JACKKNIFE L-ESTIMATORS: AFFINE STRUCTURE AND ASYMPTOTICS

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For L-estimators, a representation of the jackknife statistic, based on an inherent reverse martingale structure of jackknifing, is incorporated in the study of the asymptotic properties of the estimator as well as the allied jackknife estimator of the standard error. Some applications to sequential analysis are also discussed.

1. Introduction. Let $\{X_i;\ i\geq 1\}$ be a sequence of independent and identically distributed random variables (i.i.d.r.v.) with a continuous distribution function (d.f.) F, defined on the real line R. For every n (≥ 1) let $X_{n:1} \leq \cdots \leq X_{n:n}$ be the ordered r.v. (corresponding to $X_1,\ldots,X_n$); by virtue of the assumed continuity of F, ties among the $X_{n:\alpha}$ are neglected, in probability. For suitable g: $R\to R$ and *scores* $\{a_n(i),\ 1\leq i\leq n\}$, consider an L-estimator of the form

$$L_n = n^{-1}\Sigma_{i=1}^{n} a_n(k) g(X_{n:i}) \quad . \tag{1.1}$$

On the ground of robustness and efficiency, various forms of L_n are often advocated in problems of statistical inference [see Serfling (1980, Ch. 8) and Huber (1981)]; the asymptotic theory plays a vital role in this context.

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Whenever, for every (fixed) $u(0<u<1)$, $a_n([nu]+1) \to \phi(u)$, as $n\to\infty$, where $\phi(F(x))g(x)$ is (at least) square integrable and some additional regularity conditions hold [see Serfling (1980, Ch. 8)], then as $n\to\infty$,

$$n^{\frac{1}{2}}(L_n-\mu)/\sigma_L \xrightarrow{\mathcal{D}} N(0,1) \quad , \tag{1.2}$$

where

$$\mu = \int_{-\infty}^{\infty} \phi(F(x))g(x)dF(x) \tag{1.3}$$

and

$$\sigma_L^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{F(x\wedge y) - F(x)F(y)\}\phi(F(x))\phi(F(y))dg(x)dg(y) \quad . \tag{1.4}$$

Stronger results in the form of weak as well as strong invariance principles are contained in Sen (1981, Ch.7). In order to make full use of these results in problems of inference, one usually needs to estimate σ_L^2; in a non-sequential setup, usually, the weak consistency of this estimator suffices, while strong consistency is generally needed in sequential analysis. In this context, jackknifing is found to be very useful. Let $L_{n-1}^{(i)}$ be the L-estimator based on $(X_1,\ldots,X_{i-1},X_{i+1},\ldots,X_n)$ (i.e., on a sample of size $n-1$ when X_i has been removed), and let

$$L_{n,i} = nL_n - (n-1)L_{n-1}^{(i)} \,, \quad 1\le i\le n \quad . \tag{1.5}$$

Then, the jackknife L-statistic is

$$L_n^* = n^{-1}\Sigma_{i=1}^n L_{n,i} \quad . \tag{1.6}$$

Also, the (Tukey) estimator of σ_L^2 is defined by

$$s_n^{*2} = (n-1)^{-1}\Sigma_{i=1}^n (L_{n,i}-L_n^*)^2 \quad . \tag{1.7}$$

We shall see later on that S_n^{*2} is structurally very close to the alternative estimator considered in Sen (1978). Properties of L_n^* and S_n^{*2} have not been studied, so far, in full generality. For finite $\int g^2 dF$ and for ϕ of bounded variation (on (0,1)), some work in this area is due to Babu and Singh (1982), Efron (1982), and others. Our primary interest lies in the development of a general theory where ϕ need not be bounded (or of bounded variation) and/or Eg^2 may not be finite. In this context, a reverse martingale structure inherent in jackknifing [viz., Sen (1977)] has been exploited. Unlike other approaches, traditional decomposition into i.i.d.r.v.'s and a residual term has not been attempted. Rather, linear and quadratic functions of order statistics are employed and the reverse martingale approach of Sen (1978) is incorporated to study asymptotic results on L_n^* and S_n^{*2}. Along with the basic regularity conditions, the representations are considered in Section 2. Section 3 is devoted to the derivation of the main results. The last section deals with some applications in sequential analysis.

