ON TRANSIENT REGENERATIVE PROCESSES

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§1 INTRODUCTION

The theory of regenerative and cumulative processes was introduced by Smith (1955,1958) and has found numerous applications in operations research. We note that the main emphasis of this theory is somewhat different from that of Kingman's work on regenerative phenomena (1972). The usual basis for these regenerative and cumulative processes is a renewal process $\{X_i\}_{i=1}^{\infty}$ of iid positive random variables with a proper distribution function $F(x)$. However, various applications arise where the notion of a regenerative or cumulative process seems appropriate but, unfortunately, $F(\infty) < 1$ (i.e. the X's are improper). The present paper is concerned with adapting the existing theory of such processes to this improper case.

Consider a stochastic process built up from processes of random length called <u>tours</u> in the following way. A random tour consists of an ordered pair $(X(\omega), \underset{\sim}{\xi}(t,\omega))$ defined on a probability space (Ω,β,p) where X is positive, $\underset{\sim}{\xi}(\cdot,\omega)$ is a function defined for $0 < t \leq X(\omega)$ taking its values in $\mathbb{R}^d$, and $(X(\omega), \underset{\sim}{\xi}(t,\omega))$ is jointly measurable with respect to β. Throughout this paper, d will represent the dimensionality of $\underset{\sim}{\xi}$. One could allow the graph $\underset{\sim}{\xi}(\cdot,\omega)$ to take its values in some abstract space X and then consider the properties of the function $V(\xi(t,\omega)) \in \mathbb{R}^d$; the particular definition of V and the dimension d would be determined by the aspects of the graph $\underset{\sim}{\xi}$ which are of interest in particular applications. But this generalization is one of appearance rather than reality. We write F for the distribution of X; and shall assume that F is nonlattice, $F(0^+) = 0$, and $F(\infty) = \omega < 1$.

Let (Ω_i,β_i,p_i) be independent copies of the probability space and let P be the product measure defined on $\prod_{i=1}^{\infty} \beta_i$. Suppose $\theta_1,\theta_2,\ldots$ is a sequence of tours in which each $\theta_i = (X(\omega_i), \underset{\sim}{\xi}(t,\omega_i))$ has domain Ω_i and is chosen in accordance with the probability measure p_i.

Let $X_i = X(\omega_i)$, set $S_o = 0$, and $S_n = X_1+\ldots+X_n$. $N(t)$, the number of complete tours observed by time t, is defined to be that integer k such that $S_k \le t < S_{k+1}$. $H(t) \equiv EN(t)$ is called the renewal function. As in ordinary renewal theory,

$$H(t) = \sum_{j=1}^{\infty} F^{(j)}(t) \tag{1.1}$$

and

$$H(t) = F(t) + \int_0^t H(t-z)\, F(dz) \quad ; \tag{1.2}$$

$F^{(j)}(t)$ denotes the j^{th} Stieltjes convolution of F with itself.

Were F a proper distribution, N(t) would diverge as t approaches infinity. However, when F is improper, N(t) approaches a finite limit with probability one and $H(\infty) = \omega(1-\omega)^{-1}$. Because we expect to observe only $\omega(1-\omega)^{-1}$ complete tours, $\{X_i\}$ is called a transient renewal process.

We can construct a transient cumulative process as follows:

$$\underset{\sim}{W}(t) = \begin{cases} \underset{\sim}{\xi}(t,\omega_1), \quad 0 < t \le X_1 \\ \sum_{i=1}^{N(t)} \underset{\sim}{\xi}(X_i,\omega_i) + \underset{\sim}{\xi}(t-S_{N(t)}, \omega_{N(t)+1}), \; S_{N(t)} \le t < S_{N(t)+1} < \infty \\ \sum_{i=1}^{N(t)} \underset{\sim}{\xi}(X_i,\omega_i), \quad S_{N(t)+1} = \infty \end{cases} \tag{1.3}$$

Thus so long as the lifetimes X_j are finite, there is no difference between the usual cumulative process and the transient one. However, once an infinite lifetime arises, the process is assumed to have "died" and $\underset{\sim}{W}(t)$ then remains constant for all $t \ge S_{N(t)}$.

