STAGGERING ENTRY, RANDOM WITHDRAWAL AND PROGRESSIVE CENSORING SCHEMES: SOME NONPARAMETRIC PROCEDURES

by

Agam Nath Sinha

and

Pranab Kumar Sen

Department of Biostatistics
University of North Carolina at Chapel Hill

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STAGGERING ENTRY, RANDOM WITHDRAWAL AND PROGRESSIVE CENSORING SCHEMES: SOME NONPARAMETRIC PROCEDURES*

By AGAM NATH SINHA

American Cyanamid Co., Princeton, N.J.

and

PRANAB KUMAR SEN

University of North Carolina, Chapel Hill, N.C.

SUMMARY. In a life testing experiment (or a clinical trial), for staggering entry of units (including some batch arrival models), some nonparametric testing procedures (based on a general class of rank statistics and weighted empirical processes) under progressive censoring schemes are considered, for some multiple regression model. Adjustments needed to incorporate dropouts are also discussed. The relevance of some weak convergence results provides the desired large sample theory of the porposed procedure, and some simulation studies of the critical values of the allied test statistics are made.

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Key Words & Phrases: Batch arrivals, Bessel sheets, dropouts, Kiefer-Bessel processes, multisample problem, progressively censoring scheme, rank test, staggering entry, weighted empirical process.

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1. INTRODUCTION

For life testing experiments, the *logrank test* (Mantel, 1966; Cox, 1972; Peto and Peto, 1972) and modified versions of the *Wilcoxon test* (Gehan, 1965a,b; Halperin and Ware, 1974) provide fixed-sample comparison of two *survival distributions* in the presence of arbitrary *right censoring*. Recently, Fleming et. al. (1980) have developed one- and two-sample Kolmogorov-Smirnov type tests for arbitrary right-censored data. Sequential versions of some of these tests are due to Gehan (1965a), Curnow (1972) and Armitage (1975, Ch. 7), among others. Under *progressively censoring schemes* (PCS), *time-sequential* nonparametric tests based on a general class of *linear rank statistics* (LRS) were developed, for the simple regression model, by Chatterjee and Sen (1973) and extended further to grouped data and multiple regression models by Majumdar and Sen (1977, 1978a); the choice of asymptotically optimal score function for PCS rank tests was studied by Sen (1976b). These tests are designed for possible early termination of experimentation, where time and cost are crucial factors. Davis (1978) and Koziol and Petkau (1978) have used the theory of Chatterjee and Sen (1973) to illustrate some useful applications of PCS tests. DeLong (1980) has studied asumptotic power properties and expected stopping times for PCS rank tests. Sinha and Sen (1979a,b) have considered another class of PCS tests based on some *weighted empirical processes*, for the simple as well as multiple regression models and made a comparative study of their performance characteristics relative to the rank procedures, with respect ot both *asymptotic power* and *stopping times*.

One common feature of all these procedures is that the subjects all enter into the scheme at a common point of time, i.e., we have a *non-staggering* (or single point) entry plan. For staggering entry plans, a PCS gets complicated with varying number of subjects and their differential exposure times. Majumdar and Sen (1978b) using LRS and Sinha and Sen (1982) using weighted empirical processes have considered some PCS tests for the simple regression model in life testing experiments allowing staggering entries as well as withdrawals (dropouts) of the subjects. The object of the present investigation is to extend their results to the multiple regression model, which includes the several sample problem as a special case.

Along with the basic setup, the proposed PCS tests are outlined in Section 2. The related asymptotic theory is then presented in Section 3. Section 4 deals with some apecific test procedures. Section 5 is devoted to the incorporation of (random) withdrawals in a staggering entry plan. The q (≥ 2)-sample problem, in a possible batch-arrival model is briefly presented in Section 6. The concluding section is devoted to some numerical studies of the critical values of the proposed test statistics obtained through simulation.

2. THE PROPOSED PCS TEST PROCEDURES

Consider a life testing experimentation involving n subjects (units) which may not all arrive at a common point of time. Let E_i be the *entry time-point* of the ith subject (into the scheme), and suppose that it remains in the scheme until the *failure* (or the response) occurs at time T_i ($= E_i + X_i^0$) or it drops out of the scheme at time W_i ($= S_i + Y_i$), which ever comes first, for $i=1,\ldots,n$. Then X_i^0 and Y_i are the actual *failure* and *withdrawal* times, respectively, and the observable random variables are

$$X_i = \min(X_i^0, Y_i) \ , \qquad \delta_i = \begin{cases} 1, & X_i = X_i^0 \\ 0, & X_i = Y_i \end{cases} \ , \text{ for } i=1,\ldots,n \ . \quad (2.1)$$

In a life testing situation, one would keep track of the experimentation and record the entries, failures or withdrawals over time as they occur. It may be noted that in a staggering entry plan, where the E_i are not the same, the ordered values of the T_i, as recorded, do not necessarily reflect the ordering of the X_i (or X_i^0, even if there is no withdrawal).

Following the general approach of Majumdar and Sen (1978b) and Sinha and Sen (1982) and extending their model to a general multiple regression one, we assume that $X_1^0,\ldots,X_n^0$ are independent random variables (r.v.) with continuous distribution functions (d.f.) $F_1,\ldots,F_n$, respectively, all defined on the real line R, where

$$F_i(x) = F(x - \beta_0 - \underset{\sim}{\beta}'\underset{\sim}{c}_i),\ x \in R,\ i=1,\ldots,n, \qquad (2.2)$$

where $\beta_0, \underset{\sim}{\beta}' = (\beta_1,\ldots,\beta_p)$ (for some $p \geq 1$) are unknown parameters, $\underset{\sim}{c}_i = (c_{i1},\ldots,c_{ip})'$, $i=1,\ldots,n$ are vectors of specified regression constants (not all equal) and F is an unknown (but continuous) d.f. For simplicity of presentation, we consider first the case of staggering entry without dropouts i.e., where $X_i = X_i^0$ and $\delta_i = 1$, for every $i=1,\ldots,n$. Later on, in Section 5, modifications for incorporating withdrawals will be considered. Now, based on $\underset{\sim}{X}_n = (X_1,\ldots,X_n)$ in (2.1), for the model (2.2), we desire to test in a life testing setup (under preogressive censoring) for

$$H_0:\ \underset{\sim}{\beta} = \underset{\sim}{0} \text{ against } H_1:\ \underset{\sim}{\beta} \neq \underset{\sim}{0}\ . \qquad (2.3)$$

Without any loss of generality, we may assume that

$$E_1 \leq \ldots \leq E_n \text{ with } E_n > E_1\ , \qquad (2.4)$$

(as otherwise, we have a non-staggering entry plan where the results of Majumdar and Sen (1978a) and Sinha and Sen (1981b) would apply). Let us review the picture at a time point T ($> E_1$). Only those units for which $E_i < T$ are in the scheme at that time point. For a subject entering at time point E_i ($< T$) the monitoring upto the time T covers a period of length $T - E_i$ and during that time period $[E_1, T]$, a failure occurs only when $T_i \le T$. For the n subjects, we define the *T-exposure times* by $T - E_i$ or 0, according as E_i is $\le T$ or not, for every $T \ge E_1$ and $i=1,\ldots,n$. Essentially these T-exposure times for $T \ge E_1$, viewed progressively over time, along with the failure events provide the basic data for the PCS testing procedures.

