ON SOME RECURSIVE RESIDUAL RANK TESTS FOR CHANGE-POINTS

by

Pranab Kumar Sen

Department of Biostatistics
University of North Carolina at Chapel Hill

Institute of Statistics Mimeo Series No. 1378

March 1982

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PRANAB KUMAR SEN

University of North Carolina, Chapel Hill

A general class of recursive residuals is incorporated in the formulation of suitable (aligned) rank tests for change-points pertaining to some simple linear models. The asymptotic theory of the proposed tests rests on some invariance principles for recursively aligned signed rand statistics, and these are developed. Along with the asymptotic properties of the proposed tests, allied efficiency results are studied.

1. INTRODUCTION

Let $X_1,\ldots,X_n$ be n independent random variables (r.v.), taken at time-points $t_1<\ldots<t_n$, respectively, where X_i has an unknown, continuous distribution function (d.f.) F_i, defined on the real line E (= $(-\infty,\infty)$), for $i=1,\ldots,n$. In the simplest model, one may assume that $F_1=\ldots=F_n=F$ (unknown), and, based on $X_1,\ldots,X_n$, one may then like to draw statistical inference on suitable parameters (functionals) of the d.f.F. There are, however, problems in which a change of the d.f. (and hence, the parameters) may occur at an unknown time-point [viz., Page (1957)], so that it may be of some interest to test for such a possible change occurring at some unknown time point in (t_1,t_n). In a somewhat more general setup, one may conceive of the usual linear model:

$$F_i(x) = F(x - \underset{\sim}{\beta}_i'\underset{\sim}{c}_i),\ x \in E,\ i=1,\ldots,n\ , \qquad (1.1)$$

AMS Subject Classification: 62E20, 62G10, 62L99

Key Words & Phrases: Aligned rank statistics; asymptotic relative efficiency; CUSUM tests; invariance principles; rank estimators.

*Work supported by the National Heart, Lung and Blood Institute, Contract NIH-NHLBI-71-2243-L from the National Institutes of Health. This research is dedicated to Professor Herman Chernoff on the occasion of his 60th birthday.

where the $\underset{\sim}{c}_i$ are q-vectors ($q \geq 1$) of known regression constants, the $\underset{\sim}{\beta}_i$ are unknown regression parameters and F is an unspecified, continuous d.f.; the location model is a special case of (1.1) with q=1 and $\underset{\sim}{c}_i=1$, $\forall\ i \geq 1$. One may then conceive of the null hypothesis H_0 of the constancy of the regression relationships over time, i.e.,

$$H_0: \quad \underset{\sim}{\beta}_1 = \ldots = \underset{\sim}{\beta}_n = \underset{\sim}{\beta} \text{ (unknown)} , \qquad (1.2)$$

and based on the given $\underset{\sim}{c}_i$ and the observed X_i, one may then proceed to estimate the common $\underset{\sim}{\beta}$ or to draw other statistical conclusions on it. But, a constancy of the regression relationships may not hold, and a change may occur at some unknown tome-point, i.e., one may have

$$\underset{\sim}{\beta}_1 = \ldots = \underset{\sim}{\beta}_m \neq \underset{\sim}{\beta}_{m+1} = \ldots = \underset{\sim}{\beta}_n, \text{ for some m: } 1 \leq m < n . \qquad (1.3)$$

As such, one may desire to test for the null hypothesis H_0 in (1.2) against the composite alternatives in (1.3), where m is unknown.

Testing procedures for a possible change in location or regression realtionshipe occurring at an unknown time-point between consecutively taken observations have been proposed and srudied by a host of workers; a recent bibliography by Hinkley (1980) and a somewhat specialized monograph by Hackl (1980) provide some detailed accounts of these developments. In the parametric case, test statistics are constructed from either residuals based on the terminal estimator of $\underset{\sim}{\beta}$ or recursive residuals based on sequential estimators of $\underset{\sim}{\beta}$. An excellent account of this work is available with Brown, Durbin and Evans (1975). Some further recent studies in this direction (allowing F to be possible unspecified) are due to Deshayes and Picard (1981) and Sen (1982a, b), among others. In the nonparametric case, the developments are mostly restricted to the location model ($q=1$, $c_i=1$, $\forall\ i \geq 1$) where recursive ranking [viz., Bhattacharya and Frierson (1981)] is adaptable, as is also the pseudo reduction to the two sample problem [viz., A. Sen and Srivastava

(1975) and Sen (1978)]. Some ad hoc procedures are also due to Bhattacharyya and Johnson (1968), while some aligned rank tests based on terminal estimates are due to Sen (1977). Most of these procedures encounter difficulties when applied to the general model in (1.1)-(1.3); for such models, some non-recursive residual rank procedures are discussed in Sen (1982b).

The object of the present study is to incorporate a general class of recursive residuals in the formulation of suitable rank order test statistics for some change-point problems pertaining to (1.1)-(1.3). For the sake of simplicity of presentation, these procedures are considered first for the location model in Section 2; the case of the general model in (1.1)-(1.3) is then treated in Section 3. Asymptotic properties of the proposed tests are studied in Section 4. Section 5 deals with some allied asymptotic efficiency results. The Appendix is devoted to the derivation of some results on recursive estimates.

2. RECURSIVE RESIDUAL RANK TESTS FOR THE LOCATION MODEL

We confine ourselves to the location model, for which, in (1.1), we have

$$F_i(x) = F(x-\theta_i), \quad x \in E, \ i=1,\ldots,n , \qquad (2.1)$$

where $\theta_1,\ldots,\theta_n$ are the location parameters, and we want to test for

$$H_0: \quad \theta_1=\ldots=\theta_n=\theta \text{ (unknown)}, \qquad (2.2)$$

against the composite alternative

$$H: \quad \theta_1=\ldots=\theta_m\neq\theta_{m+1}=\ldots=\theta_n, \text{ for some } m: \quad 1\leq m<n . \qquad (2.3)$$

We assume that F is symmetric about 0, so that θ_i is the median of the d.f. F_i, for $i=1,\ldots,n$. First, we proceed to estimate θ recursively as follows.

