset{\sim}{A}_j = [\underset{\sim}{A}, \underset{\sim}{Y}_{(j-1)}], \quad \underset{\sim}{\eta}_j = \underset{\sim}{\beta}_j - \underset{\sim}{\beta}_{(j-1)} \underset{\sim}{\gamma}_j, \quad \underset{\sim}{\theta}_j = [\underset{\sim}{\eta}_j, \underset{\sim}{\gamma}_j'] ,$$

we may write [see Roy <u>et. al.</u> (1971, page 42)]

(4.15) $$H_0 = \bigcap_{j=1}^{p} H_{0j}; \quad H_{0j}: \ \underset{\sim}{C}\underset{\sim}{\eta}_j = \underset{\sim}{C}_j \underset{\sim}{\theta}_j = \underset{\sim}{0}, \quad 1 \le j \le p ,$$

(where $\underset{\sim}{C}_j = [\underset{\sim}{C}, \underset{\sim}{0}_j]$, $\underset{\sim}{0}_j$ being a null matrix of order $r \times (j-1)$, $1 \le j \le p$) and (4.12) may be written as $\underset{\sim}{Y}_j = \underset{\sim}{A}_j \underset{\sim}{\theta}_j + e_j^*$ $(1 \le j \le p)$, on which the classical ANOVA theory yields the following test statistics:

(4.16) $$F_j = r^{-1}(n-m-j+1) S_H^{(j)} / S_E^{(j)} ,$$

where

(4.17) $$S_E^{(j)} = \underset{\sim}{Y}_j' \underset{\sim}{Q}_j \underset{\sim}{Y}_j; \quad \underset{\sim}{Q}_j = \underset{\sim}{I}_n - \underset{\sim}{A}_j (\underset{\sim}{A}_j' \underset{\sim}{A}_j)^{-1} \underset{\sim}{A}_j ,$$

(4.18) $$S_H^{(j)} = \underset{\sim}{Y}_j' \underset{\sim}{A}_j (\underset{\sim}{A}_j' \underset{\sim}{A}_j)^{-1} \underset{\sim}{C}_j' [\underset{\sim}{C}_j (\underset{\sim}{A}_j' \underset{\sim}{A}_j)^{-1} \underset{\sim}{C}_j']^{-1} \underset{\sim}{C}_j (\underset{\sim}{A}_j' \underset{\sim}{A}_j)^{-1} \underset{\sim}{A}_j' \underset{\sim}{Y}_j ,$$

for $j=1,\ldots,p$. Under H_{0j}, F_j has the variance-ratio distribution with DF's $(r, n-m-j+1)$, so that rF_j has asymptotically (as $n \to \infty$) the central chi square distribution with r DF, for $j=1,\ldots,p$. Note that the $\underset{\sim}{Q}_j$ are all indempotent matrices, with

(4.19) $$\text{Rank of } \underset{\sim}{Q}_j = \text{Trace of } \underset{\sim}{Q}_j = n - m - j + 1, \quad 1 \le j \le p .$$

Further, conditional on $\underset{\sim}{Y}_{(j-1)}$, when H_{0j} may not hold, F_j has a non-central F-distribution with non-centrality parameter

(4.20) $$r^{-1}\nu_j^{-2}\{\underset{\sim}{\eta}_j' A' \underset{\sim}{B}_{nj} \underset{\sim}{A}(\underset{\sim}{A}'\underset{\sim}{A})^{-1}\underset{\sim}{C}'[\underset{\sim}{C}(\underset{\sim}{A}'\underset{\sim}{A})^{-1}\underset{\sim}{A}'\underset{\sim}{Q}_j\underset{\sim}{A}(\underset{\sim}{A}'\underset{\sim}{A})^{-1}\underset{\sim}{C}']^{-1} \cdot$$
$$\underset{\sim}{C}(\underset{\sim}{A}'\underset{\sim}{A})^{-1}\underset{\sim}{A}'\underset{\sim}{B}_{nj}'\underset{\sim}{A}\underset{\sim}{\eta}_j\}$$

where

(4.21) $$B_{nj} = I_n - (I_n - Q_1)(Y_{(j-1)}(Y'_{(j-1)}Y_{(j-1)})^{-1}Y'_{(j-1)}, \quad 1 \le j \le p \quad .$$

Note that $Y_{(j-1)} = A\beta_{(j-1)} + e_{(j-1)}$, where $e_{(j-1)} \sim N(0, I_n \otimes \Sigma_{(j-1)})$, so that $Y_{(j-1)}(Y'_{(j-1)}Y_{(j-1)})^{-1}Y'_{(j-1)}$ is indempotent of rank (j-1), almost surely, for $1 \le j \le p$. As such, by some standard steps, it follows that as $n \to \infty$, for each j (=1,...,p), under (4.8)