Let n_T $(= \Sigma_{i=1}^n I(E_i \le T)$, with $I(A)$ as the indicator function for the set $A)$ be the number of subjects entering the scheme before time T, so that n_T is $\nearrow$ in T $(\ge E_1)$. In general, the entry-points $E_1,\ldots,E_n$ are random, so that n_T is also a random (step-) function obtaining its maximum value n at time E_n. Because of this stochastic nature of n_T, formulation of testing procedures under PCS for a staggering entry plan encounters some additional difficulties.

To fix notations, we define for every $k(\ge 1)$,

$$\bar{\underset{\sim}{c}}_k = k^{-1} \Sigma_{i=1}^k \underset{\sim}{c}_i, \quad \underset{\sim}{C}_k = \Sigma_{i=1}^k (\underset{\sim}{c}_i - \bar{\underset{\sim}{c}}_k)(\underset{\sim}{c}_i - \bar{\underset{\sim}{c}}_k)' \ , \tag{2.5}$$

and, conventionally, let $\bar{\underset{\sim}{c}}_0 = \underset{\sim}{0}$, $\underset{\sim}{C}_0 = \underset{\sim}{0}$. Note that for every $k \ge 1$, $\underset{\sim}{C}_k$ is symmetric and positive semi-definite (p.s.d.) and $\underset{\sim}{C}_{k+1} - \underset{\sim}{C}_k$ is also p.s.d. Thus, without any loss of generality, we may assume that there exists an integer n_0 $(\ge q+1)$, such that

$$\underset{\sim}{C}_k \text{ is positive-definite (p.d.)}, \ \forall\ k \ge n_0 \ . \tag{2.6}$$

Also, for every k (≥ 1), we define

$$\underset{\sim}{M}_k(x) = \Sigma_{i=1}^k (\underset{\sim}{c}_i - \bar{\underset{\sim}{c}}_k) I(X_i \le x), \ x \in R \tag{2.7}$$

$$S_k(x) = k^{-1} \Sigma_{i=1}^k I(X_i \le x), \ x \in R$$

$$R_{ki} = \Sigma_{j=1}^k I(X_j \le X_i), \ i=1,\ldots,k \ , \tag{2.9}$$

where the X_i are defined by (2.1). Further, for every k (≥ 1), we consider a set $\{a_k(1),\ldots,a_k(k)\}$ of scores (to be defined more precisely in Section 3) and define the *linear rank statistic* (LRS) $\mathbf{L}_k$ by

$$\mathbf{L}_k = \Sigma_{i=1}^{k} (\mathbf{c}_i - \bar{\mathbf{c}}_k) a_k(R_{ki}),\ k \geq 1,\ L_0 = 0\ . \tag{2.10}$$

By the assumed continuity of $F_1, \ldots, F_n$, ties among the X_i may be neglected with probability 1, so that $\mathbf{R}_k = (R_{k1}, \ldots, R_{kk})$ is some permutation of $(1, \ldots, k)$. We may define the *anti-ranks* Q_{ki} by

$$R_{kQ_{ki}} = Q_{kR_{ki}} = i,\ \text{for } i=1,\ldots,k \tag{2.11}$$

and rewrite $\mathbf{L}_k$ as

$$\mathbf{L}_k = \Sigma_{i=1}^{k} (\mathbf{c}_{Q_{ki}} - \bar{\mathbf{c}}_k) a_k(i)\ ,\quad k \geq 1\ . \tag{2.12}$$

In the event that only the smallest q observations among $X_1, \ldots, X_k$ are observed and the remaining $k-q$ are censored, one may modify $\mathbf{L}_k$, as in Chatterjee and Sen (1973) and Majumdar and Sen (1978a), as

$$\mathbf{L}_{k,q} = \Sigma_{i=1}^{a} (\mathbf{c}_{Q_{ki}} - \bar{\mathbf{c}}_k)[a_k(i) - a_k^*(q)],\ 1 \leq q \leq k-1\ , \tag{2.13}$$

while $\mathbf{L}_{k,0} = \mathbf{0}$ and $\mathbf{L}_{k,k} = \mathbf{L}_{k,k-1} = \mathbf{L}_k$, where

$$a_k^*(q) = \begin{cases} (k-q)^{-1}\, \Sigma_{j=q+1}^{k}\, a_k(j), & 0 \leq q \leq k-1 \\ 0\ , & q=k\ . \end{cases} \tag{2.14}$$

Finally, we denote by

$$A_{k,q}^2 = \begin{cases} 0, \quad q=0 \\ (k-1)^{-1}\{\Sigma_{i=1}^{q}\, a_k^2(i) + (k-q)[a_k^*(a)]^2 - k\bar{a}_k^2\}, & 1 \leq q \leq k-1 \\ A_k^2 = (k-1)^{-1}\{\Sigma_{i=1}^{k}\, a_k^2(i) - k\bar{a}_k^2\} & q = k-1,\ k\ , \end{cases} \tag{2.15}$$

where

$$\bar{a}_k = k^{-1}\, \Sigma_{i=1}^{k}\, a_k(i)\ ,\quad k \geq 1\ . \tag{2.16}$$

Let us now examine the T-exposure times and related failure events as we progressively move with T $(\geq E_1)$; these will be employed in the formulation of the *history processes* based on the $\underset{\sim}{M}_k(x)$ and $\underset{\sim}{L}_{k,q}$ and the proposed PCS testing procedures will be based on these history porcesses. At time point T $(\geq E_1)$, there are already n_T $(\leq n)$ entries. Now for every $E_1 < t \leq T$, there are n_t units which have entered the scheme at time points not later than t, so that these have T-exposure times at least equal to $T - t$. Thus, for every $t \in [E_1, T]$, we are in a position to compute $\underset{\sim}{M}_{n_t}(x)$ and $S_{n_t}(x)$, for every x: $0 \leq x \leq T-t$, and then allow t to vary between E_1 and T. Thus, at time point T, we obtain the process

$$\{\underset{\sim}{M}_{n_t}(x),\ S_{n_t}(x):\ \ 0 \leq x \leq T-t,\ E_1 \leq t \leq T\}. \qquad (2.17)$$

Similarly, we may note that by (2.8), $q_{t,T} = n_t S_{n_t}(T-t)$ is the number of failures (among the n_t entries prior to time-point t) of magnitude $\leq T-t$. Then, by reference to (2.13)-(2.14), we may note that the remaining $n_t - q_{t,T}$ responses of magnitude $> T-t$ may be regarded as censored at time $(T-t)^+$ and we have the LRS $\underset{\sim}{L}_{n_t,q_{t,T}}$. Thus, at the time-point T, we obtain the process

$$\{\underset{\sim}{L}_{n_t,q_{t,T}}:\ \ E_1 \leq t \leq T\}\ . \qquad (2.18)$$

We intend to incorporate (2.17) and (2.18) in the formulation of suitable test statistics, which we would employ in the setup of repeated significance testing (RST) as we allow T to vary between E_1 and T_c, where T_c $(> E_n)$ is some preassigned censoring time for the experimentation, set in advance in accordance with other side conditions of the experimentation. When the $\underset{\sim}{c}_i$ are scalar constants, such tests have been formulated in Majumdar and Sen (1978b) and Sinha and Sen (1982). These will be generalized here to the vector case.