For every k ($\geq$ 1), let $U_{k1}<\ldots<U_{kk}$ be the ordered r.v.'s of a sample of size k from the uniform (0,1) d.f., $\phi^+=\{\phi^+(u),\ 0<u<1\}$ be a non-constant, nondecreasing and square-integrable *score function*, generated by a Skew-symmetric $\phi=\{\phi(u),\ 0<u<1\}$ (i.e., $\phi(u) + \phi(1-u) = 0,\ 0<u<1$) in the following way:

$$\phi^{+}(u) = \phi((1+u)/2),\ 0<u<1,\ (\phi^{+}(0)=0) \tag{2.4}$$

and let

$$a_k(i) = E\phi^{+}(U_{ki}),\ i=1,\ldots,k;\ k\geq 1\ . \tag{2.5}$$

Some other regularity conditions on ϕ will be introduced in Section 4. For every k ($\geq$ 1) and b ($\in$ E), let $R^{+}_{ki}(b)$ be the rank of $|X_i-b|$ among $|X_1-b|,\ldots,|X_n-b|$, for i=1,...,k, and let

$$T_k(b) = T(X_1-b,\ldots,X_k-b) = \textstyle\sum_{i=1}^{k}\operatorname{sign}(X_i-b)a_k(R^{+}_{ki}(b))\ . \tag{2.6}$$

Note that $T_k(b)$ is ↘ in b ($\in$ E), and under (2.2), $T_k(\theta)$ has a specified distribution, symmetric about 0. Hence, based on $X_1,\ldots,X_k$, $\hat{\theta}_k$, the usual rank order estimator of θ, may be defined as

$$\hat{\theta}_k = \tfrac{1}{2}(\sup\{b:\ T_k(b)>0\} + \inf\{b:\ T_k(b)<0\})\ . \tag{2.7}$$

At the kth stage, we may define the (recursive) residuals as

$$\hat{X}_{ki} = X_i - \hat{\theta}_{k-1},\ i=1,\ldots,k,\ \text{for } 2\leq k\leq n, \tag{2.8}$$

while $\hat{X}_{11}$ is conventionally taken as equal to 0. Let $\hat{R}^{+}_{ki} = R_{ki}(\hat{\theta}_{k-1})$ be the rank of $|\hat{X}_{ki}|$ among $|\hat{X}_{k1}|,\ldots,|\hat{X}_{kk}|$, for i=1,...,k, and let

$$\hat{u}_k = \operatorname{sign}\hat{X}_{kk}a_k(\hat{R}^{+}_{kk}),\ \text{for } k\geq 2;\ \hat{u}_1=0\ . \tag{2.9}$$

We define the cumulative sums (CUSUM) for the recursive residual rank statistics in (2.9) by

$$\hat{U}_r = \textstyle\sum_{k\leq r}\hat{u}_k,\ \text{for } 1\leq r\leq n\ . \tag{2.10}$$

It may be noted that in (2.8), it may not be necessary to employ the rank order estimates $\{\hat{\theta}_k,\ k\leq n\}$, in (2.7), for defining these residuals. We may, under fairly general regularity conditions, employ other recursive estimators of θ as well. This point will be elaborated in Section 4. Let

$$A^2 = \int_0^1 \phi^2(u)du = \int_0^1 \{\phi^{+}(u)\}^2 du\ , \tag{2.11}$$

so that by assumption, $0<A<\infty$. Define then

$$D_n^+ = n^{-\frac{1}{2}}A^{-1}\{\max_{1\leq r\leq n} \hat{U}_r\} \text{ and } D_n = n^{-\frac{1}{2}}A^{-1}\{\max_{1\leq r\leq n}|\hat{U}_r|\} \qquad (2.12)$$

The proposed test for (2.2) against (2.3) is based on the statistic D_n^+ (for the one-sided alternative $\theta_m<\theta_{m+1}$) or D_n (for the two-sided one: $\theta_m\neq\theta_{m+1}$). Unlike the procedures considered by A. Sen and Srivastave (1975), Sen (1978) and Bhattacharya and Frierson (1981), the proposed tests may not be genuinely distribution-free under H_0 in (2.2). Nevertheless, they are asymptotically distribution-free and are easily extendable to the general model in (1.1)-(1.3), where the other procedures run into obstacles. This will be considered in the next section. The distribution theory of D_n^+ and D_n, under the null hypothesis as well as (local) alternatives, needed for the study of the (asymptotic) properties of the proposed tests will be considered in Section 4. It may be remarked that instead of the Kolmogorov-Smirnov type statistics in (2.12) one may also consider some Cramér-von Mises' type statistics based on the CUSUM's in (2.10), viz.,

$$V_n = n^{-2}A^{-2}\sum_{r=1}^{n}(\hat{U}_r^2) , \qquad (2.13)$$

or some weighted version of the same. In view of the invariance principles for the CUSUMs in (2.10), to be developed in Section 4, distributional results on such statistics would follow under the same set of regularity conditions, and hence, these details are omitted. Generally, D_n^+ (or D_n) has better (asymptotic) performance than V_n and intuitively more appealing too. Moreover, for small values of m, V_n may not perform that well.

3. RECURSIVE RESIDUAL RANK TESTS FOR THE REGRESSION MODEL

We consider here the general model in (1.1) and proceed to test for (1.2) against (1.3). As in Section 2, we assume that the d.f. F is symmetric about 0, and define the scores $\{a_k(i)\}$ as in (2.4)-(2.5). Also, we would employ here recursive estimates of $\underset{\sim}{\beta}$ [under (1.2)] in the construction of residuals and aligned rank statistics.

