

EFFECTS OF FALSE AND INCOMPLETE IDENTIFICATION OF DEFECTIVE ITEMS  
ON THE RELIABILITY OF ACCEPTANCE SAMPLING

by

Samuel Kotz

University of Maryland  
College Park, Maryland

Norman L. Johnson

University of North Carolina  
Chapel Hill, North Carolina

April 1982

ABSTRACT

The effects of false and incomplete identification of nonconforming items on the properties of two-stage acceptance sampling procedures are studied. Numerical tables are presented, and there is some discussion of sensitivity to inspection errors. Methods of taking into account extra costs needed to implement better inspection techniques, when initial grading is inconclusive, are described.

## 1. INTRODUCTION

Recently, we have considered a number of distributions arising from inspection sampling, when inspection may fail to identify a defective item, or may erroneously classify a nondefective item as 'defective'. (Johnson et al. (1980), Johnson & Kotz (1981), Kotz & Johnson (1982a)). Our interest in these papers was mainly in the distributions (of numbers of items classified as defective) themselves. We now consider some consequences, with special regard to properties of acceptance sampling schemes. Although this is the main purpose of the present paper, we will incidentally encounter some further compound distributions which are of interest on their own account.

We also consider a simple grading situation, allowing for a possible second inspection when first inspection fails to decide whether an item is or is not defective, and introducing some cost functions.

We will suppose sampling is carried out, without replacement, from a lot of size  $N$  which contains  $D$  defective items. The symbol  $Y$  (possibly with subscripts) will denote the number of defective items included in a random sample (without replacement) and  $Z$  (with subscripts) the number of items classified as 'defective' after inspection.

## 2. SINGLE-STAGE ACCEPTANCE SAMPLING

Single-stage acceptance sampling schemes have the following simple rule:

"If the number of (alleged) defective items in a sample of size  $n$  exceeds  $a$ , reject the lot; otherwise accept it."

Formally:

"Reject if  $Z > a$ ; accept if  $Z \leq a$ ".

In order to assess the properties of this procedure, we need only the distribution of  $Z$ , which was obtained in Johnson & Kotz (1981) - namely

$$\begin{aligned} \Pr[Z=z | p, p'; D] &= \binom{N}{n}^{-1} \sum_y \binom{D}{y} \binom{N-D}{n-y} \sum_{j=0}^z \binom{y}{j} \binom{n-y}{z-j} p^j (1-p)^{y-j} p'^{z-j} (1-p')^{n-y-z+j} \\ &= \binom{N}{n}^{-1} \sum_y \binom{D}{y} \binom{N-D}{n-y} b(z; y, n-y; p, p') \end{aligned} \quad (1)$$

where  $p$  = probability that a defective item is detected on inspection  
 and  $p'$  = probability that a nondefective item is classified as 'defective',  
 and  $\max(0, n-N+D) \leq y \leq \min(n, D)$ .

In the construction of acceptance sampling schemes (that is, choosing the values of  $n$  and  $a$ ) it is (usually) assumed that inspection is faultless, that is  $p = 1$  and  $p' = 0$ . The values of  $n$  and  $a$  are then chosen to make

$$\Pr[Z > a | 1, 0; D_0] \doteq \alpha \text{ (the 'Producer's Risk')}$$

while  $\Pr[Z \leq a | 1, 0; D^*] \doteq \beta$  (the 'Consumer's Risk')

where  $\alpha$ ,  $\beta$ ,  $D_0$  and  $D^*$  are parameters chosen in accordance with the specific circumstances.

3. TWO-STAGE ACCEPTANCE SAMPLING

These procedures (see e.g. Dodge & Romig (1959)) are of form:

"Take a random sample (without replacement) of size  $n_1$ , and observe the number of apparently defective items,  $Z_1$ .

If  $Z_1 \leq a_1$  accept the lot; if  $Z_1 > a_1'$ , reject the lot; if  $a_1 < Z_1 \leq a_1'$ , take a further random sample, from the remaining items in the lot, of size  $n_2$  and observe the number of apparently defective items in it,  $Z_2$ .

If  $Z_1 + Z_2 \leq a_2$  accept the lot; if  $Z_1 + Z_2 > a_2$  reject it."