Using the motivation in Sinha and Sen (1979b), we may let

$$m^0_{n_t}(x) = \underset{\sim}{M}'_{n_t}(x)\underset{\sim}{C}^-_n \underset{\sim}{M}_{n_t}(x) \ , \tag{2.19}$$

$$m^*_{n_t}(x) = \{\underset{\sim}{M}'_{n_t}(x)\underset{\sim}{C}^-_{n_t}\underset{\sim}{M}_{n_t}(x)\}/\{S_{n_t}(x)[1-S_{n_t}(x)]\} \ , \tag{2.20}$$

so that at time T, we may set the test statistics

$$K^0_n(T) = \sup\{m^0_{n_t}(x): \ x \le T-t, \ t < T\} \ , \tag{2.21}$$

$$K^*_n(T) = \sup\{m^*_{n_t}(x): \ x < T-t, \ t < T\} \ . \tag{2.22}$$

Similarly, as in Majumdar and Sen (1978a), we may set

$$\ell^0_{n_t}(x) = (\underset{\sim}{L}'_{n_t,q_{t,T}}\underset{\sim}{C}^-_n \underset{\sim}{L}_{n_t,q_{t,T}})/A^2_{n_t} \ , \tag{2.23}$$

$$\ell^*_{n_t}(x) = (\underset{\sim}{L}'_{n_t,q_{t,T}}\underset{\sim}{C}^-_{n_t}\underset{\sim}{L}_{n_t,q_{t,T}})/A^2_{n_t,q_{t,T}} \ , \tag{2.24}$$

so that at time-point T, we may set the test statistics

$$L^0_n(T) = \sup\{\ell^0_{n_t}(x): \ x \le T-t, \ t < T\} \ , \tag{2.25}$$

$$L^*_n(T) = \sup\{\ell^*_{n_t}(x): \ x \le T-t, \ t < T\} \ . \tag{2.26}$$

Statistics with the superscript 0 and * are termed the *unweighted* and *weighted* ones, respectively. For the weighted statistics, we may remark that in (2.20), whenever $S_{n_t}(x) = 0$ or in (2.24), $q_{t,T} = 0$, there may be an operational difficulty in these definitions. As a result, the range of x and t in (2.22) and (2.26) be such that S_{n_t} is strictly > 0 (and < 1) and $q_{t,T} \ge 1$. This also demands that n_t be strictly positive in these domains. Keeping this in mind, (and to be able to adapt some simple asymptotic theory), we modify (2.22) and (2.26), by choosing an ε (> 0), usually quite small, and letting

$$K_n^*(T) = \sup\{m_{n_t}^*(x):\ n^{-1}n_t \geq \varepsilon,\ \varepsilon \leq S_{n_t}(x) \leq 1-\varepsilon,\ x \leq T-t,\ t<T\}\ , \tag{2.27}$$

$$L_n^*(T) = \sup\{\ell_{n_t}^*(x):\ n^{-1}n_t \geq \varepsilon,\ S_{n_t}(x) \geq \varepsilon,\ x \leq T-t,\ t<T\}\ . \tag{2.28}$$

Finally, we define

$$K_n^0 = \sup\{K_n^0(T):\ T \leq T_c\}\ ,\quad K_n^* = \sup\{K_n^*(T):\ T \leq T_c\} \tag{2.29}$$

$$L_n^0 = \sup\{L_n^0(T):\ T \leq T_c\}\ ,\quad L_n^* = \sup\{L_n^*(T):\ T \leq T_c\}\ . \tag{2.30}$$

[In (2.27)-(2.28), the supreme over a null set (if so) is taken as 0.]

Suppose now that there exist real numbers $k_{n\alpha}^0$, $k_{n\alpha,\varepsilon}^*$, $\lambda_{n\alpha}^0$ and $\lambda_{n\alpha,\varepsilon}^*$, such that under H_0: $\underset{\sim}{\beta} = \underset{\sim}{0}$,

$$P\{K_n^0 > k_{n\alpha}^0 | H_0\} \leq \alpha \leq P\{K_n^0 \geq k_{n\alpha}^0 | H_0\} \tag{2.31}$$

and similarly for the other three statistics in (2.29)-(2.30), where α $(0 < \alpha < 1)$ is the desired *level of significance* of the test. Then, operationally, the proposed PCS test may be described as follows. Monitor experimentation from the first entry point E_1. At each $T(> E_1)$, compute $K_n^0(T)$ [or $K_n^*(T)$ or $L_n^0(T)$ or $L_n^*(T)$]. If $K_n^0(T)$ is $< k_n^0$ [or $K_n^*(T) < k_{n\alpha,\varepsilon}^*$ or $L_n^0 < \lambda_{n\alpha}^0$ or $L_n^*(T) < \lambda_{n\alpha,\varepsilon}^*$], continue experimentation, while, if for the first time at $T = T^*(\leq T_c)$, the opposite inequality holds, stop experimentation at time T^* along with the rejection of H_0: $\underset{\sim}{\beta} = \underset{\sim}{0}$. If no such T^* $(\leq T_c)$ exists, then esperimentation is curtailed at the preplanned timepoint T_c, along with the acceptance of H_0. T^* may be defined as the *stopping time* for the proposed PCS testing procedure. Note that by definition

$$T^* \leq T_c, \text{ with probability 1,} \tag{2.32}$$

and it may be smaller than T_c with a positive probability. Further, by

(2.29)-(2.31), the proposed procedure has the prescribed level of significance α $(0 < \alpha < 1)$. Since for different T, the statistics $K_n^0(T)$ [or $K_n^*(T)$ or $L_n^0(T)$ or $L_0^*(T)$] are not independent, nor $K_n^0(T)$ etc. are processes of independent or homogeneous increments, determination of the exact critical values in (2.31) may pose a challenging problem. However, certain invariance principles for LRS and weighted empirical processes can be incorporated to provide suitable large sample approximation (or bounds) for these critical values. This will be discussed in detail in Section 3. A discussion on the relative merits and demerits of the weighted and unweighted statistics will be made in Section 4.