Assuming (1.2) to be true, based on $X_1,\ldots,X_k$, let $\hat{\underset{\sim}{\beta}}_k$ be some suitable estimator of $\underset{\sim}{\beta}$. The estimator $\hat{\underset{\sim}{\beta}}_k$ may be quite arbitrary (e.g., least squares estimator, rank order estimator or some other robust estimator) and will be defined more formally in Section 4. Then, at the kth stage, we define the (recursive) residuals as

$$\hat{X}_{ki} = X_i - \hat{\underset{\sim}{\beta}}'_{k-1}\underset{\sim}{c}_i, \ 1\leq i\leq k, \text{ for } k=1,\ldots,n \ . \qquad (3.1)$$

[Usually, for $k\leq q$, the $\hat{X}_{ki}$ are all equal to 0.] Let then $\hat{R}^+_{ki}$ be the rank of $|\hat{X}_{ki}|$ among $|\hat{X}_{k1}|,\ldots,|\hat{X}_{kk}|$, for $i=1,\ldots,k$; $k\geq 1$. Further, as in (2.9), we define the residual signed rank scores by

$$\hat{u}_k = \text{sign}(\hat{X}_{kk})a_k(\hat{R}^+_{kk}), \ k=q+1,\ldots,n \ , \qquad (3.2)$$

and, conventionally, we let $\hat{u}_k=0$ for $k\leq q$. Then the CUSUM's for the residual rank scores in (3.2) are

$$\hat{u}_r = \textstyle\sum_{k\leq r}\hat{u}_k, \text{ for } r=1,\ldots,n \ . \qquad (3.3)$$

Finally, we define A^2 as in (2.11), and parallel to (2.12), we let

$$D^+_n = (n-q)^{-\frac{1}{2}}A^{-1}\{\max_{r\leq n} \hat{U}_r\} \ , \qquad (3.4)$$

$$D_n = (n-q)^{-\frac{1}{2}}A^{-1}\{\max_{r\leq n}|\hat{U}_r|\} \ . \qquad (3.5)$$

The proposed tests are based on the statistics D^+_n and D_n. It may be noted that unlike the location model, here, the use of D^+_n may only be advocated for certain cases where under the alternative hypothesis, the $\underset{\sim}{\beta}'_k\underset{\sim}{c}_k$ are monotone. This may not generally be the case, and the two-sided statistic D_n is more generally applicable. The necessary distribution theory of D^+_n and D_n will be studied in Section 4.

It may be noted that for the location model (2.1), under H_0 in (2.2), the X_i are independent and identically distributed (i.i.d.) r.v.'s, so that one may also use the sequential ranking scheme as in Bhattacharya and Frierson (1981), where the ranks R_{kk} (of X_k among $X_1,\ldots,X_k$), for different k, are stochastically independent, and R_{kk} assumes the values $1,\ldots,k$ with the equal probability k^{-1}, for $k\geq 1$. On the other hand, for the general linear model in (1.1), even under H_0 in (1.2), the X_i, though independent, are not identically distributed (unless $\beta=0$ or the c_i are all equal), and hence, the stochastic independence and uniformity of the distributions of the R_{kk} may not hold. Thus, the procedure suggested by Bhattacharya and Frierson (1981) may not be generally applicable for the testing problem in (1.2)-(1.3). Further, the residuals in (3.1) are, in general, neither independent, nor (marginally) identically distributed, so that the exact distribution of D_n^+ or D_n may be difficult to obtain even for small n and simple scores; in fact, the same generally depends on the underlying F. For this reason, for general linear models, exact distribution-free tests for change-points may not exist and one may have to be satisfied with ADF tests. We shall see in Section 4 that under fairly general regularity conditions (on F, the c_i and the scores), some invariance principles hold for the CUSUM's in (2.10) or (3.3), and these provide the ADF structure of D_n^+ or D_n, when H_0 in (1.2) holds.

4. ASYMPTOTIC PROPERTIES OF D_n^+ AND D_n

We consider first some invariance principles for the CUSUM's in (2.10) or (3.3), when H_0 in (1.2) or (2.2) may or may not hold. First, consider the case where H_0 in (1.2) holds, and define $Y_i=X_i=\beta' c_i$, $i\geq 1$. Then, the Y_i are i.i.d.r.v. with the common d.f. F. Also, let R_{ki}^+ be the rank of $|Y_i|$ among $|Y_1|,\ldots,|Y_k|$, for $i=1,\ldots,k$; $k\geq 1$. Define then

$$u_k = \mathrm{sign}(Y_k)a_k(R_{kk}^+),\quad k\geq 1\ , \tag{4.1}$$

$$U_r = \textstyle\sum_{k\leq r} u_k,\ \text{for } r=1,\ldots,n\ . \tag{4.2}$$

Note that under H_0 in (1.2), (i) sign Y_k and R_{kk}^+ are independent, (ii) signY_k assumes the values ± 1 with equal probability ½, (iii) R_{kk}^+ assumes the values $1,\ldots,k$ with equal probability k^{-1}, and (iv) for different k, (sign Y_k, R_{kk}^+)