2. Representations for L_n^* and S_n^{*2}. Jackknifing rests on the construction of the n subsamples $\{(X_1,\ldots,X_{i-1},\ldots,X_n),\ 1\le i\le n\}$ of size $n-1$ [from $(X_1,\ldots,X_n)$]. We characterize this, equivalently, in terms of the order statistics $X_{n:\alpha}$, $1\le\alpha\le n$. For every α ($1\le\alpha\le n$), let $\underset{\sim}{X}_{n-1}^{(\alpha)} = (X_{n-1:1}^{(\alpha)},\ldots,X_{n-1:n-1}^{(\alpha)})$ be the vector of order statistics corresponding to the sample of size $n-1$ formed by deleting $X_{n:\alpha}$ from the set $(X_{n:1},\ldots,X_{n:n})$; the L-estimator corresponding to the sample of size $n-1$ relating to $\underset{\sim}{X}_{n-1}^{(\alpha)}$ is denoted by $L_{n-1}^{(\alpha)}$, for $\alpha = 1,\ldots,n$. Note that

$$(2.1)\qquad X_{n-1:i}^{(\alpha)} = \begin{cases} X_{n:i}, & \text{for } 1\le i\le\alpha-1, \\ X_{n:i+1}, & \text{for } \alpha\le i\le n-1, \end{cases}$$

so that

$$(2.2)\qquad L_{n-1}^{(\alpha)} = \frac{1}{n-1} \Sigma_{i=1}^{n-1} a_{n-1}(i) X_{n-1:i}^{(\alpha)}$$

$$= \frac{1}{n-1} \{\Sigma_{i=1}^{\alpha-1} a_{n-1}(i) X_{n:i} + \Sigma_{i=\alpha+1}^{n} a_{n-1}(i-1) X_{n:i}\} \quad ,$$

for $\alpha = 1,\ldots,n$. With the same re-indexing $(i \to \alpha)$, we have, for every α: $1 \leq \alpha \leq n$,

$$(2.3)\qquad L_{n,\alpha} = nL_n - (n-1)L_{n-1}^{(\alpha)}$$

$$= a_n(\alpha) g(X_{n:\alpha}) + \Sigma_{i=1}^{\alpha-1}[a_n(i) - a_{n-1}(i)] g(X_{n:i})$$

$$+ \Sigma_{i=\alpha+1}^{n}[a_n(i) - a_{n-1}(i-1)] g(X_{n:i})$$

$$= \Sigma_{i=1}^{n} b_{n\alpha}(i) g(X_{n:i}) \quad , \quad \text{say.}$$

As a result, for the jackknife estimator L_n^*, we have

$$(2.4)\qquad L_n^* = n^{-1}\Sigma_{\alpha=1}^{n} L_{n,\alpha}$$

$$= n^{-1}\Sigma_{\alpha=1}^{n}\Sigma_{i=1}^{n} b_{n\alpha}(i) g(X_{n:i})$$

$$= \Sigma_{i=1}^{n} c_n(i) g(X_{n:i}) \quad , \quad \text{say,}$$

where

$$(2.5)\qquad c_n(i) = a_n(i) - \frac{1}{n}[(n-i)a_{n-1}(i) + (i-1)a_{n-1}(i-1)] \quad , \quad 1 \leq i \leq n \quad .$$

This representation of L_n^* will be exploited in our subsequent manipulations. Similarly, from (1.7), (2.3) and (2.4), we have

$$(2.6)\qquad S_n^{*2} = (n-1)^{-1}\{\Sigma_{\alpha=1}^{n}(\Sigma_{i=1}^{n}\{b_{n\alpha}(i) - c_n(i)\} g(X_{n:i}))^2\}$$

$$= (n-1)^{-1}\Sigma_{\alpha=1}^{n}\Sigma_{i=1}^{n}\Sigma_{j=1}^{n}\{b_{n\alpha}(i) - c_n(i)\}\{b_{n\alpha}(j) - c_n(j)\} g(X_{n:i}) g(X_{n:j})$$