3. ASYMPTOTIC THEORY

The exact distribution theory of the proposed test statistics [even under H_0: $\underset{\sim}{\beta} = \underset{\sim}{0}$] depends on the $\underset{\sim}{c}_i$, the entry-points $E_1, \ldots, E_n$, β_0 and T_c, through $F(T_c - E_i - \beta_0)$, and becomes quite involved as the sample size n increases. In this section, we intend to provide good approximations for large sample sizes, valid under quite general regularity conditions. In this context, we assume that the following holds:

(I) There exists a p.d. matrix $\underset{\sim}{C}_0$, such that

$$m^{-1}\underset{\sim}{C}_m \to \underset{\sim}{C}_0 \quad \text{as} \quad m \quad \text{increases} , \tag{3.1}$$

and (II) if $\text{ch}_1(\underset{\sim}{A})$ stands for the largest characteristic root of $\underset{\sim}{A}$, then

$$\max_{1 \le i \le m} \{\text{ch}_1(\underset{\sim}{C}_m^{-1}(\underset{\sim}{c}_i - \bar{\underset{\sim}{c}}_m)(\underset{\sim}{c}_i - \bar{\underset{\sim}{c}}_m)')\} \to 0 \quad \text{as} \quad m \to \infty . \tag{3.2}$$

For rank based procedures, we assume further that for every k (≥ 1), the set $\{a_k(1), \ldots, a_k(k)\}$ of scores is generated by a *score-function* $\phi = \{\phi(u):\ 0 < u < 1\}$ in the following way

$$a_k(\ ku \ + 1) = \phi_k(u),\ 0 < u < 1\ , \tag{3.3}$$

$$\lim_{k\to\infty} \int_0^1 \{\phi_k(u) - \phi(u)\}^2 du = 0\ , \tag{3.4}$$

where $\phi(u) = \phi_{(1)}(u) - \phi_{(2)}(u),\ 0 < u < 1$ and the $\phi_{(j)}$ are non-decreasing and square integrable inside [0, 1]. A typical case for (3.3) is $a_k(i) = \phi(i/(k+1)),\ 1 \le i \le k$.

To introduce the asymptotic theory, first, we consider some basic processes. Let $W_j = \{W_j(s,t):\ 0 \le s < \infty,\ 0 \le t < \infty\},\ j=1,\dots,p$ be $p\ (\ge 1)$ independent copies of a standard *Brownian sheet.* Also, consider the related *Kiefer processes* $W_j^0 = \{W_j^0(s,t) = W_j(s,t) - tW_j(s,1):\ 0 \le s < \infty,\ 0 \le t \le 1\},\ 1 \le j \le p$. Define then the (squared) *Bessel sheets* by letting

$$B_p = \{B_p(s,t) = \Sigma_{j=1}^p W_j^2(s,t):\ \underset{\sim}{0} \le (s,t) \le \underset{\sim}{\infty}\} \tag{3.5}$$

and the (squared) *Kiefer-Bessel sheets* by letting

$$B_p^0 = \{B_p^0(s,t) = \Sigma_{j=1}^p W_j^{02}(s,t):\ 0 < s < \infty,\ \ 0 \le t \le 1\}\ . \tag{3.6}$$

Finally, we introduce the processes

$$B_p^* = \{B_p^*(s,t) = t^{-1}s^{-1}B_p(s,t):\ 0 < s < \infty,\ 0 < t < \infty\}\ , \tag{3.7}$$

$$B_p^{0*} = \{B_p^{0*}(s,t) = s^{-1}t^{-1}(1-t)^{-1}B_p^0(s,t):\ 0 < s < \infty,\ 0 < t < 1\}. \tag{3.8}$$

We shall see later on that the boundary crossing probabilities for these processes provide the desired asymptotic theory for the PCS tests. Note that by the time and scale transformation on the Brownian sheets and (3.5), we have for every $0 < \eta_1 < \eta_2 < \infty$ and $0 < \varepsilon_1 < \varepsilon_2 < \infty$, on letting $\eta = \eta_2/\eta_1\ (> 1)$ and $\varepsilon = \varepsilon_2/\varepsilon_1\ (> 1)$,

$$\left\{\sup_{\substack{\eta_1 \le s \le \eta_2 \\ \varepsilon_1 \le t \le \varepsilon_2}} B_p^*(s,t)\right\} \overset{\mathcal{D}}{=} \left\{\sup_{\substack{1 \le s \le \eta \\ 1 \le t \le \varepsilon}} B_p^*(s,t)\right\} . \tag{3.9}$$

Further, note that on using the identity that $W_j(s,t) = (t+1)W_j^0(s, \frac{t}{t+1})$, $0 \le s < \infty$, $0 \le t < \infty$, we obtain by a few routine steps that for every $0 < \eta_1 < \eta_2 < \infty$ and $0 < \varepsilon_1 < \varepsilon_2 < 1$, on letting $\eta = \eta_2/\eta_1$ (> 1) and $\varepsilon_j^0 = \varepsilon_j/(1-\varepsilon_j)$, $j=1,2$; $\varepsilon^0 = \varepsilon_2^0/\varepsilon_1^0$ (> 1),

$$\left\{\sup_{\substack{\eta_1 \le s \le \eta_2 \\ \varepsilon_1 \le t \le \varepsilon_2}} B_p^{0*}(s,t)\right\} \overset{\mathcal{D}}{=} \left\{\sup_{\substack{1 \le s \le \eta \\ 1 \le t \le \varepsilon^0}} B_p^*(s,t)\right\} . \tag{3.10}$$

Thus, for the standardized processes B_p^* adn B_p^{0*}, with suitable adjustments on the range spaces, the boundary crossing probabilities may be obtained from one another.

Let us now define the $\underset{\sim}{L}_{k,q}$ as in (2.13) and let

$$B_{(n)} = \{B_{(n)}(s,t): \ 0 \le s \le 1, \ 0 \le t \le 1\}$$
$$= \{A_n^{-2} \underset{\sim}{L}'_{[ns],q(s,t)} \underset{\sim}{C}_n^{-} \underset{\sim}{L}_{[ns],q(s,t)}: \ \underset{\sim}{0} \le (s,t) \le \underset{\sim}{1}\} \tag{3.11}$$

where $q(s,t)$ is defined by

$$q(s,t) = \max\{q: \ A^2_{[ns],q} \le tA^2_{[ns]}, \ 0 \le t \le 1\}, \ s \ge 0 , \tag{3.12}$$

and A_m^2 and $A_{m,q}^2$ are defined by (2.15). Then, in the same manner as Majumdar and Sen (1978a) extended the invariance principle (for the non-staggering entry case) of Chatterjee and Sen (1973) to the vector $\underset{\sim}{c}_i$ case, we may extend the theory in Sen (1976a) to the vector case and obtain that under H_0: $\underset{\sim}{\beta} = \underset{\sim}{0}$ in (2.2)-(2.3) and the assumed regularity conditions, as $n \to \infty$,

$$B_{(n)} \xrightarrow{\mathcal{D}} B_p = \{B_p(s,t):\ \underset{\sim}{0} \le (s,t) \le \underset{\sim}{1}\}\ , \tag{3.13}$$

where $B_p(s,t)$ is defined by (3.5). For intended brevity, the details are omitted. Let us also denote by $\underset{\sim}{M}_k^0(t) = \underset{\sim}{M}_k(x)$ when $F(x-\beta_0) = t$ (refer to (2.2) and (2.7)) and define

$$\begin{aligned} B^0_{(n)} &= \{B^0_{(n)}(s,t):\ \underset{\sim}{0} \le (s,t) \le \underset{\sim}{1}\} \\ &= \{(\underset{\sim}{M}^0_{[ns]}(t))' C_n^-(\underset{\sim}{M}^0_{[ns]}(t)):\ \underset{\sim}{0} \le (s,t) \le \underset{\sim}{1}\}\ . \end{aligned} \tag{3.14}$$