(and hence, u_k) are stochastically independent of each other. Thus,

$$E(u_k|H_0) = 0 \text{ and } E(u_k^2|H_0) = A_k^2 = \frac{1}{k}\sum_{i=1}^{k} a_k^2(i) , \tag{4.3}$$

$$E(U_r|H_0) = 0 \text{ and } E(U_r^2|H_0) = \sum_{k=1}^{r} A_k^2, \ \forall\ r\geq 1, \tag{4.4}$$

where $A_k^2 \to A^2$ as $k \to \infty$. Hence, if we consider a stochastic process $Z_n^o = \{Z_n^o(t),\ 0\leq t\leq 1\}$, by letting

$$Z_n^o(t) = n^{-\frac{1}{2}}A^{-1}U_k \text{ for } k\leq nt<k+1,\ k=0,\dots,n , \tag{4.5}$$

then under H_0 in (1.2), by the stochastic independence of the u_k and (4.3)-(4.4),

$$Z_n^o \underset{\mathcal{D}}{\to} Z, \text{ in the } J_1\text{-topology on } D[0,1], \tag{4.6}$$

where $Z=\{Z(t):\ 0\leq t\leq 1\}$ is a standard Wiener process on [0,1]. Side by side, we introduce the stochastic process $Z_n=\{Z_n(t):\ 0\leq t\leq 1\}$ by letting

$$Z_n(t) = (n-q)^{-\frac{1}{2}}A^{-1}\hat{U}_k, \text{ for } \frac{k-q}{n-q} \leq t < \frac{k-q+1}{n-q} , \quad k=q,\dots,n . \tag{4.7}$$

Note that by (3.4), (3.5) and (4.7),

$$D_n^+=\sup\{Z_n(t):\ 0\leq t\leq 1\} \text{ and } D_n=\sup\{|Z_n(t)|:\ 0\leq t\leq 1\} . \tag{4.8}$$

Let us then define

$$D^+=\sup\{Z(t):\ 0\leq t\leq 1\} \text{ and } D=\sup\{|Z(t)|:\ 0\leq t\leq 1\} . \tag{4.9}$$

It is well known that for every $\lambda\geq 0$,

$$P\{D^+\leq\lambda\} = 2\Phi(\lambda) - 1 , \tag{4.10}$$

$$P\{D\leq\lambda\} = \sum_{k=-\infty}^{\infty}(-1)^k\{\Phi((2k+1)\lambda) - \Phi((2k-1)\lambda)\} , \tag{4.11}$$

where Φ is the standard normal d.f. We shall show under appropriate regularity conditions, when H_0 holds, as $n\to\infty$,

$$\rho(Z_n,Z_n^o) = \sup\{|Z_n(t)-Z_n^o(t)|:\ 0\leq t\leq 1\} \underset{p}{\to} 0, \tag{4.12}$$

so that by (4.6) and (4.12),

$$Z_n \underset{\mathcal{D}}{\to} Z, \text{ in the } J_1\text{-topology on } D[0,1] , \tag{4.13}$$

and hence, by (4.8), (4.9), (4.13) and (4.10)-(4.11), as $n\to\infty$,

$$P\{D_n^+ \le \lambda | H_0\} \to P\{D^+ \le \lambda\} = 2\Phi(\lambda) - 1 \ , \tag{4.14}$$

$$P\{D_n \le \lambda | H_0\} \to P\{D \le \lambda\} = \sum_{k=-\infty}^{\infty} (-1)^k \{\Phi((2k+1)\lambda) - \Phi((2k-1)\lambda)\}. \tag{4.15}$$

Thus, ADF tests for H_0 in (1.2) against (1.3) may be based on D_n^+ or D_n, using the critical values of D^+ and D, respectively.

Looking at (3.2), (3.3), (4.1), (4.2), (4.5), (4.7) and (4.12), we gather that for proving (4.12), it suffices to show that under H_0,

$$\max_{k \le n}\{n^{-\frac{1}{2}} | \sum_{i \le k} (\hat{u}_i - u_i) | \} \xrightarrow{p} 0, \text{ as } n \to \infty \ . \tag{4.16}$$

For this purpose (as well as for studying the asymptotic nonnull distribution theory of D_n^+ and D_n), we introduce the following regularity conditions on F, the $\underset{\sim}{c}_i$ and the score function ϕ. The d.f. F is assumed to have bounded and continuous first and second order derivatives [f(x) and f'(x), respectively] almost everywhere, and

$$I(f) = \int_{-\infty}^{\infty} \{f'(x)/f(x)\}^2 dF(x) < \infty. \tag{4.17}$$

Also, we assume that there exists a positive definite (p.d.) and finite matrix $\underset{\sim}{C}_0$, such that

$$n^{-1}\underset{\sim}{C}_n = n^{-1}\sum_{i=1}^n \underset{\sim}{c}_i \underset{\sim}{c}_i' \to \underset{\sim}{C}_0, \text{ as } n \to \infty \ , \tag{4.18}$$

$$\max_{1 \le k \le n}\{\underset{\sim}{c}_k' \underset{\sim}{C}_0^{-1} \underset{\sim}{c}_k\} = O((\log n)^2) \ . \tag{4.19}$$

[For the location model, C_0=1 and the left hand side of (4.15) is also equal to 1.] Note that (4.19) is weaker than the Hájek (1968) condition, but is more stringent than the classical Noether condition. Further, let $\phi^{(1)}$ and $\phi^{(2)}$ be the first and second derivatives of ϕ, and assume that there exist a generic constant K ($<\infty$) and a δ ($< 1/6$), such that

$$|\phi^{(r)}(u)| \le K[u(1-u)]^{-r-\delta}, \ 0<u<1, \ r=0,1,2. \tag{4.20}$$

As in Sen (1980b), it is possible to replace $\delta < 1/6$ by $\delta < 1/4$, provided we assume that

$$\sup_x f(x)\{F(x)[1-F(x)]\}^{-\frac{1}{2}+\eta} < \infty \text{ for some } \eta<\infty . \qquad (4.21)$$

Also, if $\phi^{(2)}$ is bounded a.e., then (4.19) may be replaced by the Noether condition: $\max\{\underset{\sim}{c}_k' \underset{\sim}{C}_n^{-1} \underset{\sim}{c}_k : 1\leq k\leq n\} \to 0$ as $n\to\infty$. We may note that (4.20) holds for the Wilcoxon, Normal as well as all the other commonly adapted score functions. Finally, concerning the estimators $\{\hat{\underset{\sim}{\beta}}_k\}$ employed in (3.1), we assume that under H_0 in (1.2), for every $\varepsilon>0$, there exists an integer k_0 (≥ 1), such that

$$P\{\max_{k_0\leq k\leq n}(\log k)^{-1}k^{\frac{1}{2}}||\hat{\underset{\sim}{\beta}}_k-\underset{\sim}{\beta}||\geq 1\} < \varepsilon, \ \forall\ n \geq k_0 . \qquad (4.22)$$

Later on (in the appendix), we shall see that (4.22) holds under fairly general conditions.