$$= \Sigma_{i=1}^{n}\Sigma_{j=1}^{n} d_n(i,j)g(X_{n:i})g(X_{n:j}) \quad , \quad \text{say,}$$

where

$$(2.7) \qquad d_n(i,j) = (n-1)^{-1}\Sigma_{\alpha=1}^{n}\{b_{n\alpha}(i) - c_n(i)\}\{b_{n\alpha}(j) - c_n(j)\} \quad ,$$

for $i,j=1,\ldots,n$. This quadratic representation of s_n^{*2} will be utilized in the study of its asymptotic properties. In the rest of this section, we introduce the regularity conditions (on g and the scores) under which we shall pursue the study of the properties of L_n^* and S_n^*.

We denote by $b(u) = g(F^{-1}(u))$, $0<u<1$ and assume that for every $\theta \in (0,\frac{1}{2})$, $b(u)$ is of bounded variation on $(\theta,1-\theta)$, and, for some real a and finite, positive K,

$$(2.8) \qquad |b(u)| \leq K\{u(1-u)\}^{-a} \quad , \quad \forall\ u\in(0,1) \quad .$$

Also, we consider a *score generating function* $\phi = \{\phi(u): \ 0<u<1\}$ and relate the scores $a_n(i)$ by letting

$$(2.9) \qquad a_n(i) = \phi(i/(n+1)) \quad , \quad 1\leq i\leq n;\ n\geq 1 \quad ;$$

$$(2.10) \qquad \phi_n(u) = \phi(\tfrac{i}{n+1}) \quad \text{for} \quad \frac{i-1}{n} < u \leq \frac{i}{n} \ , \quad 1\leq i\leq n \quad .$$

We could have defined the scores in some asymptotically equivalent alternative forms; this would not make any difference in the asymptotic results to follow. We assume that ϕ has a continuous first order derivative ϕ' $(= \{\phi'(u): \ 0<u<1\})$ almost everywhere, where

$$(2.11) \quad |\phi(u)| \leq K\{u(1-u)\}^{-b} \ , \quad |\phi'(u)| \leq K\{u(1-u)\}^{-b-1} \ , \quad \forall\ 0<u<1 \quad ,$$

where b is real and, in conjunction with (2.8),

$$(2.12) \qquad a + b = \tfrac{1}{2} - \delta \ , \quad \text{for some} \quad \delta>0 \quad .$$

Note that in this setup, we need not assume that $Eg^2(x) < \infty$ and/or that ϕ is of bounded variation, though $\int_0^1 b^2(u)\phi^2(u)du < \infty$. However, in this setup, jump discontinuities of ϕ has been excluded; it is, of course, known that for such singular components, the L-statistics may not readily amend to jackknifing.

3. Asymptotics on jackknifing. First, we consider the first order asymptotics. Note that by (1.1), (1.4) and (2.5),

$$L_n^* - L_n = n^{-1}\Sigma_{i=1}^n\{(n-1)a_n(i) - (n-i)a_{n-1}(i) - (i-1)a_{n-1}(i-1)\}g(X_{n:i})$$

$$= n^{-1}\Sigma_{i=1}^n a_n^*(i)g(X_{n:i}) \ , \quad \text{say}, \tag{3.1}$$

where by the first order Taylor expansion and (2.9),

$$a_n^*(i) = -i(n-i)n^{-2}\phi'(\xi_{i1}^{(n)}) + (i-1)(n-i+1)n^{-2}\phi'(\xi_{i2}^{(n)}) \quad , \tag{3.2}$$

where

$$\frac{i}{n+1} < \xi_{i1}^{(n)} < \frac{i}{n} \text{ and } \frac{i-1}{n} < \xi_{i2}^{(n)} < \frac{i}{n+1}\ , \quad 1\le i\le n \quad . \tag{3.3}$$

Note that for $i=1$ (or n), the second (or first) term on the right hand side of (3.2) vanishes, while for $2\le i\le n-1$, we may rewrite (3.2) as

$$\frac{i(n-i+1)}{n^2}\{\phi'(\xi_{i2}^{(n)}) - \phi'(\xi_{i1}^{(n)})\} + \frac{i}{n^2}\phi'(\xi_{i1}^{(n)}) - \frac{n-i+1}{n^2}\phi'(\xi_{i2}^{(n)}) \quad . \tag{3.4}$$