Then, as a direct (vector-) extension of the results of Sinha and Sen (1982), we obtain (on omitting the details) that under H_0: $\underset{\sim}{\beta} = \underset{\sim}{0}$ and (3.1)-(3.2), as $n \to \infty$,

$$B^0_{(n)} \xrightarrow{\mathcal{D}} B^0_p = \{B^0_p(s,t):\ \underset{\sim}{0} \le (s,t) \le \underset{\sim}{1}\}\ , \tag{3.15}$$

where the $B^0_p(s,t)$ are defined by (3.6).

Now, in a staggering entry plan with distinct entry points $E_1 < \ldots < E_n$ ($< T_c$), in (2.21), (2.23) and (2.29)-(2.30), as t is allowed to vary between E_1 and E_n, n_t assumes all possible values between 1 to n. On the other hand, the domain of x,t in (2.21), (2.23) and (2.29)-(2.30) depends on the entry-point $E_1,\ldots,E_n$, censoring point T_c and the d.f. F. The maximum value of $F(x-\beta_0)$ is of course $F(T_c - E_1 - \beta_0)$ which is a point on the unit line $[0,1]$ and the minimum value is $F(T_c - E_n - \beta_0)$. Thus, in the staggering entry plan under consideration, the processes $B_{(n)}$ and $B^0_{(n)}$ in (3.11) and (3.14) are defined only a lower subspace of $I^2 = [0,1]^2$ and the structure of this subspace depends on $F(T_c - E_i - \beta_0)$, $1 \le i \le n$. Let us denote these subspaces for the two cases by $I_{(n)}$ and $I^0_{(n)}$, respectively, and note that, in general, both of these are stochastic in nature. In some cases with some simple entry-pattern (e.g., uniform spacing of the E_i), $I_{(n)}$ and $I^0_{(n)}$ may be specified or may even be bounded by non-random subsets (I_* and I_*^0) of I^2. In any case, $I_* \subset I^2$ and $I_*^0 \subset I^2$, so that some dominated results can be obtained by using (3.13) and (3.15) for the entire domain I^2. Hence, we are confronted with the problem of finding the distribution of the statistics

$$\sup\{B_p(s,t):\ (s,t) \in I_*\} \text{ and } \sup\{B_p^0(s,t):\ (s,t) \in I_*^0\}, \qquad (3.16)$$

for suitable I_* and I_*^0, including the possibility for $I_* = I_*^0 = I^2$. If I_* is a lower rectangle of I^2, then, of course, the scale-time transformation may be used to express the first statistic in (3.16) as a constant multiple of $\sup\{B_p(s,t):\ (s,t) \in I^2\}$ and a similar transformation on the time parameter s in $B_p^0(s,t)$ is possible. We shall discuss more about these in Section 7. For a batch arrival model, the n_t may not take on all values between 1 and n, it increases to its asymptotic value n in finitely many jumps only. Typically, there may be only b (≥ 1) entry points $E_1^*<\ldots<E_b^*$ and n_j subjects enter the scheme at timepoint E_j^*, so that the possible values of n_t are $\Sigma_{s=1}^r n_s$, for $r=1,\ldots,b$. Thus, in (2.19)-(2.30), the n_t can only assume b distinct values, so that in (3.14) and (3.14), on the first time parameter scale, we will have only b points, say $s_1<\ldots<s_b\leq 1$. As a result, in (3.16), the subspace I_* or I_*^0 will consist of b contours on I^2. Nevertheless, the suprema over I^2 will dominate the ones over I_* or I_*^0 and when b is not very small, the difference will be negligible. Thus, the use of (3.16) with appropriate I_*, I_*^0 or I^2, will result in a slightly higher critical value and the test will thereby be somewhat conservative. One advantage of using these critical values will be to make them valid irrespective of any batch-frequency patterns. In a similar manner, for the test statistics K_n^* and L_n^* in (2.27)-(2.30), to study the asymptotic theory, we confront with the problem of finding the distributions of

$$\sup\{B_p^*(s,t):\ (s,t) \in I_*\} \quad \text{and} \quad \sup\{B_p^{0*}(s,t):\ (s,t) \in I_*^0\} \qquad (3.17)$$

where $I_* \subset [\varepsilon,1]^2$ and $I_*^0 \subset [\varepsilon,1] \times [\varepsilon, 1-\varepsilon]$, $\varepsilon > 0$. In this context, we can use, with advantage, (3.9) and (3.10), and hence, the problem reduces to that of finding out the distribution of

$$\sup\{B_p^*(s,t):\ 1 \le s \le a,\ 1 \le t \le b\}, \text{ for suitable } (a,b) > \underset{\sim}{1}\ . \qquad (3.18)$$

For some numerical studies (by simulation) of (3.18), we may refer to Section 7.

For the particular case of $p=1$ (i.e., Scalar $\underset{\sim}{c}_i$), weak invariance principles for $B_{(n)}$ and $B_{(n)}^0$, under suitable sequences of local (contiguous) alternatives have been studied by Sen (1976a), Majumdar and Sen (1978b) and Sinha and Sen (1982). Their results extend to the case of vector $\underset{\sim}{c}_i$ (as under review), and under such local alternatives [viz., $\{H_n\}$, where under H_n: (2.2) holds with $\underset{\sim}{\beta} = n^{-\frac{1}{2}}\underset{\sim}{\gamma}$ for some $\underset{\sim}{\gamma} \in R^p$], the asymptotic power function of the proposed tests can be expressed in terms of the boundary crossing probabilities of some drifted Bessel sheets or Kiefer-Bessel processes. These drift functions are not, in general, linear in t (or s) and, even if they are so, there is no precise analytical expressions for these probabilities. As such, such results do not lead to any useful tool for studying the asymptotic efficiency of these competing tests. Empirical study of the power properties (by Monte Carlo simulation techniques) seems to be a more practical alternative. For some numerical studies of this nature, we may refer to Sinha (1979).