Formally:

"Accept if  $Z_1 \leq a_1$ , or if  $a_1 < Z_1 \leq a_1'$  and  $Z_1 + Z_2 \leq a_2$ ; otherwise reject."

(Popular special cases are  $n_2 = n_1$ , or  $n_2 = 2n_1$  and/or  $a_2 = a_1'$ )

To assess the properties of this procedure we need the joint distribution of  $Z_1$  and  $Z_2$ . Conditionally on the actual numbers  $Y_1, Y_2$  of defective items in the two samples,  $Z_1$  and  $Z_2$  are independent, and (for  $i = 1,2$ )  $Z_i$  is distributed as the sum of two independent binomial variables with parameters  $(Y_i, p)$  and  $(n_i - Y_i, p')$  corresponding to items correctly and incorrectly classified as defective, respectively. Formally

$$Z_i | Y_1, Y_2 \sim \text{Binomial}(Y_i, p) * \text{Binomial}(n_i - Y_i, p') \tag{2}$$

(\* denotes convolution.)

The joint distribution of  $Y_1$  and  $Y_2$  is a bivariate hypergeometric with parameters  $(n_1, n_2; D, N)$  and

$$\Pr[Y_1 = y_1; Y_2 = y_2] = \frac{\binom{n_1}{y_1} \binom{n_2}{y_2} \binom{N - n_1 - n_2}{D - y_1 - y_2}}{\binom{N}{D}} \tag{3}$$

$(0 \leq y_i \leq n_i; D - N + n_1 + n_2 \leq y_1 + y_2 \leq D)$ .

The unconditional distribution of  $(Z_1, Z_2)$  is a mixture of (2) with mixing distribution (3).

Formally, then

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \sim \left( \begin{array}{l} \text{Binomial}(Y_1, p) * \text{Binomial}(n_1 - Y_1, p') \\ \text{Binomial}(Y_2, p) * \text{Binomial}(n_2 - Y_2, p') \end{array} \right) \wedge \text{Biv. Hypg}(n_1, n_2; D, N) \quad (4)$$

( $\wedge$  denotes the compounding operator (e.g. Johnson & Kotz (1969, p. 184)).)

It would be straightforward to generalize this formula to allow  $p$  and  $p'$  to vary from sample to sample. (see Johnson & Kotz (1982b)). This will not be done here, as it appears reasonable to suppose  $p$  and  $p'$  are the same for both the first and second sample.

Explicitly

$$\Pr[Z_1 = z_1, Z_2 = z_2 | p, p'; D] = \sum_{y_1} \sum_{y_2} \frac{\binom{n_1}{y_1} \binom{n_2}{y_2} \binom{N-n_1-n_2}{D-y_1-y_2}}{\binom{N}{D}} b(z_1; y_1, n_1 - y_1; p, p') b(z_2; y_2, n_2 - y_2; p, p') \quad (5)$$

(Limits for  $y_1, y_2$  as in (3)).

The expected number of items inspected is

$$n_1 + n_2 \Pr[a_1 < Z_1 \leq a_1'] .$$

This can be evaluated using the distribution of  $Z_1$ , which is of the same form as (1), with subscript '1' attached to  $n$  and  $z$ . The probability of acceptance at first sample is

$$\sum_{z=0}^{a_1} \sum_{y_1} \frac{\binom{n_1}{y_1} \binom{N-n_1}{D-y_1}}{\binom{N}{D}} b(z; y_1, n_1 - y_1; p, p'). \quad (5)'$$

The probability of acceptance at second sample is the sum of probabilities (5) over  $a_1 < Z_1 \leq a_1'$  and  $Z_1 + Z_2 \leq a_2$ . The distribution of  $Z_1 + Z_2$  is

$$Z_1 + Z_2 \sim \prod_{i=1}^{a_1'} \text{Binomial}(Y_i, p) \prod_{i=1}^{a_2'} \text{Binomial}(n_i - Y_i, p') \wedge \text{Biv.Hypg}(n_1, n_2; D, N) \quad (6)$$

but it is not directly applicable to calculation of this probability. The acceptance probability is calculated directly as the sum of

$$\sum_{z_1=a_1'+1}^{a_1'} (1+a_2-z_1) = \frac{1}{2}(a_1'-a_1)(2a_2-a_1-a_1'+1) \text{ terms of type (5).}$$

Acceptance probabilities for four sampling schemes, with lot sizes  $N = 100, 200$  and defective fractions  $D/N = 0.05, 0.1, 0.2$  are shown in Table 1 for  $p = 1.00, 0.98, 0.95, 0.90, 0.75$  and  $p' = 0.00, 0.01, 0.02, 0.05, 0.10$ . The sampling schemes have  $n_1 = n_2$ , the common value corresponding to sample size codes D-G of Military Standard 105D for double sampling (see Duncan (1974)).