We require each component of $\underset{\sim}{W}(t)$ to be a function of bounded variation in every finite t interval with probability one and that the random vectors

$$\underset{\sim}{Y}_i^* = \int_{S_{i-1}}^{S_i} |\underset{\sim}{W}(dt)| \text{ also be iid.}$$

Let $\underset{\sim}{Y}_i \equiv \underset{\sim}{\xi}(X_i, \omega_i)$; then so long as $S_{N(t)+1} < \infty$, $\underset{\sim}{W}(t) = \sum_{i=1}^{N(t)} \underset{\sim}{Y}_i + \underset{\sim}{\xi}(t-S_{N(t)}, \omega_{N(t)+1})$ and therefore is the sum of N(t) iid random vectors and an extra term which depends on the tour in progress at time t.

At this point it may be helpful to mention some of the principle results from the conventional theory of cumulative processes when $F(\infty) = 1$. We give results for d=1, but extension to general cases is easy.

Write $\kappa_r = EY_i^{\ r}$ and $\kappa_r^* = EY_i^{*r}$ when these moments exist. Let $\sigma_x^2 = Var(X_i)$, $\sigma_y^2 = Var(Y_i)$, and $\rho_{xy} = Cov(X_i, Y_i)$. Smith (1955) has shown that

$$\lim_{t\to\infty} \frac{1}{t} W(t) = \frac{\kappa_1}{\mu_1} \text{ a.s. if } \mu_1 < \infty, \kappa_1^* < \infty \tag{1.4}$$

$$EW(t) = \frac{\kappa_1}{\mu_1} t + o(t) \text{ if } \mu_1 < \infty, \kappa_1^* < \infty . \tag{1.5}$$

Define $\gamma = \sigma_y^2 - 2\rho_{xy}\sigma_x\sigma_y(\frac{\kappa_1}{\mu_1}) + \sigma_x^2(\frac{\kappa_1}{\mu_1})^2 = E(Y_i - \frac{\kappa_1}{\mu_1} X_i)^2$. Then

$$Var(W(t)) = \frac{t}{\mu_1} \gamma + o(t) \text{ if } \mu_2 < \infty, \kappa_2^* < \infty \tag{1.6}$$

$$\lim_{t\to\infty} P\{\frac{W(t)-\kappa_1 N(t)}{\sigma_y\sqrt{\frac{t}{\mu_1}}} \le \alpha\} = \Phi(\alpha) \text{ if } \kappa_2^* < \infty, \mu_1 < \infty \tag{1.7}$$

and

$$\lim_{t\to\infty} P\{\frac{W(t) - \frac{\kappa_1 t}{\mu_1}}{\sqrt{\frac{\gamma t}{\mu_1}}} \le \alpha \} = \Phi(\alpha) \text{ if } \kappa_2^* < \infty, \mu_2 < \infty . \tag{1.8}$$

Our goal is to show that under certain general conditions analogous results hold in the transient setting.

Before closing this section, we introduce a critical hypothesis. Suppose there is a $\sigma > 0$ making

$$\int_0^\infty e^{\sigma x} F(dx) = 1.$$

When such a σ exists we say we have the exponential case. It is easy to construct improper distributions for which no such σ can be found; this case will be discussed in a later paper.

§2. EXAMPLES OF TRANSIENT RENEWAL AND CUMULATIVE PROCESSES

1. Time until ruin in collective risk theory

Let the renewal process $\{X_i\}$ represent times between claims against an insurance company and let Y_i be the value of the i^{th} claim. Assume that in the absence of claims, reserves increase at the constant rate $c > 0$ and that the company has initial assets u. The risk reserve at time t is

$$R(t) = u + ct - \sum_{j=1}^{N(t)} Y_j .$$

The company is "ruined" at time t if $R(t) < 0$. Define

$$\tau^* \equiv \inf\{t: R(t) < 0\} . \tag{2.1}$$

Cramer (1955), von Bahr (1974), and Siegmund (1975) have studied τ^*, the time of ruin.