4. SOME SPECIFIC TESTS

In Section 2, we have proposed both the unweighted and weighted test statistics based on the PCS weighted empirical processes and LRS. From the point of view of repeated significance testing (RST), the weighted statistics in (2.27), (2.28) [and (2.29)-(2.30)] are naturally appealing. If we consider the collection of the statistics in (2.20) [or (2.24)], for all permissible x and t, then the union-intersection principle leads us to the test statistics in (2.29)-(2.30) for the weighted case. Pointwise (i.e., given x,t), under H_0 in (2.3), the statistics in (2.20) [or (2.24)] has asymptotically chi-square distribution [see Majumdar and Sen (1978a) and Sinha and Sen (1979b)] and they may even be judged locally optimal against appropriate alternatives. Thus, the union-intersection principle applied to such desirable test statistics leads to the weighted ones in (2.27)-(2.28) and (2.29)-(2.30). However, in practice, for these weighted test statistics, one needs to choose some appropriate $\varepsilon > 0$, and, in general, the critical values in (2.31) etc. may depend on this choice of ε. On the other hand, for the unweighted test statistics in (2.19), (2.21), (2.23), (2.25) and (2.29)-(2.30), one does not need to restrict the domain of (x,t) by suitable choice of ε and the critical values in (2.31) etc. are also independent of this ε. From the computational point of view, the unweighted ones appear to be a lot simpler too. For tests based on the weighted empirical processes, Sinha (1979) has made some numerical studies of the relative powers of the weighted and unweighted statistics (mostly, for exponential distributions), by Monte Carlo studies, and observed that there is not much difference in their performances; the unweighted one may even perform somewhat better than the weighted one in some situations. A very similar picture holds for the rank procedures too. As such, we generally tend to recommend the use of the simpler unweighted procedures based on K_n^0 and L_n^0, respectively.

For rank procedures, we have considered a general class of LRS, staisfying (3.3)-(3.4). Among various possibilities, two particular scores are especially recommended for survival analysis. These are (i)the *Wilcoxon scores*, where $a_k(i) = i/(k+1)$, $1 \le i \le k$ and (ii)the *logrank scores* where

$a_k(i) = 1 - \Sigma_{j=1}^{i} (k-j+1)^{-1}$, $1 \le i \le k$. The first set is suitable on the grounds on computational simplicity and overall robustness, while the second set, on the ground of local optimality when the underlying d.f.'s are all exponential. For non-staggering entry plans, the asymptotic opitmality of PCS rank tests has been studied by Sen (1976b). Though the picture essentially remains the same for a staggering entry plan, analytical comparisons may be very difficult to make.

For all the procedures proposed in Section 2, the asymptotic theory works out well when the increments of $n^{-1}n_t$ $(t \le S_n)$ are no where very large, i.e., the batch frequencies (relative to n) are all small. When they are not so, we are faced with a somewhat different situation (e.g., the batch arrival model), which will be discussed in Section 6.

5. INCORPORATION OF RANDOM WITHCRAWALS

The modifications needed to incorporate dropouts or random withdrawals of subjects (during $[E_1, T_c]$) are similar to those described in Section 5 of Majumdar and Sen (1978b) and Section 5 of Sinha and Sen (1982) and carry over to the vector case treated here. Referred to the model (2.1)-(2.2), we assume that for each i, X_i^0 and Y_i are independent r.v.'s with d.f.'s F_i and G, respectively, where G (does not depend on i) is some unknown (but continuous) d.f. Then, the X_i in (2.1) are independent r.v.'s with d.f.'s F_i^*, given by

$$1 - F_i^*(x) = [1-G(x)][1-F_i(x)], \; i \ge 1 \tag{5.1}$$

and hence, under (2.1)-(2.3) and (5.1), $\underset{\sim}{\beta} = \underset{\sim}{0} \Rightarrow F_1^* = \ldots = F_n^*$. As such, the test procedures described in Section 2 remains valid in this case too. However, we may note that by (5.1),

$$F_i^*(x) - F_j^*(x) = [1-G(x)][F_i(x)-F_j(x)],\ x \in R,\ i\neq j=1,\dots,n\ , \quad (5.2)$$

so that the actual distance between F_i^* and F_j^* is smaller than that of F_i and F_j, for every $i\neq j=1,\dots,n$, so that the test based on the X_i (instead of X_i^0) becomes less powerful and the loss of power depends on the damping factor $1-G(x)$, $0 \le x \le T_c-E_1$. In the above development we have assumed that G is continuous. If G is not necessarily continuous, (5.1) remains true, but, the ties among the Y_i (and hence, X_i) may occur with a positive probability. In such a case, the definition of the ranks in (2.9) needs some adjustments (i.e., mid-ranks for tied observations), and, generally, this will make the test more conservative. So far, we have tacitly assumed that X_i^0 and Y_i are independent and G does not depend on i $(=1,\dots,n)$. In the negation of either of these (5.1) may not hold, and hence, some other adjustments may be necessary to apply the proposed tests. As in Cox (1972), use of some partial likelihood or empirical processes is possible, but will be generally very complicated.

6. MULTISAMPLE CASE AND BATCH ARRIVAL MODELS

Consider now the q $(=p+1)$ smaple problem which can be characterized by the model (2.2) with a simple structure on the $\underset{\sim}{c}_i$. Suppose that the individual sample sizes are $n_1,\dots,n_q$, respectively, so that $n = n_1+\dots+n_q$. We write $\lambda_{ni} = n^{-1}n_i$, $1 \le i \le n$ and assume that

$$\underset{\sim}{\lambda}_n = (\lambda_{n1},\dots,\lambda_{nq})' \to \underset{\sim}{\lambda} = (\lambda_1,\dots,\lambda_q):\ \lambda_j > 0,\ 1 \le j \le q\ . \quad (6.1)$$

Then, if at the ith entry point E_i an unit from the kth population enters the scheme, we let

$$\underset{\sim}{c}_i = \begin{cases} \underset{\sim}{0}\ , & \text{if } k=1 \\ (\delta_{k-11},\dots,\delta_{k-1q}), & 2 \le k \le q,\ \ 1 \le i \le n, \end{cases} \quad (6.2)$$

where the δ_{rs} are the usual Kronecker delta. Therefore,

$$\bar{\underset{\sim}{c}}_n = (\lambda_{n2},\ldots,\lambda_{nq})' \quad \text{and} \quad \underset{\sim}{C}_n = ((\delta_{jj'}n_{j+1} - n_{j+1}n_{j'+1})). \qquad (6.3)$$

If we let $n_t = n_{t1}+\ldots+n_{tq}$, where n_{tk} is the number of subjects in the kth sample entering the scheme at times on or before t $(E_1 \le t < E_n < T_c)$, and let $\underset{\sim}{\lambda}_{n_t} = n_t^{-1}(n_{t1},\ldots,n_{tq})' = (\lambda_{n_t 1},\ldots,\lambda_{n_t q})'$ then we assume that

$$n^{-1}n_t \to \theta: \quad 0 < \theta \le 1 \Rightarrow \underset{\sim}{\lambda}_{n_t} \to \underset{\sim}{\lambda}, \text{ as } n \to \infty . \qquad (6.4)$$

Technically this means a proportional entry plan for the q samples, so that PCS does not hamper the design of the experimentation relative to the q samples. In this case, in (2.8), we have

$$S_{n_t}(x) = \Sigma_{j=1}^{q} \lambda_{n_t j} S_{n_t j}(x) , \qquad (6.5)$$

where the $S_{n_t j}$ are the empirical d.f. for the jth sample n_{tj} observations. Also, in (2.7), we have

$$\underset{\sim}{M}_{n_t}(x) = (n_{t2}S_{n_t 2}(x),\ldots,n_{tq}S_{n_t q}(x))' - S_{n_t}(x)\cdot(n_{t2},\ldots,n_{tq})'. \qquad (6.6)$$

Thus, the weighted empirical processes can be expressed as linear combinations of the individual sample empirical distribution processes. Similarly, the $\underset{\sim}{L}_{n_t, q_{t,T}}$ in (2.13) and (2.18), are linear combinations of the censored multi-(sub-)sample rank statistics, where the $R_{n_t i}$ will stand for the ranks of the observations among the n_t entries prior to time t. The computations of the test statistics poses no problem.