As is to be expected, the acceptance probability increases as  $p$  decreases, and decreases as  $p'$  increases. The latter effect is relatively greater, for the values of  $p$  and  $p'$  used (which correspond to the situations most likely to be encountered). For a given defective fraction ( $D/N$ ) probabilities of acceptance for lot sizes  $N = 100$  and  $N = 200$  do not differ much. It is noteworthy that the change with increasing  $N$  is sometimes positive and sometimes negative.

When  $D$  is small, variation in  $p$  has less effect, because it is only the  $D$  defectives that are affected. For converse reasons, variation in  $p'$  has greater effect when  $D$  is small. Effects of changes in  $p$  and  $p'$  become more marked as the sample size increases.

Roughly speaking, it appears that values of  $p$  as low as 95% do not have drastic effect on acceptance probability, but values of  $p'$  even as small as 1% do have a noticeable effect.

4. COST CONSIDERATIONS IN GRADING INDIVIDUAL ITEMS

The topic of *grading* was discussed by Kotz and Johnson (1982). This differs from acceptance sampling in that we are primarily concerned with the classification assigned to *each* item individually, rather than using the apparent total number of defective items in a sample as a criterion for accepting or rejecting the lot from which it was drawn.

The simplest possible situation to consider is when a single individual is chosen at random and assigned to one of two classes "defective" or "nondefective". (This decision is restricted to the particular item at hand - it is not extended to the whole lot.) A natural extension is obtained by allowing for the possibility that on first inspection, no clear decision will be reached - but that this can be resolved, one way or the other, by a second more careful (and probably more efficient and more costly) inspection.

We now introduce  $\pi$ ,  $\pi'$  to denote the probability of no decision on first inspection for a defective, nondefective item respectively. Also let  $p_E$ ,  $p'_E$  (E for "expensive") denote the probability that a defective, or non-defective item respectively is classified as 'defective' at the second inspection. Then the probability of a defective item being correctly classified is  $(p + \pi p_E)$ , and the probability of a nondefective being incorrectly classified as defective is  $(p' + \pi' p'_E)$ . (Note that all the formulae in Section 2 and 3 are still applicable, with  $p$  replaced by  $(p + \pi p_E)$  and  $p'$  by  $(p' + \pi' p'_E)$ .)

Some new points arise if cost is taken into consideration. If  $c_1$  is the cost of the first inspection and  $c_2$  that of the second, the expected cost of inspection for an individual chosen at random from a lot of  $N$  items, of which  $D$  are defective, is

$$C = c_1 + \left\{ \frac{D}{N} \pi + \left( 1 - \frac{D}{N} \right) \pi' \right\} c_2 \quad . \quad (7)$$

If  $\rho$  denotes the cost of failing to detect a defective item, and  $\rho'$  the cost of classifying a nondefective item as 'defective' then the expected cost of the procedure, per item is

$$R = c_1 + \left\{ \frac{D}{N} \pi + \left(1 - \frac{D}{N}\right) \pi' \right\} c_2 + (1 - p - \pi p_E) \frac{D}{N} \rho + (p' + \pi' p_E) \left(1 - \frac{D}{N}\right) \rho' . \quad (8)$$

If there is some choice in regard to the amount of effort devoted to second inspections, we say be able to regard  $p_E$  and  $p'_E$  as functions of  $c_2$ . We would expect  $p_E$  to increase and  $p'_E$  to decrease with  $c_2$ . We would also expect to have

$$c_2 > c_1, \quad p_E > p \text{ and } p'_E < p' .$$

If we also able to give reasonably relevant values to  $\rho$  and  $\rho'$  we can try to minimize  $R$  by appropriate choice of  $c_2$ , by using the value of  $c_2$  satisfying

$$\frac{\partial R}{\partial c_2} = 0, \quad \text{that is}$$

$$\frac{D}{N} \pi + \left(1 - \frac{D}{N}\right) \pi' = \rho \pi \frac{D}{N} \cdot \frac{\partial p_E}{\partial c_2} + \rho' \pi' \left(1 - \frac{D}{N}\right) \frac{\partial p'_E}{\partial c_2} . \quad (9)$$

If  $\partial p_E / \partial c_2 > 0$  and  $\partial p'_E / \partial c_2 < 0$ , as is to be expected, this equation can have no more than one root in  $c_2$ .