Let

$$W(t) = \sum_{j=1}^{N(t)} Y_j - ct;$$

$W(t)$ is a cumulative process with increments $Y_j - cX_j$. Ruin can occur only at some regeneration point S_n when $W(t)$ exceeds all of its previous values. Let $L_1, L_2, \ldots$ be the iid positive ladder random variables constructed from the increments $Y_j - cX_i$ and let $Z_1, Z_2, \ldots$ be the renewal process underlying the L's. That is, if I_1 is the smallest integer making

$$L_1 \equiv \sum_{k=1}^{I_1} (Y_k - cX_k)$$

positive, then

$$Z_1 = \sum_{k=1}^{I_1} X_k .$$

Subsequent vectors (L_i, Z_i) are defined similarly.

Let $M(t)$ be the renewal count for the process $\{Z_i\}$ and let

$$W_z(t) = \sum_{k=1}^{M(t)} L_k$$

be the increasing cumulative process built from the ladder variables.

$$P\{\tau^* \le t\} = P\{ \sum_{k=1}^{M(t)} L_k > u\} . \tag{2.2}$$

If $E(Y_1 - cX_1) < 0$, $\sum_{k=1}^{n} (Y_k - cX_k)$ attains a maximum with probability one and then drifts toward $-\infty$. This means $\{Z_i\}$ and $W_z(t)$ are transient processes.

2. Waiting times for long gaps

Suppose the renewal process $\{X_i\}$ having proper distribution F is stopped at the first appearance of an interval longer than L which is free of renewals. Define W to be the waiting time for such an interval and let $V(t) \equiv P\{W \le t\}$.

$$V(t) = \begin{cases} 0, & t \le L \\ 1 - F(L) + \int_0^L V(t-\tau)F(d\tau) & , \quad t > L \end{cases} \tag{2.3}$$

by a standard renewal argument.

We can write

$$V(t) = v(t) + \int_0^t V(t-\tau)G(d\tau) \tag{2.4}$$

where G is a defective distribution defined by

$$G(t) = \begin{cases} F(t), & t \le L \\ F(L), & t > L \end{cases}$$

and

$$v(t) = \begin{cases} 0 & , t \le L \\ 1-F(L), & t > L. \end{cases}$$

Hence

$$V(t) = v(t) + \int_0^t v(t-\tau)H_G(d\tau) \ . \qquad (2.5)$$

Feller (1971) uses the notion of waiting for a large gap to model the problem of a pedestrian trying to cross a stream of traffic. Let the renewal process $\{X_i\}$ be the gaps between successive cars. In order to cross the street with safety, the pedestrian must wait for a gap of more than L seconds, say. The distribution of his waiting time is given by (2.5).

V(t) may also be interpreted as the probability that the maximum lifetime or partial lifetime observed by time t exceeds L. Lamperti (1961) has studied the problem from this point of view.

3. Lost telephone calls

Suppose that calls arriving at a telephone trunkline form a Poisson process with intensity λ. A call is placed at time 0. The lengths of conversations are independent random variables with common distribution F. Calls arriving during a busy period are lost; we are interested in the waiting time W for the first lost call. We may consider the renewal process $\{X_i\}$ where X_i is the time between the i-1st and ith calls so long as no calls arrive during the busy period caused by the i-1st call; otherwise $X_i = \infty$ and the process stops.

The busy periods associated with the process have distribution

$$G(x) = \int_0^x e^{-\lambda\tau}F(d\tau) \ . \qquad (2.6)$$

The event $\{X_1 \le x\}$ occurs when the call begun at time 0 lasts for some time $\tau \le x$ and a new call is received in the remaining time $x - \tau$.

Hence

$$J(x) = P\{X_1 \le x\} = \int_0^x [1-e^{-\lambda(x-\tau)}]e^{-\lambda\tau} F(d\tau) \qquad (2.7)$$

$$\lim_{x\to\infty} J(x) = \int_0^\infty e^{-\lambda\tau} F(d\tau) = F^*(\lambda) = \omega < 1. \qquad (2.8)$$

We can study W, the waiting time for the first lost call, by studying the transient process $\{X_i\}$. Let q(t) be the probability the X process is alive at time t. Then $q(t) = P\{W > t\}$; in Lemma 3.1 we find the rate at which this converges to zero. This example is from problem 17, Chapter 6.13 of Feller (1971).