Reherning to the proof of (4.16), we may note first that by (4.20) and (2.4)-(2.5), for every k ($\geq$ 1),

$$\max_{1\leq i\leq k}|a_k(i)| = o(k^{\delta}) , \qquad (4.23)$$

so that by (2.9), (3.1), (3.2), (4.1) and (4.2), $|\hat{u}_k-u_k|=o(k^{\delta})$, with probability 1. Hence, we may always choose a sequence $\{k_n\}$ of positive integers, such that $k_n \nearrow \infty$ but $k_n^{1+\delta}n^{-\frac{1}{2}} \searrow 0$, as $n\to\infty$, and to prove (4.16), it suffices to show that for every $\varepsilon>0$,

$$P_0\{n^{-\frac{1}{2}}\textstyle\sum_{k_n\leq i\leq n}|\hat{u}_i-u_i|>\varepsilon\} \to 0, \text{ as } n\to\infty . \qquad (4.24)$$

Let the $\hat{X}_{ki}$, Y_k, R_{ki}^{+} and $\hat{R}_{ki}^{+}$ be defined as in Sections 3 and 4, and let

$$B_{nk} = \{\max_{k\leq i\leq n}(\log i)^{-1}i^{\frac{1}{2}}||\hat{\underset{\sim}{\beta}}_i-\underset{\sim}{\beta}||\leq 1\} \ (k\leq n) \qquad (4.25)$$

and B_{nk}^{c} be the complementary event to B_{nk}. Then, by (3.2), (4.1) and (4.25), for every $\varepsilon>0$,

$$\begin{aligned} &P_0\{n^{-\frac{1}{2}}\textstyle\sum_{i=k_n}^{n}|\hat{u}_i-u_i|>\varepsilon\} \qquad (4.26)\\ &\leq P_0\{n^{-\frac{1}{2}}\textstyle\sum_{k=k_n}^{n}|(\text{sign}\hat{X}_{kk}-\text{sign}Y_k)a_k(R_{kk}^{+})|>\varepsilon/2,\ B_{nk_n}\},\\ &+ P_0\{n^{-\frac{1}{2}}\textstyle\sum_{k=k_n}^{n}|a_k(\hat{R}_{kk}^{+})-a_k(R_{kk}^{+})|>\varepsilon/2,\ B_{nk_n}\} + P_0\{B_{nk_n}^{c}\} , \end{aligned}$$

where by (4.22), the last term on the right hand side of (4.26) converges to 0 as $n\to\infty$. Note that by (4.18), (4.19) and (4.25), when B_{nk_n} holds, $|\text{sign}\hat{X}_{kk}-\text{sign}Y_k|$ may only be different from zero, when $|Y_k| \le dk^{-1/2}(\log k)^2$, for some finite d. Further, by the well-known results on the empirical d.f. (for the $|Y_i|$), we have on denoting by $F_k^*(x)=k^{-1}\sum_{i=1}^k I(|Y_i|\le x)$, $x\ge 0$, the sample d.f. and $F^*(x)-P\{|Y_k|\le x = F(x)-F(-x)=2F(x)-1$, $x\ge 0$, that

$$\begin{aligned} &P_0\{|k^{-1}R^+_{kk}-F^*(|Y_k|)| \ge (2k^{-1}\log k)^{1/2}\} \\ &\le P_0\{\sup_{x\ge 0}|F_k^*(x)-F^*(x)| \ge (2k^{-1}\log k)^{1/2}\} \\ &\le 2\exp(-2k(2k^{-1}\log k)) = 2k^{-4},\ \forall\ k\ge 2\ . \end{aligned} \tag{4.27}$$

Also, $\phi^+(0)=0$ and by the assumed boundedness of f, $F^*(dk^{-1/2}(\log k)^2)= O(k^{-1/2}(\log k)^2)$, $\forall$ $k\ge 2$. Hence, by (4.20) and some routine steps, we obtain that for every r: $0\le r/k\le c<1$, there exists a finite positive constant C, such that

$$\max_{1\le i\le r}|a_k(i)| \le C(r/k),\ \forall\ k\ge k_0. \tag{4.28}$$

From (4.23), (4.28) and (4.27), we obtain by some standard steps that as $n\to\infty$,

$$\begin{aligned} &E_0\{n^{-1/2}\sum_{k=k_n}^n I_{B_{nk_n}}|(\text{sign}\hat{X}_{kk}-\text{sign}Y_k)a_k(R^+_{kk})|\} \\ &\le n^{-1/2}\sum_{k=k_n}^n\{[O(k^{-1/2}(\log k)^2)]^2 + [o(k^\delta)O(k^{-4})]\} \\ &=O(n^{-1/2}(\log n)^5) \to 0\ , \end{aligned} \tag{4.29}$$

so that by (4.29) and the Chebyshev inequality, the first term on the right hand side of (4.26) converges to 0, as $n\to\infty$.