Thus, if we write $a_n^*([nu] + 1) = \psi_n^*(u)$, then for every (fixed) $u \in (0,1)$, as $n\to\infty$, by (2.11) and (3.4),

$$\psi_n^*(u) \to \psi^*(u) = 0 \ , \quad 0<u<1 \quad , \tag{3.5}$$

while by (3.2), (3.3) and (2.11),

$$|a_n^*(i)| \le C\{i(n-i+1)n^{-2}\}^{-b} \quad , \quad \forall\ 1\le i\le n \quad , \tag{3.6}$$

where $C(<\infty)$ is a finite positive constant. Hence, by an appeal to Theorem 7.6.2 of Sen (1981), we conclude that under (2.8) through (2.12), as $n\to\infty$,

$$L_n^* - L_n \to 0 \text{ almost surely (a.s.)} \quad . \tag{3.7}$$

Let us now define the L_n, as in (2.3), and let

$$\begin{aligned} S_n^2 &= (n-1)^{-1}\Sigma_{\alpha=1}^{n}(L_{n,\alpha}-L_n)^2 \\ &(= S_n^{*2} + n(n-1)^{-1}(L_n^*-L_n)^2) \quad . \end{aligned} \tag{3.8}$$

Then, by (3.7) and (3.8),

$$S_n^2 - S_n^{*2} \to 0 \text{ a.s., as } n\to\infty \quad . \tag{3.9}$$

Also, if $\mathcal{F}_n = \mathcal{F}(X_{n:1},\ldots,X_{n:n};\ X_{n+j},\ j\geq 1)$ be the sigma-field generated by $(X_{n:1},\ldots,X_{n:n})$ and X_{n+j}, $j\geq 1$, for every $n\geq 1$, then $\mathcal{F}_n$ is a non-increasing sequence of sigma-fields, and

$$\begin{aligned} &n(n-1)E\{(L_{n-1}-L_n)^2|\mathcal{F}_n\} \\ &= n(n-1)\{\frac{1}{n}\Sigma_{\alpha=1}^{n}(L_{n-1}^{(\alpha)}-L_n)^2\} \\ &= (n-1)^{-1}\Sigma_{\alpha=1}^{n}(L_{n,\alpha}-L_n)^2 = S_n^2 \ , \quad \forall\ n\geq 1 \ , \end{aligned} \tag{3.10}$$

where the penultimate step follows by using (2.3). On the other hand, by Theorem 7.6.3 of Sen (1981), under (2.8) through (2.12), as $n\to\infty$,

$$n(n-1)E\{(L_{n-1}-L_n)^2|\mathcal{F}_n\} \to \sigma_L^2 \text{ a.s.,} \tag{3.11}$$

where σ_L^2 is defined in (1.4). Consequently, by (3.9), (3.10) and (3.11), under (2.8) through (2.12), as $n\to\infty$,

$$S_n^{*2} \to \sigma_L^2 \text{ a.s.} \tag{3.12}$$

To exploit the full utility of jackknifing, we need to consider the second order asymptotics. First, we notice that by virtue of (3.1)-(3.6) (where by (3.5), the variance function $\sigma^2_{L^*-L} = 0$) and Theorem 4 of Wellner (1977),

$$n^{\frac{1}{2}}|L_n^* - L_n| = o((\log\log n)^{\frac{1}{2}}) \quad \text{a.s., as } n\to\infty \quad . \tag{3.13}$$

This result, though of some interest, is not good enough to provide the full utility of jackknifing. Towards this, in view of the fact that, in (3.4), the score function rests on ϕ', we make an additional assumption that ϕ' has a derivative ϕ'' (a.e.) on $(0,1)$, where defining b by (2.11)-(2.12),

$$|\phi''(u)| \leq K\{u(1-u)\}^{-2-b} \quad , \quad \forall\ 0<u<1 \quad . \tag{3.14}$$