4. Generalized type II Geiger counters

Particles arriving at a generalized type II Geiger counter constitute a Poisson process with intensity λ. The nth particle locks the counter for a time T_n and annuls the after-effects of all preceding particles. Suppose the T's have common distribution B and let Y_i be the length of the ith locked period. Define $Z(t) = P\{Y_1 > t\}$. The event $\{Y_1 > t\}$ can result either because $T_1 > t$ and no particles arrive in (0,t] or because a particle arrives at some time $\tau < T_1$, $\tau \le t$ and the locked period begun at τ exceeds $t - \tau$. Thus

$$Z(t) = e^{-\lambda t}[1-B(t)] + \int_0^t \lambda e^{-\lambda\tau}[1-B(\tau)]Z(t-\tau)d\tau\ . \qquad (2.9)$$

We may regard

$$F(t) = \int_0^t [1-B(\tau)]\lambda e^{-\lambda\tau}d\tau \qquad (2.10)$$

as a defective distribution and write

$$Z(t) = e^{-\lambda t}[1-B(t)] + \int_0^t Z(t-\tau)F(d\tau) \qquad (2.11)$$

which implies

$$Z(t) = e^{-\lambda t}[1-B(t)] + \int_0^t e^{-\lambda(t-\tau)}[1-B(t-\tau)]\ H_F(d\tau). \qquad (2.12)$$

To investigate the distribution of the locked periods, we must deal with the transient renewal function $H_F(t)$. Also, if we are interested in the renewal process $\{X_i\}$ where X_i represents the time between the beginning of the i-1st and ith blocked periods, we see that

$$P\{X_1 \le x\} = \int_0^x \lambda e^{-\lambda\tau}[1-Z(x-\tau)]d\tau \qquad (2.13)$$

because X_1 is the sum of the length of a locked period and the waiting time for the arrival of the first particle after the counter has become unlocked. Study of $\{X_i\}$ requires coping with F and H_F. This example is from problem 15, Chapter 11.10 of Feller (1971).

5. Age dependent branching processes

A particle born at time 0 lives some random time and then splits into k new particles with probability q_k, k = 0, 1, ... Its lifetime has distribution G; assume $G(0^+) = 0$ and $G(\infty) = 1$. The new particles develop independently of one another and of their time of birth; they have the same lifetime distribution and splitting probabilities as the first one.

Let Z(t) be the number of particles at time t and $p_r(t) = P\{Z(t) = r\}$. If Z(t) = 0, then Z(t+s) = 0 for all s ≥ 0; the branching process becomes extinct. Suppose that $\alpha = \sum_{k=1}^{\infty} kq_k < \infty$ and let A(t) be the expected number of particles at time t. An integral equation of Bellman and Harris (1948) yields

$$A(t) = \alpha \int_0^t A(t-y)\ G(dy) + 1-G(t) \qquad (2.14)$$

Bondarenko (1960) has shown that if $\alpha \le 1$, $p_0(t) \to 1$ as $t \to \infty$. That is, the process becomes extinct with probability one. Consider the case $\alpha < 1$. We may regard $\alpha\, G(y)$ as an improper distribution with corresponding renewal function $H_\alpha(y)$. Hence

$$A(t) = 1-G(t) + \int_0^t [1-G(t-\tau)]H_\alpha(d\tau)\ . \qquad (2.15)$$

If there exists a $\sigma > 0$ making

$$\alpha \int_0^\infty e^{\sigma t} G(dt) = 1,$$

the methods of the present paper lead to asymptotic estimates of $A(t)$ which agree with those of Chistyakov (1964). Vinogradov (1964) refers to the nonexponential case and estimates corresponding to his will be obtained in a future paper.

§3 MAIN RESULTS

$$\text{Let } A(t) = \begin{cases} 1 & \text{if } S_{N(t)+1} < \infty \\ 0 & \text{otherwise.} \end{cases}$$

We say the renewal process is "alive" at time t if $A(t) = 1$; when $A(t) = 1$, we have encountered only finite waiting times by time t and anticipate the process has behaved like an ordinary renewal process thus far. We will study $\underset{\sim}{W}(t)$ by conditioning expectations of interest on the event $\{A(t)=1\}$. Note that

$$P\{A(t) = 1\} \equiv q(t) = \sum_{j=0}^{\infty} P\{A(t) = 1,\ N(t) = j\}$$

$$= \omega - F(t) + \sum_{j=1}^{\infty} \int_0^t [\omega - F(t-\tau)] F^{(j)}(d\tau) = \omega - (1-\omega)H(t) \ . \qquad (3.1)$$