The possibility of using this approach may be rather difficult in practice. In particular, assessment of values of  $\rho$  and  $\rho'$  requires a very considerable knowledge of the likely financial effects of misclassification. Generally,  $\rho$  will reflect the adverse results of accepting a defective item which will commonly have high variability consequent on the actual effects of failure when (and if) it occurs. On the other hand,  $\rho'$  corresponds to the loss incurred to the producer by rejecting an item which is really satisfactory, and is likely to be less variable.



In this section our aim has been to alert practitioners to the existence of rather straightforward procedures, which, coupled with adequate practical experience can yield helpful results in a variety of applications.

#### ACKNOWLEDGEMENT

Samuel Kotz's work was supported by the U.S. Office of Naval Research under Contract N00014-81-K-0301. Norman L. Johnson's work was supported by the National Science Foundation under Grant MCS-8021704.

#### REFERENCES

1. Dodge, H.F. and Romig, H.G. (1959) *Sampling Inspection Tables*, Wiley, New York. (Second Edition)
2. Duncan, A.J. (1974) *Quality Control and Industrial Statistics*, R.D. Irwin, Homewood, Ill. (Fourth Edition).
3. Johnson, N.L. and Kotz, S. (1969) *Distributions in Statistics - Discrete Distributions*, Wiley, New York.
4. Johnson, N.L. and Kotz, S. (1981) Faulty Inspection Distributions - Some Generalizations. *Institute of Statistics Mimeo Series #1335*, University of North Carolina at Chapel Hill. (To be published in *Proc. of ONR/ARO Reliability Workshop*, April 1981).
5. Johnson, N.L. and Kotz, S. and Sorkin, H.L. (1980) Faulty Inspection Distributions, *Commun. Statist.* A9, 917-922.
6. Kotz, S. and Johnson, N.L. (1982a) Errors in Inspection and Grading: Distributional Aspects of Screening and Hierarchical Screening, *Institute of Statistics Mimeo Series #1385*, University of North Carolina at Chapel Hill: (to appear in *Commun. Statist.* A11 (1982)).
7. Kotz, S. and Johnson, N.L. (1982b) *Some Distributions Arising in Multistage Sampling with Inspection Errors*, submitted for publication.

TABLE 1: ACCEPTANCE PROBABILITIES

(See formula (5) and (5)')

$$\underline{n_1 = n_2 = 5; a_1 = 0, a_1' = a_2 = 2}$$

p	p' = 0	0.01	0.02	0.05	0.10	0	0.01	0.02	0.05	0.10
<u>N = 100, D = 5</u>						<u>N = 200, D = 10</u>				
1.00	0.9939	0.9878	0.9796	0.9433	0.8470	0.9917	0.9854	0.9772	0.9410	0.8455
0.98	0.9942	0.9883	0.9804	0.9447	0.8493	0.9921	0.9860	0.9780	0.9425	0.8478
0.95	0.9947	0.9891	0.9815	0.9467	0.8526	0.9928	0.9870	0.9793	0.9446	0.8513
0.90	0.9955	0.9903	0.9832	0.9500	0.8582	0.9938	0.9884	0.9812	0.9481	0.8570
0.75	0.9974	0.9934	0.9877	0.9592	0.8744	0.9963	0.9923	0.9865	0.9579	0.8735
<u>N = 100, D = 10</u>						<u>N = 200, D = 20</u>				
1.00	0.9445	0.9286	0.9112	0.8502	0.7279	0.9395	0.9240	0.9068	0.8470	0.7266
0.98	0.9473	0.9319	0.9148	0.8548	0.7336	0.9425	0.9274	0.9106	0.8517	0.7323
0.95	0.9514	0.9366	0.9201	0.8616	0.7420	0.9469	0.9323	0.9161	0.8586	0.7407
0.90	0.9579	0.9440	0.9285	0.8726	0.7559	0.9538	0.9401	0.9248	0.8698	0.7546
0.75	0.9741	0.9633	0.9507	0.9030	0.7960	0.9713	0.9606	0.9481	0.9009	0.7949
<u>N = 100, D = 20</u>						<u>N = 200, D = 40</u>				
1.00	0.7007	0.6763	0.6518	0.5784	0.4605	0.6987	0.6749	0.6509	0.5789	0.4625
0.98	0.7126	0.6884	0.6640	0.5906	0.4721	0.7104	0.6867	0.6629	0.5909	0.4739
0.95	0.7303	0.7063	0.6821	0.6089	0.4896	0.7277	0.7043	0.6807	0.6089	0.4911
0.90	0.7589	0.7355	0.7118	0.6393	0.5191	0.7561	0.7332	0.7099	0.6388	0.5201
0.75	0.8383	0.8175	0.7959	0.7280	0.6089	0.8350	0.8146	0.7935	0.7266	0.6089