One of the advantages of the proposed test procedures is their adaptability when the units enter into the scheme in batches and/or the monitoring is made systemetically at regular time intervals instead of continuously over time. To illustrate this, we consider the multi-sample model in batch arrivals and distinct time-interval inspection plans. A detailed account of such schemes relating to ordered categorical data is given by Majumdar and Sen (1980). Suppose that the subjects enter into the scheme in b (≥ 1) batches at time points $E_1^* < \ldots < E_b^*$. Also, suppose that the process is statistically monitored at each of these E_j^* and their after (if needed) at points $E_{b+1}^*, \ldots, E_c^*$ ($= T_c$), where T_c is a preassigned censoring time point. Consider then the (ordered) time intervals $J_j = [E_j^*, E_{j+1}^*)$, $1 \leq j \leq c$ where $E_{c+1}^* = +\infty$. We assume that the lengths of these intervals are all the same, so that $E_{j+1}^* = E_1^* + ja$, $a > 0$, for $j=1,\ldots,c-1$, though J_c is of infinite length. Let n_{jk} be the number of subjects from the jth sample which enter the scheme at time point E_k^*, for $1 \leq k \leq b$, $1 \leq j \leq q$ ($= p+1$). Thus, $n_j = \Sigma_{k=1}^b n_{jk}$, $1 \leq j \leq q$. Suppose that out of these n_{jk} subjects, $N_{j\ell}^k$ failures occur in the time interval J_ℓ, for $1 \leq j \leq q$, $1 \leq k \leq b$, $k \leq \ell \leq c$; note that N_{jk}^k need not be equal to 0. For notational simplicity, we net $n_{jk} = 0$, $\forall\, k > b$ and define

$$n_{j\ell}^* = \Sigma_{s=1}^{\ell} n_{js}, \quad n_\ell^* = \Sigma_{j=1}^{q} n_{j\ell}^*, \quad 1 \leq \ell \leq c \,, \tag{6.7}$$

$$N_{jkr}^* = \Sigma_{s=1}^{k} N_{jr+s-1}^{s}, \quad 1 \leq j \leq q, \; 1 \leq k \leq c, \; r \geq 1 \,, \tag{6.8}$$

$$N_{kr}^{**} = \Sigma_{j=1}^{q} N_{jkr}^*, \quad 1 \leq k \leq c, \; r \geq 1 \;. \tag{6.9}$$

Note that at time point E_k^*, $k \geq 1$, there are n_k^* units already in the scheme. Thus, according to (2.7)-(2.8), for every $r \geq 1$,

$$S_{n_k^*}(ar) = N_{kr}^{**}/n_k^*, \quad 1 \leq k \leq c \,, \tag{6.10}$$

$$\underset{\sim}{M}_{n_k^*}(ar) = (N_{jkr}^* - S_{n_k^*}(ar) n_{jk}^*/n_k^*, \quad 2 \leq j \leq q), \; 1 \leq k \leq c; \tag{6.11}$$

$$\underset{\sim}{C}_{n_k^*} = ((\delta_{jj'} n_{j+1k}^* - n_{j+1k}^* n_{j'+1k}^*/n_k^*)), \quad 1 \leq k \leq c \;. \tag{6.12}$$

Thus, by (2.20), (6.10), (6.11) and (6.12), we obtain that

$$m^*_{n^*_k}(ar) = \sum_{j=1}^{q} n^*_{jk}\left\{\frac{N^*_{jkr}}{n^*_{jk}} - \frac{N^{**}_{kr}}{n^*_k}\right\} \Big/ \left\{\frac{N^*_{kr}(n^*_k - N^{**}_{kr})}{(n^*_k)^2}\right\}, \qquad (6.13)$$

for $k=1,2,\ldots,c-1$, $r=1,2,\ldots$. Clearly, at time point E^*_ℓ, one has the bunch of $\{m^*_{n^*_k}(ar),\ k < \ell,\ r \ge 1,\ k+r = \ell\}$. Thus, if statistical monitoring is made only at the points $E^*_1,\ldots,E^*_c$ $(= T_c)$, there are in all $\binom{c}{2} = c^*$ points in the set I^0_* in (3.16), so that unless c^* is fairly large, replacing this discrete set by a subrectangle of I^2 may result in some conservative property of the test.

For rank based procedures, in this batch-arrival model, we will have tied observations, as it is only known that $E^*_i + X_i$ belongs to some interval J_ℓ, for $\ell=1,\ldots,c$ and $i \ge 1$. Adjustments for ties for non-staggering entry plans (relating to grouped data) were discussed in detail by Majumdar and Sen (1977) (for scalar c_i) and Majumdar (1977) (for vector $\underset{\sim}{c}_i$). As in (6.7) through (6.13), one can obtain the expressions for the $L_{n_t,q_t,T}$ in (2.19) for $T = E^*_2,\ldots,E^*_c$ and $t = E^*_1,\ldots,E^*_{c-1}$; we refer to Majumdar (1977) for these details. For intended brevity we do not reproduce these notations. Here also, we will have $c^* = \binom{c}{2}$ points in the set I_* in (3.16) and the smae conservative character of the test prevails.

In a nonstaggering entry plan, based on some simulation studies, it has been observed by Sinha (1979) that the two PCS testing procedures based on the $m^*_{n^*_k}(ar)$ and $L_{n^*_k,q}$ behave very similarly with respect to their significance levels as well as empirical powers. While this picture is expected to remain the same in a staggering entry plan, a verification needs an extensive amount of simulation work not only for various underlying distributions, but also, for various entry patterns and withdrawal schemes.

7. SIMULATION STUDIES OF THE CRITICAL VALUES OF THE PCS TEST STATISTICS

In practical applications of the PCS testing procedures, we need to know about the critical values of K_n^0, K_n^*, L_n^0 and L_n^*. As has been discussed in Section 3, boundary crossing probabilities for the Bessel sheets and the Kiefer-Bessel processes provide suitable large sample approximations to these critical values. On the other hand, for such Bessel sheets or Kiefer-Bessel processes, generally, no analytical expressions for these boundary crossing probabilities are available, though some bounds are available in the literature. However, Monte Carlo techniques can readily be employed for this purpose and this is pursued in this section. The basic idea ia to incorporate the weak invariance principles for multi-dimensional array of r.v.'s to generate these processes and employ them to provide suitable simulated results for the desired percentile points.