Let $G(x,\underset{\sim}{y}$ $P\{X_i \le x,\ \underset{\sim}{Y}_i \le \underset{\sim}{y})$ and for $x > \tau$ let $T(x,\underset{\sim}{w},\tau) =$ $P\{\tau < X \le x,\ \underset{\sim}{\xi}(\tau) \le \underset{\sim}{w}\}$. Throughout the remainder of this paper we assume

$$\int_0^\infty e^{\sigma x} F(dx) = 1.$$

Hence we may define the proper distribution functions $\tilde{F}(x)$ and $\tilde{G}(x,\underset{\sim}{y})$ by

$$\tilde{F}(dx) = e^{\sigma x} F(dx) \qquad (3.2)$$

and

$$\tilde{G}(dx,\ d\underset{\sim}{y}) = e^{\sigma x} \tilde{G}(dx,\ d\underset{\sim}{y}) \ . \qquad (3.3)$$

We also define $\tilde{T}(x,\underset{\sim}{w},\tau)$ for $x > \tau$ by

$$\tilde{T}(dx,d\underset{\sim}{w},\tau) = e^{\sigma x} T(dx,d\underset{\sim}{w},\tau) \ . \qquad (3.4)$$

$\tilde{T}(x,\underset{\sim}{w},z)$ summarizes the joint behavior of a lifetime X having proper distribution $\tilde{F}$ and the value of the graph function $\underset{\sim}{\xi}(\cdot)$ at some time $\tau < X$. Let us write $\tilde{E}$ and $\tilde{P}$ for expectations and probabilities when the random vectors $(X_i,\underset{\sim}{Y}_i)$ have proper distribution $\tilde{G}$ and when $\tilde{T}$ governs the joint distribution of $(X_i,\underset{\sim}{\xi}_i(\tau))$. We represent the j^{th} component in $\underset{\sim}{Y}_i$ as Y_{ij}, $1 \le j \le d$, and write $\tilde{\mu}_{rs}(j) = \tilde{E}X_i^r Y_{ij}{}^s$ when this product moment exists. Of course, when $s = 0$, $\tilde{\mu}_{ro}(j) \equiv \tilde{\mu}_r$ is independent of j; we also let $\tilde{\kappa}_s(j) = \tilde{\mu}_{os}(j)$. When $d = 1$ we suppress j altogether.

Definition 1. A stochastic process $\underset{\sim}{G}(\cdot)$ taking its values in $\mathbf{R}^m$ is a C process if, for every $t > 0$, $\underset{\sim}{G}(t)$ depends only on $t,(\underset{\sim}{X}_1,\underset{\sim}{Y}_1),\ldots,(\underset{\sim}{X}_{N(t)},\underset{\sim}{Y}_{N(t)})$, and $(\underset{\sim}{X}_{N(t)+1},\underset{\sim}{\xi}(t-S_{N(t)}))$.

Cumulative processes are C processes with $m = d$. $\underset{\sim}{G}(t)$ is a transient process if the lifetimes underlying $\underset{\sim}{G}$ have distribution F; it is proper if they have distribution $\tilde{F}$. The transient and proper processes are said to be homothetic and their expected values are related.

In the work that follows, $e^{-\sigma S_{N(t)+1}}\underset{\sim}{G}(t)$ is the m-dimensional vector whose j^{th} component is $e^{-\sigma S_{N(t)+1}}G_j(t)$. We note that

$$\tilde{E}e^{-\sigma S_{N(t)+1}}\underset{\sim}{G}(t) = \tilde{E}e^{-\sigma S_{N(t)+1}}\chi(S_{N(t)+1} < \infty)\underset{\sim}{G}(t)$$

since $\tilde{E}$ refers to a proper process in which the lifetimes are finite. Hence