We then define δ as in (2.12) and consider

$$n^{(1+\delta)/2}(L_n^* - L_n) = n^{-1}\Sigma_{i=1}^n \{n^{(1+\delta)/2} a_n^*(i)\} g(X_{n:i}) \quad , \tag{3.15}$$

where by (2.8), (2.11), (2.12) and (3.2), the contribution of the first (or the last) term in the sum in (3.15) is $-n^{-1} \cdot n^{(1+\delta)/2} n^{-1} \phi'(\xi_{i1}^{(n)}) g(X_{n:1})$ (or $n^{-2+(1+\delta)/2}\phi'(\xi_{n2}^{(n)}) g(X_{n:n})$) which is $O(n^{-\delta/2})$ a.s., as $n\to\infty$, while for $2\leq i\leq n-1$, we write [by using (3.4) and (3.14)],

$$\begin{aligned} n^{(1+\delta)/2} a_n^*(i) &= \{i(n-i+1)n^{-2}\} n^{(1+\delta)/2} (\xi_{i2}^{(n)} - \xi_{i1}^{(n)}) \phi''(\xi_{i\cdot}^{(n)}) \\ &\quad + (in^{-1}) n^{-(1-\delta)/2} \phi'(\xi_{i1}^{(n)}) - ((n-i+1)n^{-1}) n^{-(1-\delta)/2} \phi'(\xi_{i2}^{(n)}) \quad , \end{aligned} \tag{3.16}$$

where $\xi_{i\cdot}^{(n)} \in (\xi_{i2}^{(n)}, \xi_{i1}^{(n)})$. Note that by (3.3), $n^{(1+\delta)/2}|\xi_{i2}^{(n)} - \xi_{i1}^{(n)}| \leq n^{-(1-\delta)/2} \leq \{i(n-i+1)/n^2\}^{(1-\delta)/2}$, $\forall\ 2\leq i\leq n-1$. Hence, using (2.11), (2.12), (3.14) and (3.16), we have

$$|n^{(1+\delta)/2} a_n^*(i)| \leq C\{i(n-i+1)/n^2\}^{-\frac{1}{2}-\delta/2-b} \quad , \tag{3.17}$$

for every $2 \le i \le n-1$, $n \ge 2$, while for every (fixed) $u \in (0,1)$,

$$(3.18) \qquad n^{(1+\delta)/2} a_n^*([nu]+1) \to 0, \quad \text{as} \quad n \to \infty \quad .$$

Hence, by (2.8), (2.12), (3.15), (3.17), (3.18) and Theorem 7.6.2 of Sen (1981), we conclude that as $n \to \infty$,

$$(3.19) \qquad n^{(1+\delta)/2} |L_n^* - L_n| \to 0 \quad \text{a.s.}$$

Note that under (2.8) through (2.12) and (3.14), by virtue of Theorem 7.5.1 of Sen (1981),

$$(3.20) \qquad L_n - \mu = n^{-1} \Sigma_{i=1}^n Z_i + R_n \quad ,$$

where

$$(3.21) \qquad Z_i = \int_{-\infty}^{\infty} [I(X_i \le x) - F(x)] \phi(F(x)) dx \;, \quad i \ge 1 \quad ,$$

$$(3.22) \qquad R_n = 0(n^{-\frac{1}{2}-\eta}) \quad \text{a.s.} \;, \quad \text{as} \quad n \to \infty \quad ,$$

and $\eta > 0$. By virtue of (3.19) and (3.20), we have

$$(3.21) \qquad L_n^* - \mu = n^{-1} \Sigma_{i=1}^n Z_i + R_n^* \quad ,$$

where μ is defined by (1.3) and

$$(3.22) \qquad R_n^* = 0(n^{-\frac{1}{2}-\eta}) \quad \text{a.s., as} \quad n \to \infty \quad ,$$

for some $\eta > 0$. Since the Z_i are i.i.d.r.v. with mean 0 and variance σ_L^2, defined by (1.4), the Skorophod-Strassen embedding of Wiener process holds for $\{\sigma_L^{-1} \Sigma_{i=1}^n Z_i, \; n \ge 1\}$ and this along with (3.21)-(3.22) provide us with all the desired asymptotic results on the jackknife statistics L_n^*. Further, by (3.8) and (3.19), we have