Consider now the sample d.f.'s $\hat{F}_k^*(x)=k^{-1}\sum_{i=1}^k I(|\hat{X}_{ki}|\le x)$, $x\ge 0$, $k\ge 1$. Then. we may virtually repeat the proof of Theorem 3.1 of Ghosh and Sen (1972) and obtain that for every γ ($0<\gamma<1/4$) and h (which we take > 1), there exist positive constants K_1 and K_2 and an integer k_0 (≥ 1), such that under H_0 in (1.2), for every $k\ge k_0$,

$$P_0\{\sup_{x\geq 0} k^{\frac{1}{2}}|\hat{F}_k^*(x)-F_k^*(x)| > K_1 k^{-\gamma}(\log k)^2|B_{kk}\} \leq K_2 k^{-h}, \quad (4.30)$$

which for h>1, insures that

$$P_0\{\max_{k_n\leq k\leq n}\sup_{x\geq 0} k^{\frac{1}{2}+\gamma}(\log k)^{-2}|\hat{F}_k^*(x)-F_k^*(x)| > K_1|B_{nk_n}\} \to 0, \quad (4.31)$$

as $n\to\infty$. Also, note that $\{[F_k^*(x)-F^*(x): 0\leq x<\infty]; k\geq 1\}$ is a reverse martingale (process) sequence, so that for every $\varepsilon>0$, $\{\sup_{x\geq 0}(F^*(x)[1-F^*(x)])^{-\frac{1}{2}+\varepsilon}|F_k^*(x)-F^*(x)|;$ $k\geq 1\}$ is a reverse sub-martingale; by the use of the Hájek-Rényi-Chow inequality, we obtain that

$$P\{\max_{k_n\leq k\leq n}\sup_{x\geq 0} k^{\frac{1}{2}}(\log k)^{-1}|F_k^*(x)-F^*(x)|\{F^*(x)[1-F^*(x)]\}^{-\frac{1}{2}+\varepsilon} \geq 1\}$$

$$\leq k_n^{-1}(\log k_n)^{-2}E_0\{\sup_{x\geq 0} k_n[F_{k_n}^*(x)-F^*(x)]^2\{F^*(x)[1-F^*(x)]\}^{-1+2\varepsilon}\}$$

$$+ \sum_{k=k_n+1}^{n}\{(\log k)^{-2}-((k-1)/k)(\log \overline{k-1})^{-2}\}E_0\{\sup_{x\geq 0} k[F_k^*(x)-F^*(x)]^2\{F^*(x)[1-F^*(x)]\}^{-1+2\varepsilon}\}$$

$$= k_n^{-1}(\log k_n)^{-2}\; 0(1) + \sum_{i=k_n+1}^{n} 0(\frac{1}{k((\log k)^2})\quad (4.32)$$

$$\to 0, \text{ as } n\to\infty,$$

where the penultimate step follows from the fact that the expectations in the preceding step are all bounded (uniformly in $k\geq k_n$) [see (7.4.54)-(7.4.55) of Sen (1981) in this respect]. Let us now define δ (> 0) as in (4.20), take ε in (4.31)-(4.32) or $\frac{1}{2}\delta$ and let

$$J_k = \{x: F^*(x)[1-F^*(x)] \geq k^{-1+\delta}(\log k)^2\},\ k\geq 2 . \quad (4.33)$$

Then, from (4.32) and (4.33), we obtain that for every $\eta>0$, as $n\to\infty$,

$$P_0\{\max_{k_n\leq k\leq n}\sup_{x\in J_k}|F_k^*(x)/F^*(x)-1|\geq\eta\} \to 0 , \quad (4.34)$$

$$P_0\{\max_{k_n\leq k\leq n}\sup_{x\in J_k}|\{1-F_k^*(x)\}/\{1-F^*(x)\}-1|\geq\eta\} \to 0 . \quad (4.35)$$

Now, the second term on the right hand side of (4.26) is bounded by

$$P_0\{n^{-\frac{1}{2}}\sum_{k=k_n}^{n}(2\max_{1\le i\le k}|a_k(i)|)I(|Y_k|\notin J_k) > \varepsilon/4\} + \qquad (4.36)$$

$$P_0\{n^{-\frac{1}{2}}\sum_{k=k_n}^{n}I(|Y_k|\in J_k)|a_k(\hat{R}_{kk}^{+})-a_k(R_{kk}^{+})| > \varepsilon/4,\ B_{nk_n}\}.$$

By (4.23) and (4.33), as $n\to\infty$,

$$E_0\{n^{-\frac{1}{2}}\sum_{k=k_n}^{n}(2\max_{1\le i\le k}|a_k(i)|)I(|Y_k|\notin J_k)\}$$

$$= 2n^{-\frac{1}{2}}\sum_{k=k_n}^{n}(\max_{1\le i\le k}|a_k(i)|)P\{|Y_k|\notin J_k\}$$

$$= 2n^{-\frac{1}{2}}\sum_{k=k_n}^{n}[o(k^{\delta})][O(k^{-1+\delta}(\log k)^2)]$$

$$= o(n^{-\frac{1}{2}+2\delta}(\log n)^2) \to 0, \quad (\text{as } \delta < \tfrac{1}{4}) \qquad (4.37)$$

so that by (4.37) and the Chebyshev inequality, the first term of (4.36) converges to 0, as $n\to\infty$. For the second term, we make use of (4.20), (4.31), (4.32), (4.33) and (4.35), and obtain that when B_{nk_n} holds, with probability converging to 1 (as $n\to\infty$),

$$n^{-\frac{1}{2}}\sum_{k=k_n}^{n}I(|Y_k|\in J_k)|a_k(\hat{R}_{kk}^{+})-a_k(R_{kk}^{+})|$$

$$\le cn^{-\frac{1}{2}}\sum_{k=k_n}^{n}I(|Y_k|\in J_k)k^{-\frac{1}{2}-\gamma}(\log k)^2[1-F^*(|Y_k|)]^{-1-\delta}, \qquad (4.3*0$$

where C (< ∞) is a generic constant. Note that for $|Y_k|\in J_k$, $1-F^*(|Y_k|)\ge k^{-1+\delta}(\log k)^2$, so that on letting $\gamma=\delta+\eta$, $\eta>0$, we have

$$k^{-\gamma}(\log k)^2[1-F^*(|Y_k|)]^{-1-\delta}I(|Y_k|\in J_k)$$

$$\le k^{-\gamma}(\log k)^2[1-F^*(|Y_k|)]^{-1+\eta}(k^{-1+\delta}(\log k)^2)^{-\delta}$$

$$= k^{-\eta-\delta^2}(\log k)^{2(1-\delta)}[1-F^*(|Y_k|)]^{-1+\eta},\ \ \forall\ k\ge k_n\ , \qquad (4.39)$$