$$\tilde{E}e^{-\sigma S_{N(t)+1}}\underset{\sim}{G}(t) =$$

$$\sum_{n=0}^{\infty}\int_{\{N(t)=n\}\cap\{S_{n+1} < \infty\}}\underset{\sim}{G}(t,x_1,\ldots,x_{n+1},\underset{\sim}{y}_1,\ldots,\underset{\sim}{y}_n,\underset{\sim}{\xi})e^{-\sigma S_{n+1}}\tilde{G}(dx_1,\underset{\sim}{dy}_1)\ldots$$
$$\ldots\tilde{G}(dx_n,\underset{\sim}{dy}_n)\tilde{T}(dx_{n+1},d\underset{\sim}{\xi},t-s_n)$$

$$= \sum_{n=0}^{\infty}\int_{\{N(t)=n\}\cap\{S_{n+1} < \infty\}}\underset{\sim}{G}(t,x_1,\ldots,x_{n+1},\underset{\sim}{y}_n,\underset{\sim}{\xi})\ G(dx_1,\underset{\sim}{dy}_1)\ldots$$
$$\ldots G(dx_n,\underset{\sim}{dy}_n)\ T(dx_{n+1},d\underset{\sim}{\xi},t-s_n)$$

$$= E\underset{\sim}{G}(t)A(t) \qquad (3.5)$$

Let us write $S_{N(t)+1} = t + \zeta_t$; the random variable ζ_t is called the forward delay. When F is proper and $\mu_1(F) < \infty$, ζ_t has limiting distribution

$$K(x) = \mu_1^{-1}\int_0^x [1-F(u)]du \text{ as } t \to \infty . \tag{3.6}$$

Thus we can rewrite (3.5) as

$$e^{-\sigma t}\, \tilde{E}e^{-\sigma\zeta_t} \underset{\sim}{G}(t) = E\underset{\sim}{G}(t)A(t) \tag{3.7}$$

At this point we need:

<u>Lemma 3.1</u> If $\tilde{\mu}_1, < \infty$, then $q(t) \sim \dfrac{(1-\omega)e^{-\sigma t}}{\sigma\tilde{\mu}_1}$.

Proof: Let $G(t) \equiv 1$ in (3.7); obviously, this $G(t)$ is a C process. Then

$$e^{-\sigma t}\, \tilde{E}e^{-\sigma\zeta_t} = q(t) \tag{3.8}$$

and from (3.6)

$$\lim_{t\to\infty} \tilde{E}e^{-\sigma\zeta_t} = \tilde{K}^*(\sigma) = \frac{1-\omega}{\sigma\tilde{\mu}_1} . \qquad \square$$

Hence equations (3.7) and (3.8) yield

$$E[\underset{\sim}{G}(t)|A(t)=1] = \frac{\tilde{E}e^{-\sigma\zeta_t}\underset{\sim}{G}(t)}{\tilde{E}e^{-\sigma\zeta_t}} \tag{3.9}$$

We are therefore led to consider the circumstances when

$$\frac{\tilde{E}e^{-\sigma\zeta_t}\underset{\sim}{G}(t)}{\tilde{E}e^{-\sigma\zeta_t}} - \tilde{E}\, \underset{\sim}{G}(t) \to \underset{\sim}{0}_{m\times 1} \text{ as } t \to \infty , \tag{3.10}$$

for when this holds, we have

$$E[\underset{\sim}{G}(t)|A(t)=1] - \tilde{E}\, \underset{\sim}{G}(t) \to \underset{\sim}{0} \text{ as } t \to \infty , \tag{3.11}$$

and we can apply our considerable knowledge about $\tilde{E}\, \underset{\sim}{G}(t)$ to the transient process.

The notions of sluggish events and processes are helpful in finding conditions guaranteeing that (3.11) holds.

<u>Definition 2</u>. An event $A(t)$ is <u>sluggish</u> if $P\{\chi(A(t)) \neq \chi(A(t+T))\} \to 0$ as $t \to \infty$ for all fixed $T > 0$. $A(t)$ is a <u>type C</u> event if $\chi(A(t))$ is a C process.

The following theorem is a step toward the desired conditions assuring the asymptotic independence of $\underset{\sim}{G}(t)$ and ζ_t.