$$(3.23) \qquad n|S_n^2 - S_n^{*2}| \to 0 \quad \text{a.s., as} \quad n \to \infty \quad ,$$

so that (3.12) remains intact. This leads us to

$$\sqrt{n}(L_n^*-\mu)/S_n^* = (S_n^*/\sigma_L)^{-1}\left\{\frac{\sqrt{n}}{\sigma_L}\left[\frac{1}{n}\sum_{i=1}^{n} Z_i+R_n^*\right]\right\} \tag{3.24}$$

$$= \frac{1}{\sigma_L\sqrt{n}}\sum_{i=1}^{n} Z_i + o(1) \quad \text{a.s., as} \quad n\to\infty \quad .$$

Actually, whenever in (2.12), δ is restricted to be $\geq \frac{1}{4}$, then we may use appropriate rates of convergence of S_n to σ_L and replace [in (3.24)] $o(1)$ by $O(n^{-\eta})$ a.s., for some $\eta>0$. Thus, for jackknifing of L-Statistics, not only S_n^* is a consistent estimator of σ_L (it is strongly so), but also the asymptotic normality results of the jackknife statistics extend to a strong invariance principle, as in (3.24). We conclude this section with the remark that under (2.8) through (2.12), with $\delta>\frac{1}{4}$, from Gardiner and Sen (1979), it follows that

$$n^{\frac{1}{2}}(S_n^2-\sigma_L^2) \sim N(0,\gamma^2) \quad , \tag{3.25}$$

where

$$\gamma^2 = \int_0^1\int_0^1 (s\wedge t-st)L_0(s)L_0(t)dg(F^{-1}(s))dg(F^{-1}(t)) \quad , \tag{3.26}$$

$$L_0(t) = L_1(t)\{\phi(t)+t\phi'(t) \quad + L_2(t)\{(1-t)\phi'(t) - \phi(t)\} \quad , \tag{3.27}$$

$$L_1(t) = 2\int_t^1 (1-s)\phi(s)dg(F^{-1}(s)) \tag{3.28}$$

$$L_2(t) = 2\int_0^t s\phi(s)dg(F^{-1}(s)) \quad . \tag{3.29}$$

As a result, by (3.19), the same asymptotic normality result holds for the jackknife estimator S_n^{*2}, when (3.14) is assumed to hold. In passing, it may be remarked that (3.14) (or the existence of ϕ'') may not be really needed. It is possible to replace this by a local Lipschitz condition that for every $u \in (0,1)$ and $\alpha(u)$, $\beta(u)$ such that $0 < \alpha(u) < \beta(u) < \alpha(u) + n^{-1} < 1$,

$$|\phi'(\alpha(u))-\phi'(\beta(u))| \leq |\alpha(u)-\beta(u)|^{\gamma}\cdot\max\{|\phi'(\alpha(u))| , |\phi'(\beta(u))|\} ,$$

for some $\gamma>\frac{1}{2}$. This condition will insure (3.17), and hence, (3.19) will remain intact.

4. Some general remarks. In the literature, the L-statistics have been advocated for efficient estimation of μ [in (1.3)], which may often be expressed as a parameter (location/scale etc.) of the underlying d.f.F. Also, in problems of testing hypotheses concerning μ, the L-statistics are often found to be good robust competitors of some classical tests. In both of these contexts, jackknifing is quite useful. When one wants to have a bounded width confidence interval for μ, based on jackknife L-statistics, then one encounters a sequential model, where (3.12) and (3.24) provide the necessary tool for studying the asymptotic consistency of the procedure. Since these are very analogous to that in Section 5 of Sen (1977), the details are omitted. We may, however, add that (3.25) [as extended to random sample sizes in Gardiner and Sen (1979)] provides the asymptotic normality of the stopping time for this sequential procedure. Similarly, for sequential tests based on L-statistics, the embedding of Wiener process in (3.24), provides the basic tool for the study of the asymptotic OC function, while the Pitman-efficiency results are by-product of this representation. Since the details are analogous to those in Section 6 of Sen (1977), they are not reproduced here.

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