$$\underline{n_1 = n_2 = 8; a_1 = 0, a_1' = a_2 = 2}$$

p	p'=0	0.01	0.02	0.05	0.10	0	0.01	0.02	0.05	0.10
<u>N = 100, D = 5</u>						<u>N = 200, D = 10</u>				
1.00	0.9747	0.9527	0.9258	0.8221	0.6117	0.9675	0.9459	0.9197	0.8185	0.6118
0.98	0.9760	0.9546	0.9283	0.8259	0.6163	0.9692	0.9481	0.9224	0.8224	0.6162
0.95	0.9780	0.9575	0.9320	0.8315	0.6230	0.9716	0.9514	0.9264	0.8281	0.6229
0.90	0.9811	0.9620	0.9379	0.8406	0.6343	0.9754	0.9565	0.9328	0.8375	0.6341
0.75	0.9887	0.9739	0.9539	0.8668	0.6677	0.9850	0.9701	0.9503	0.8645	0.6674
<u>N = 100, D = 10</u>						<u>N = 200, D = 20</u>				
1.00	0.8184	0.7782	0.7369	0.6110	0.4167	0.8115	0.7731	0.7334	0.6114	0.4202
0.98	0.8265	0.7868	0.7458	0.6203	0.4249	0.8195	0.7816	0.7422	0.6204	0.4281
0.95	0.8383	0.7994	0.7590	0.6342	0.4374	0.8313	0.7941	0.7552	0.6340	0.4402
0.90	0.8572	0.8198	0.7806	0.6572	0.4583	0.8503	0.8144	0.7765	0.6566	0.4607
0.75	0.9076	0.8756	0.8407	0.7248	0.5230	0.9015	0.8705	0.8366	0.7233	0.5241
<u>N = 100, D = 20</u>						<u>N = 200, D = 40</u>				
1.00	0.3683	0.3393	0.3120	0.2394	0.1480	0.3775	0.3487	0.3213	0.2481	0.1547
0.98	0.3843	0.3546	0.3264	0.2515	0.1564	0.3929	0.3634	0.3353	0.2598	0.1629
0.95	0.4090	0.3781	0.3488	0.2702	0.1696	0.4166	0.3860	0.3568	0.2781	0.1758
0.90	0.4515	0.4189	0.3878	0.3034	0.1933	0.4575	0.4254	0.3945	0.3103	0.1990
0.75	0.5872	0.5509	0.5155	0.4160	0.2777	0.5887	0.5531	0.5183	0.4199	0.2816