<u>Theorem 3.2</u> Suppose $A(t)$ is a sluggish type C event. Then if $\tilde{\mu}_1 < \infty$,

$$\tilde{P}\{\zeta_t \le x,\, A(t)\} - \tilde{K}(x)\,\tilde{P}\{A(t)\} \to 0 \text{ as } t \to \infty\,. \tag{3.12}$$

Proof: It will save trouble if we let, for example, $\tilde{P}\{A,B\}$ mean $\tilde{P}\{A \cap B\}$. Let $G_{A_t}(y) = P\{\zeta_t \le y,\, A(t)\}$ and choose $\varepsilon > 0$.

$$\begin{aligned}
\tilde{P}\{\zeta_{t+\tau} \le x, A(t)\} &= \tilde{P}\{\zeta_{t+\tau} \le x, S_{N(t)+1} \le t+\tau,\, A(t)\} \\
&\quad + P\{\zeta_{t+\tau} \le x, S_{N(t)+1} > t+\tau, A(t)\} \\
&= P\{\zeta_{t+\tau} \le x, \zeta_t \le \tau, A(t)\} + P\{\tau < \zeta_t \le \tau+x, A(t)\} \\
&= \int_0^{\tau} P\{\zeta_{\tau-y} \le x\} G_{A_t}(dy) + \int_{\tau}^{\tau+x} G_{A_t}(dy)
\end{aligned} \tag{3.13}$$

The last equality follows from the regenerative structure of the renewal process. We know that when $\tilde{\mu}_1 < \infty$, $\lim\limits_{t\to\infty} \tilde{P}\{\zeta_t \le x\} = \tilde{K}(x)$, a continuous distribution function. Thus the convergence is uniform with respect to x. Hence there is some $T(\varepsilon)$ such that

$$|\tilde{P}\{\zeta_t \le x\} - \tilde{K}(x)| < \varepsilon \text{ for all } x$$

whenever $t \ge T$. Let $\tau = 2T$ in (3.13). Then

$$\tilde{P}\{\zeta_{t+2T} \le x, A(t)\} = \int_0^{2T} \tilde{P}\{\zeta_{2T-y} \le x\} G_{A_t}(dy) + \int_{2T}^{2T+x} G_{A_t}(dy) \tag{3.14}$$

$$\begin{aligned}
&\le \int_0^{T} \tilde{P}\{\zeta_{2T-y} \le x\} G_{A_t}(dy) + \int_T^{\infty} G_{A_t}(dy) \\
&\le [\tilde{K}(x)+\varepsilon]\tilde{P}\{A(t)\} + \tilde{P}\{\zeta_t > T\} \\
&\le [\tilde{K}(x)+\varepsilon]\tilde{P}\{A(t)\} + 1-\tilde{K}(T)+\varepsilon\;\; .
\end{aligned}$$

But sufficiently large T, $1-\tilde{K}(T) < \varepsilon$ and thus

$$P\{\zeta_{t+2T} \le x, A(t)\} \le \tilde{K}(x)\tilde{P}\{A(T)\} + 3\varepsilon \ . \qquad (3.15)$$

(3.14) also yields

$$\tilde{P}\{\zeta_{t+2T} \le x, A(t)\} \ge [\tilde{K}(x)-\varepsilon]G_{A_t}(T) = [\tilde{K}(x)-\varepsilon][\tilde{P}\{A(t) - \tilde{P}\{A(t), \zeta_t > T\}]$$

$$\ge \tilde{K}(x)P\{A(t)\} - \tilde{P}\{A(t), \zeta_t > T\} - \varepsilon$$

$$\ge \tilde{K}(x)\tilde{P}\{A(t)\} - [1-\tilde{K}(T)+\varepsilon] - \varepsilon$$

$$\ge \tilde{K}(x)\tilde{P}\{A(t)\} - 3\varepsilon \ . \qquad (3.16)$$

Therefore

$$\overline{\lim_{t\to\infty}} \left|\tilde{P}\{\zeta_{t+2T} \le x, A(t)\} - \tilde{K}(x)\tilde{P}\ \{A(t)\}\right| < 3\varepsilon(T) \qquad (3.17)$$

where $\varepsilon(T) \to 0$ as $T \to \infty$. Because $A(t)$ is sluggish,

$$\tilde{P}\{A(t)\} - \tilde{P}\{A(t+2T)\} \to 0 \text{ and } \tilde{P}\{\zeta_{t+2T} \le x, A(t)\} - \tilde{P}\{\zeta_{t+2T} \le x, A(t+2T)\}$$