By (4.39), the right hand side of (4.38) is bounded by

$$Cn^{-\frac{1}{2}}\sum_{k=k_n}^{n}k^{-\frac{1}{2}-\eta-\delta^2}(\log k)^{2(1-\delta)}[1-F^*(|Y_k|)]^{-1+\eta}\ . \qquad (4.40)$$

Since (4.40) represents a positive r.v. whose expectation is

$$c\eta^{-1}n^{-\frac{1}{2}}\sum_{k=k_n}^{n} k^{-\frac{1}{2}-\eta-\delta^2}(\log k)^{2(1-\delta)} = O(n^{-\eta-\delta^2}(\log n)^2) \quad ,(4.41)$$

by the Chebyshev inequality, (4.40) converges to 0, in probability, as $n\to\infty$, which via (4.38) and (4.35)-(4.37) insures that (4.26) converges to 0 as $n\to\infty$. This completes the proof of (4.16), and hence, of (4.12).

Next, we proceed to Study the non-null distribution theory of the proposed test statistics. In this context, we assume that in (1.3), the change-point τ (= τ_n) satisfy the condition that

$$t_{m_n} \le \tau_n < t_{m_n+1} \text{ where } n^{-1}m_n \to \theta\text{: } 0<\theta<1, \quad (4.42)$$

as $n\to\infty$. That is in the asymptotic case, a change point does not occur near the beginning or the end of the time period (t_1,t_n). Under (1.3), (4.41) and the regularity conditions assumed before, for $\underset{\sim}{\beta}_{m_n+1}=\underset{\sim}{\beta}_{m_n}+\underset{\sim}{\lambda}$, $\underset{\sim}{\lambda}$ ($\ne \underset{\sim}{0}$) fixed, it can be shown that the $\hat{u}_k$, $k>m_n$ are consistently shifted from the origin, and hence, by (3.3)-(3.5), D_n^+ or D_n will be $O_p(n^{\frac{1}{2}})$, and thus, by (4.14)-(4.15), the proposed tests will be consistent against $\underset{\sim}{\lambda}\ne\underset{\sim}{0}$. Thus, to study the asymptotic power properties, we confine ourselves to some local alternatives for which the asymptotic power does not converge to 1. With this in mind, we consider a sequence $\{K_n\}$ of alternative hypothesis, where under K_n,

$$\underset{\sim}{\beta}_1=\cdots=\underset{\sim}{\beta}_{m_n}=\underset{\sim}{\beta}_{m_n+1}-n^{-\frac{1}{2}}\underset{\sim}{\lambda},\ \underset{\sim}{\beta}_{m_n+1}=\cdots=\underset{\sim}{\beta}_n \ , \quad (4.43)$$

where $\underset{\sim}{\lambda}$ ($\ne \underset{\sim}{0}$) is fixed and the m_n satisfy (4.41). Further, we strengthen (4.18) to

$$\lim_{m\to\infty} m^{-1}\underset{\sim}{C}_m = \underset{\sim}{C}_0 \text{ and } \lim_{m\to\infty}\frac{1}{m}\sum_{i=1}^{m}\underset{\sim}{c}_i = \bar{\underset{\sim}{c}} \quad (4.44)$$

both exist, where $\underset{\sim}{C}_0$ is p.d. Then

$$\underset{\sim}{C}_{[nt]}^{-1}\underset{\sim}{C}_{[ns]} \to (s/t)\underset{\sim}{I}, \text{ for every } 0\le s<t\le 1 \ . \quad (4.45)$$

Under (4.17)-(4.19), the contiguity of the sequence of probability measures under $\{K_n\}$ with respect to those under H_0, follows then along the lines of the general results in Chapter VI of Hájek and Šidák (1967), so that proceeding as in Sen (1977, 1980a), we first extend (4.12) to that under $\{K_n\}$, (4.6) to that of a drifted Brownian motion under $\{K_n\}$, and finally, obtain the same result for $\{Z_n\}$. Thus, we obtain that under the regularity conditions assumed and $\{K_n\}$ in (4.41)-(4.42),

$$Z_n \underset{\mathcal{D}}{\rightarrow} Z + \xi, \text{ in the } J_1\text{-topology on } D[0,1] \ , \tag{4.46}$$

where $\xi=\{\xi(t);\ 0\le t\le 1\}$ is specified by

$$\xi(t) = \begin{cases} 0, & 0\le t\le\theta \\ A^{-1}\gamma(\phi,F)(t-\theta)\underset{\sim}{\lambda}'\underset{\sim}{\bar{c}}, & \theta\le t\le 1, \end{cases} \tag{4.47}$$

where A is defined by (2.11) and

$$\gamma(\phi,F) = \int_{-\infty}^{\infty} (d/dx)\phi(F(x))dF(x) \quad (> 0) \ . \tag{4.48}$$

[Note that by partial integration, $\gamma^2(\phi,F)\le A^2 I(f)<\infty$.] Thus, the asymptotic power of D_n^+, under $\{K_n\}$, is given by

$$P\{Z(t)+\xi(t)\ge D_\alpha^+ \text{ for some } t:\ 0\le t\le 1\} \tag{4.49}$$

where $\Phi(D_\alpha^+)=1-\frac{1}{2}\alpha$; a similar expression holds for D_n.