(4.22) $$n^{-1}rF_j \to \nu_j^{-2}\{\eta_j'C'[CA^{-1}C']^{-1}C\eta_j\} ,$$

with probability 1. Since (3.5) holds for the central F-distributions (as well as the central chi-square distributions) with $\psi_j(t) \equiv t$, by an appeal to (3.3), (3.6) and (3.7), we claim that for the proposed Fisherian detour of the step-down procedure, the (exact Bahadur slope is equal to

(4.23) $$\begin{aligned} &\textstyle\sum_{j=1}^{p}\nu_j^{-2}\{\eta_j'C'[CA^{-1}C']^{-1}C\eta_j\} \\ &= \textstyle\sum_{j=1}^{p}\nu_j^{-2}\text{Trace}\{C'(CA^{-1}C')^{-1}C\eta_j\eta_j'\} \\ &= \textstyle\sum_{j=1}^{p}\text{Trace}\{C'(CA^{-1}C')^{-1}C(\nu_j^{-2}\eta_j\eta_j')\} \\ &= \text{Trace}\{C'(CA^{-1}C')^{-1}C(\textstyle\sum_{j=1}^{p}\nu_j^{-2}\eta_j\eta_j')\} \ . \end{aligned}$$

Finally, note that by (4.13) and (4.14)

(4.24) $$\begin{aligned} \textstyle\sum_{j=1}^{p}\nu_j^{-2}\eta_j\eta_j' &= \textstyle\sum_{j=1}^{p}\nu_j^{-2}(\beta_j - \beta_{(j-1)}\gamma_j)(\beta_j - \beta_{(j-1)}\gamma_j)' \\ &= \textstyle\sum_{j=1}^{p}\{|\Sigma_{(j-1)}|/|\Sigma_{(j)}|\}(\beta_j - \beta_{(j-1)}\gamma_j)(\beta_j' - \gamma_j'\beta_{(j-1)}') \\ &= \beta\Sigma^{-1}\beta' , \end{aligned}$$

where the last step follows from the Gram-Schmidt triangular reduction (orthogonalization) process [see Rao (1965, p. 21)]. Thus, from (4.23) and (4.24), the Bahadur slope reduces to

$$\text{(4.25)} \qquad \begin{aligned} &\mathrm{Trace}\{\underset{\sim}{C}'(\underset{\sim}{C}\underset{\sim}{\Lambda}^{-1}\underset{\sim}{C}')^{-1}\underset{\sim}{C}\underset{\sim}{\beta}\Sigma^{-1}\underset{\sim}{\beta}'\} \\ &= \mathrm{Trace}\{\underset{\sim}{\beta}'\underset{\sim}{C}'(\underset{\sim}{C}\underset{\sim}{\Lambda}^{-1}\underset{\sim}{C}')^{-1}\underset{\sim}{C}\underset{\sim}{\beta}\Sigma^{-1}\} \\ &= \mathrm{Trace}\{\Sigma^{-1}(\underset{\sim}{\beta}'\underset{\sim}{C}'(\underset{\sim}{C}\underset{\sim}{\Lambda}^{-1}\underset{\sim}{C}')^{-1}\underset{\sim}{C}\underset{\sim}{\beta}\} , \end{aligned}$$

and this agrees with (4.10). Hence, the Bahadur-efficiency of the proposed Fisherian detour with respect to the likelihood ratio test is equal to unity, so that, the proposed procedure is asymptotically efficient. For independent test statistics this property is due to Littell and Folks (1971).

It may be remarked that for general hierarchical model or incomplete multiresponse models [see Roy et. al. (1971)], the proposed Fisherian detour works out well and its Bahadur slope may also be computed. However, there the same design matrix ($\underset{\sim}{A}$) does not appear in the various step-designs, and hence, (4.23)-(4.25) may not hold. Also, for such models, (4.10) does not hold, and a more complicated expression arises. I conclude this section with the following remark on the classical step-down procedure. Under H_{0j}, rF_j has asymptotically chi square distribution with r DF, for $1 \le j \le p$. Thus, if $X^2_{m,\alpha}$ be the upper $100\alpha\%$ point of the chi square distribution with m DF, then (asymptotically) H_0 is accepted *iff*

$$\text{(4.26)} \qquad rF_j \le X^2_{m,\alpha_j} , \quad \text{for every} \quad 1 \le j \le p ,$$

where the rF_j are quasi-independent. As such, using (4.22), it follows that the Bahadur slope for the step-down procedure is

$$\text{(4.27)} \qquad \max_{1\le j\le p} \nu_j^{-2}\{\underset{\sim}{\eta}_j'\underset{\sim}{C}'(\underset{\sim}{C}\underset{\sim}{\Lambda}^{-1}\underset{\sim}{C}')^{-1}\underset{\sim}{C}\underset{\sim}{\eta}_j\} ,$$

which can not be greater than (4.25). Thus, in the light of the Bahadur-efficiency, the proposed Fisherian detour has an edge over the classical step-down procedure.

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