$$\underline{n_1 = n_2 = 13; a_1 = 0, a_1' = a_2 = 2}$$

p	p'=0	0.01	0.02	0.05	0.10	0	0.01	0.02	0.05	0.10
<u>N = 100, D = 5</u>						<u>N = 200, D = 10</u>				
1.00	0.9021	0.8374	0.7678	0.5565	0.2783	0.8862	0.8263	0.7611	0.5586	0.2838
0.98	0.9071	0.8435	0.7747	0.5638	0.2834	0.8915	0.8325	0.7679	0.5656	0.2886
0.95	0.9143	0.8525	0.7848	0.5747	0.2910	0.8993	0.8417	0.7780	0.5761	0.2959
0.90	0.9257	0.8669	0.8013	0.5929	0.3040	0.9116	0.8565	0.7945	0.5937	0.3083
0.75	0.9543	0.9055	0.8474	0.6470	0.3446	0.9437	0.8971	0.8413	0.6466	0.3473
<u>N = 100, D = 10</u>						<u>N = 200, D = 20</u>				
1.00	0.5292	0.4661	0.4081	0.2655	0.1191	0.5374	0.4768	0.4202	0.2778	0.1268
0.98	0.5446	0.4807	0.4217	0.2757	0.1244	0.5517	0.4903	0.4329	0.2874	0.1320
0.95	0.5679	0.5029	0.4424	0.2914	0.1327	0.5733	0.5110	0.4523	0.3023	0.1400
0.90	0.6069	0.5402	0.4777	0.3186	0.1475	0.6096	0.5459	0.4854	0.3282	0.1542
0.75	0.7218	0.6537	0.5872	0.4080	0.1987	0.7184	0.6535	0.5894	0.4137	0.2036
<u>N = 100, D = 20</u>						<u>N = 200, D = 40</u>				
1.00	0.0830	0.0709	0.0605	0.0372	0.0162	0.0975	0.0836	0.0714	0.0442	0.0192
0.98	0.0912	0.0781	0.0667	0.0412	0.0180	0.1060	0.0909	0.0779	0.0483	0.0211
0.95	0.1048	0.0899	0.0769	0.0477	0.0209	0.1198	0.1030	0.0884	0.0551	0.0242
0.90	0.1311	0.1128	0.0968	0.0606	0.0269	0.1463	0.1263	0.1087	0.0684	0.0304
0.75	0.2413	0.2105	0.1830	0.1183	0.0547	0.2547	0.2228	0.1943	0.1264	0.0586

$$\underline{n_1 = n_2 = 20; a_1 = 1, a_1' = 4, a_2 = 5}$$

p	p'=0	0.01	0.02	0.05	0.10	0	0.01	0.02	0.05	0.10
<u>N = 100, D = 5</u>						<u>N = 200, D = 10</u>				
1.00	0.9998	0.9919	0.9728	0.8401	0.4842	0.9948	0.9823	0.9599	0.8278	0.4862
0.98	0.9998	0.9925	0.9743	0.8453	0.4918	0.9953	0.9836	0.9622	0.8334	0.4935
0.95	0.9998	0.9932	0.9765	0.8530	0.5032	0.9960	0.9854	0.9655	0.8417	0.5045
0.90	0.9999	0.9944	0.9797	0.8652	0.5222	0.9970	0.9880	0.9705	0.8550	0.5229
0.75	1.0000	0.9970	0.9875	0.8981	0.5786	0.9988	0.9939	0.9824	0.8912	0.5783
<u>N = 100, D = 10</u>						<u>N = 200, D = 20</u>				
1.00	0.8530	0.7808	0.7041	0.4759	0.1985	0.8232	0.7594	0.6911	0.4824	0.2123
0.98	0.8649	0.7951	0.7200	0.4924	0.2088	0.8355	0.7733	0.7060	0.4976	0.2220
0.95	0.8816	0.8156	0.7430	0.5171	0.2248	0.8530	0.7934	0.7280	0.5204	0.2370
0.90	0.9064	0.8469	0.7793	0.5582	0.2528	0.8798	0.8251	0.7633	0.5587	0.2633
0.75	0.9592	0.9203	0.8703	0.6766	0.3460	0.9419	0.9035	0.8557	0.6720	0.3518
<u>N = 100, D = 20</u>						<u>N = 200, D = 40</u>				
1.00	0.1241	0.1013	0.0823	0.0432	0.0139	0.1560	0.1294	0.1067	0.0581	0.0193
0.98	0.1407	0.1153	0.0940	0.0498	0.0162	0.1724	0.1435	0.1188	0.0653	0.0220
0.95	0.1683	0.1387	0.1138	0.0613	0.0203	0.1992	0.1669	0.1389	0.0774	0.0266
0.90	0.2217	0.1849	0.1534	0.0849	0.0291	0.2500	0.2116	0.1778	0.1018	0.0362
0.75	0.4323	0.3746	0.3222	0.1965	0.0765	0.4445	0.3892	0.3379	0.2114	0.0848