5. ASYMPTOTIC RELATIVE EFFICIENCY RESULTS

For normal F, tests for change-points, relating to (1.1)-(1.3), based on recursive residuals are discussed in Brown, Durbin and Evans (1975). For F not necessarily normal, invariance principles for CUSUMs of such recursive residuals have recently been studied by Sen (1982a). It follows from the results in Section 4 of Sen (1982a) that (4.14)-(4.15) hold for the parametric procedure based on the least squares recursive residuals, and also, (4.48) holds with the drift function $\xi=\{\xi(t),\ 0\le t\le 1\}$ replaced by $\xi^*=\{\xi^*(t),\ 0\le t\le 2\}$, where

$$\xi^*(t) = \begin{cases} 0, & 0\le t\le\theta \ , \\ \sigma^{-1}(t-\theta)\underset{\sim}{\lambda}'\underset{\sim}{\bar{c}}, & \theta\le t\le 1 \ , \end{cases} \tag{5.1}$$

where σ^2, the variance of F, is assumed to be finite; a similar result holds for the two-sided case too.

The two drift functions in (4.46) and (5.1) are proportional, i.e.,

$$\xi(t)=k\xi^*(t),\ 0\le t\le 1,\ \text{where } k^2=\sigma^2\gamma^2(\phi,F)/A^2\ . \qquad (5.2)$$

Thus, as in Sen (1980, 1982b), we may justify the use of the classical Pitman-efficiency results in this context too, and k^2 represents the asymptotic relative efficiency of the rank procedure relative to the least squares procedure. This agrees with the Pitman-efficiency of the rank test with respect to the Student t-test, [discussed in detail in Chernoff and Savage (1958) and Puri and Sen (1971), among other places], and hence, the details are omitted here.

It may be noted that if instead of the recursive residuals, one would have used [as in Sen (1980a, 1982b)] aligned rank statistics based on the terminal estimator $\hat{\underset{\sim}{\beta}}_n$ of $\underset{\sim}{\beta}$, one has then the weak convergence to a Brownian bridge with a more complicated drift function (under $\{K_n\}$). In comparing such a test with the one considered here, it is, generally, not possible to adapt the measure of the Pitman-efficiency.

APPENDIX

We proceed to verify here (4.22) for some typical estimators. First, consider the location model (2.1)-(2.3) and the rank estimates $\{\hat{\theta}_k\}$ in (2.7). In this case, under the assumed regularity conditions on ϕ and F, (4.22) follows directly from (10.3.39) through (10.3.45) of Sen (1981). For the general regression model, asymptotic theory of rank estimators of $\underset{\sim}{\beta}$ rests on an asymptotic linearity property of rank statistics in regression parameters, due to Jurečková (1969, 1971). Her results relate to the weak convergence properties under weaker regularity conditions, and, in view of Theorem A.4.1 of Sen (1981, p. 389), stronger conclusions, such as (4.22), hold under our assumed regularity conditions; similar results under more stringent regularity conditions are due to Ghosh and Sen (1972). Thus, (4.22) holds for the rank estimators under the assumed regularity conditions. We proceed to show that for the conventional least squares estimators too, (4.22) holds whenever $\sigma^2<\infty$. Towards this note that

$$(\log k)^{-1}k^{\frac{1}{2}}||\hat{\beta}_k-\beta||$$
$$= (\log k)^{-1}k^{-\frac{1}{2}}||kC_k^{-1}C_k(\hat{\beta}_k-\beta)||$$
$$\leq (\log k)^{-1}k^{-\frac{1}{2}}||C_k(\hat{\beta}_k-\beta)||Ch_1(kC_k^{-1}) , \tag{A.1}$$

where Ch_1 stands for the largest characteristic root, and by (4.18), as $k\to\infty$, $Ch_1(kC_k^{-1}) \to [Ch_q C_0^{-1}] < \infty$. So, it suffices to show that under H_0 in (1.2), when $\sigma<\infty$, for every $c>0$

$$P\{\max_{k_n\leq k\leq n}(\log k)^{-1}k^{-\frac{1}{2}}||C_k(\hat{\beta}_k-\beta)|| > c\} \to 0 , \tag{A.2}$$

as $n\to\infty$. Towards this, note that under H_0 in (1.2), for $\sigma<\infty$,

$$\{C_k(\hat{\beta}_k-\beta) = \textstyle\sum_{i=1}^k c_i e_i;\ k\geq 1\} \text{ is a 0-mean martingale,} \tag{A.3}$$

where the e_i are i.i.d.r.v. with 0 mean and variance $\sigma^2<\infty$. As such, $\{Z_k=||C_k(\hat{\beta}_k-\beta)||;\ k\geq 1\}$ is a nonnegative submartingale, where

$$EZ_k = (\textstyle\sum_{i=1}^k c_i' c_i)\sigma^2 = \sigma^2 Tr(C_k),\ \forall\ k\geq 1 . \tag{A.4}$$

Therefore, by the Hájek-Rényi-Chow inequality,

$$P_0\{\max_{k_n\leq k\leq n}(\log k)^{-1}k^{-\frac{1}{2}}||C_k(\hat{\beta}_k-\beta)|| > c\}$$
$$\leq c^{-2}\sigma^{-2}\{n^{-1}(\log n)^{-2}Tr(C_n) + \textstyle\sum_{k=k_n}^{n-1}(Tr(C_k))\left(\frac{1}{k(\log k)^2} - \frac{1}{(k+1)(\log k+1)^2}\right)\}, \tag{A.5}$$

where by (4.18), the right hand side of (A.5) is $O((\log k_n)^{-1})$ and converges to 0 as $k_n\to\infty$ (i.e., $n\to\infty$). Hence, (4.22) follows from (A.2) and (A.4).

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