Let $\{X_{kij},\ 1 \le i \le n_1,\ 1 \le j \le n_2\}$, $k=1,\ldots,p$ be p independent sets of r.v.'s, where the X_{kij} are independently distributed according to a standard normal distribution. Let $S_{k00} = S_{ki0} = S_{k0i} = 0,\ \forall\ i \ge 1$, and $S_{kij} = \Sigma_{r=1}^{i} \Sigma_{s=1}^{j} X_{krs}$, for $1 \le i \le n_1$, $1 \le j \le n_2$, $k=1,\ldots,p$. Let then $W_n = \{W_n(s,t):\ 0 \le (s,t) \le 1\}$ be defined by

$$\underset{\sim}{W}_n(s,t) = \{W_{n1}(s,t),\ldots,W_{np}(s,t)\}$$

$$= (n_1n_2)^{-\frac{1}{2}}\{S_{1[n_1s][n_2t]},\ldots,S_{p[n_1s][n_2t]}\}\ , \qquad (7.1)$$

for $\underset{\sim}{0} \le (s,t) \le \underset{\sim}{1}$. Also, let $\underset{\sim}{W}_n^0 = \{\underset{\sim}{W}_n^0(s,t):\ \underset{\sim}{0} \le (s,t) \le \underset{\sim}{1}\}$ be defined by

$$\underset{\sim}{W}_n^0(s,t) = \underset{\sim}{W}_n(s,t) - t\underset{\sim}{W}_n(s,1),\quad \underset{\sim}{0} \le (s,t) \le \underset{\sim}{1}\ . \qquad (7.2)$$

Then, by virtue of the classical weak invariance principles,

$$\underset{\sim}{W}_n \overset{\mathcal{D}}{\to} \underset{\sim}{W} \quad \text{and} \quad \underset{\sim}{W}_n^0 \overset{\mathcal{D}}{\to} \underset{\sim}{W}^0, \quad \text{as} \quad n_1,n_2 \to \infty , \tag{7.3}$$

where $\underset{\sim}{W} = \{W_1,\ldots,W_p\}$ and $\underset{\sim}{W}^0 = (W_1^0,\ldots,W_p^0)$ are defined after (3.4). As a result, by (3.5), (3.6) and (7.1)-(7.3), we obtain that for large n_1,n_2,

$$\max_{\substack{1\le i\le n_1 \\ 1\le j\le n_2}} \left\{ \sum_{k=1}^{p} W_{nk}^2\left(\frac{i}{n_1}, \frac{j}{n_2}\right) \right\} \overset{\mathcal{D}}{\to} \sup_{\substack{0\le s\le 1 \\ 0\le t\le 1}} \{B_p(s,t)\} , \tag{7.4}$$

$$\max_{\substack{1\le i\le n_1 \\ 1\le j\le n_2}} \left\{ \sum_{k=1}^{p} W_{nk}^{02}\left(\frac{i}{n_1}, \frac{j}{n_2}\right) \right\} \overset{\mathcal{D}}{\to} \sup_{\substack{0\le s\le 1 \\ 0\le t\le 1}} \{B_p^0(s,t)\} . \tag{7.5}$$

We denote the upper 100α% point of the distribution of the statistics on the right hand side of (7.4) and (7.5) by $\mu_{\alpha,p}$ and $\mu^0_{\alpha,p}$, respectively, and we obtain estimates for these by simulating 500 copies of the left hand sides of (7.4) and (7.5), where we take $n_1 = 50$ and $n_2 = 100$ and thus for each of these 500 sets, we need to generate 500 standard normal variables. For p=1, these values are already reported in Majumdar and Sen (1978b) and Sinha and Sen (1982), while for some $p \ge 2$, these simulation results are given in Table 7.1.

TABLE 7.1

Simulated values of $\mu_{\alpha,p}$ and $\mu^0_{\alpha,p}$

	p=2		p=3		p=4		p=5	
α	$\mu_{\alpha,p}$	$\mu^0_{\alpha,p}$	$\mu_{\alpha,p}$	$\mu^0_{\alpha,p}$	$\mu_{\alpha,p}$	$\mu^0_{\alpha,p}$	$\mu_{\alpha,p}$	$\mu^0_{\alpha,p}$
0.01	10.712	3.209	13.018	4.061	15.064	4.334	16.109	4.881
0.05	7.788	2.578	9.470	3.077	11.155	3.438	12.394	3.956
0.10	6.180	2.036	7.930	2.676	9.770	2.935	11.166	3.448

In view of (3.9) and (3.10), we need to consider only the distribution of the statistic on the left hand side of (3.10), and, for this purpose, we note that by virtue of (7.3), for every $0 < \eta < 1$ and $0 < \varepsilon < \frac{1}{2}$

$$\max_{\substack{[n_1\eta]\le i\le n_1 \\ [n_2\varepsilon]\le j\le n_2-[n_2\varepsilon]}} \left\{ \sum_{k=1}^{p} \frac{W_{nk}^{02}(\frac{i}{n_1}, \frac{j}{n_2})}{ij(n_2-j+1)/n_1 n_2^2} \right\} \xrightarrow{\mathcal{D}} \sup_{\substack{\eta\le s\le 1 \\ \varepsilon\le t\le 1-\varepsilon}} \left\{ \frac{B_p^0(s,t)}{st(1-t)} \right\} , \quad (7.6)$$

so that the empirical distribution of the left hand side, obtained from the same set of 100 replicates (as in Table 7.1) provides the simulated values for the percentile points of the distributions of the right hand side statistics. These are denoted by $\mu_{\alpha,p}^{0*}(\varepsilon,\eta)$ and presented in Table 7.2.

TABLE 7.2

Simulated values of $\mu_{\alpha,p}^{0*}(\varepsilon,\eta)$ for $\eta = 0.02$

ε	p=2			p=3		
	α=.01	α=.05	α=.10	α=.01	α=.05	α=.10
.01	22.046	18.176	16.145	25.043	20.876	19.089
.05	22.044	17.781	15.684	24.734	20.745	18.659
.10	21.571	17.239	15.142	23.505	20.209	18.229
.15	20.583	16.103	14.570	23.378	19.122	17.474
.20	19.933	15.940	14.433	23.056	18.605	16.824
.25	19.306	15.458	14.158	22.412	18.109	16.046
	p=4			p=5		
.01	24.803	22.214	20.229	27.126	23.992	22.948
.05	24.683	21.060	19.643	26.633	23.750	21.845
.10	24.610	20.422	19.482	26.609	23.428	21.255
.15	24.331	20.265	19.326	26.227	22.881	20.961
.20	23.151	19.795	18.792	26.227	22.065	20.292
.25	22.835	19.584	18.263	25.038	21.717	19.930

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