$$= \tilde{P}\{\zeta_{t+2T} \le x,\ A(t),\ A(t+2T)^c\} - \tilde{P}\{\zeta_{t+2T} \le x,\ A(t)^c,\ A(t+2T)\} \to 0$$

$$\text{as } t \to \infty \ .$$

Therefore, from (3.17),

$$\overline{\lim_{t\to\infty}} \left| \tilde{P}\{\zeta_{t+2T} \le x,\ A(t+2T)\} - \tilde{K}(x)\tilde{P}\{A(t+2T)\}\right|$$

$$= \overline{\lim_{t\to\infty}} \left| \tilde{P}\{\zeta_{t+2T} \le x,\ A(t)\} - \tilde{K}(x)\tilde{P}\{A(t)\}\right| < 3\varepsilon(T) \ .$$

The theorem follows easily. □

To translate the asymptotic independence of ζ_t and $A(t)$ into an independence result for ζ_t and some process $\underset{\sim}{G}(t)$, we define <u>sluggish</u> processes. A real valued process $G(t)$ is <u>sluggish</u> if for all x outside a set E of Lebesgue measure 0 and for all fixed $T > 0$

$$\tilde{P}\{G(t) \le x < G(t+T)\} + \tilde{P}\{G(t) > x \ge G(t+T)\} \to 0 \text{ as } t \to \infty \ .$$

A vector valued process $\underset{\sim}{G}(t)$ is sluggish if each component $G_j(t)$ is sluggish, $1 \le j \le m$.

Lemma 3.3

If $G(t)$ is real valued and has a limiting proper distribution, then $G(t)$ is sluggish if and only if $G(t+T) - G(T) \overset{\tilde{P}}{\to} 0$ as $t \to \infty$.

Proof:

(i) Suppose $G(t)$ is sluggish and has limiting distribution J. Fix $\varepsilon > 0$. There exists some $N(\varepsilon)$ such that $\pm N$ are continuity points of J and $J(N) - J(-N) > 1-\varepsilon$. Let $A_N = (-N,N]$.

$$P\{|G(t+T)-G(t)|> \varepsilon\} \le P\{|G(t+T)-G(t)|> \varepsilon,\ G(t+T) \in A_N ,$$
$$G(t) \in A_N\} + \tilde{P}\{G(t+T) \in A_N{}^c\} + P\{G(t) \in A_N{}^c\} . \tag{3.18}$$

Let $-N = x_0 < x_1 < \cdots < x_M = N$ be a partition of A_N such that $x_{j+1}-x_j < \frac{\varepsilon}{2}$ and $x_j \in E^c$

$$P\{|G(t+T)-G(t)|> \varepsilon,\ G(t+T) \in A_N,\ \ G(t) \in A_N\}$$
$$\le \sum_{j=0}^{M-1} P\{G(t+T)\in(x_j,x_{j+1}],\ G(t) \notin (x_j,x_{j+1}]\}$$
$$\le \sum_{j=0}^{M-1} P\{G(t) \le x_j < G(t+T)\} + \sum_{j=0}^{M-1} P\{G(t+T) \le x_j < G(t)\} \to 0 \text{ as } t \to \infty$$

since G is sluggish. (3.19)

In addition, we see that

$$\overline{\lim_{t\to\infty}}\, [P\{G(t+T) \in A_N{}^c\}+ P\{G(t) \in A_N{}^c\}] \le 2\varepsilon, \tag{3.20}$$

where ε is arbitrary; thus (3.18), (3.19), and (3.20) imply

$$G(t+T) - G(t) \overset{\tilde{P}}{\to} 0 .$$

(ii) Now suppose $G(t)$ has limiting distribution J and $G(t+T) - G(t) \overset{\tilde{P}}{\to} 0$.

Define $E = \{x: J(x) \neq J(x-)\}$; E has Lebesgue measure 0. Let $x \in E^c$ and choose $\varepsilon > 0$.

$$\tilde{P}\{G(t) \le x < G(t+T)\} = \tilde{P}\{G(t) \le x\} - \tilde{P}\{G(t) \le x,\ G(t+T) \le x\}$$

$$\le \tilde{P}\{G(t) \le x\} - \tilde{P}\{G(t) \le x-\varepsilon, |G(t+T)-G(t)| < \varepsilon\}$$

$$= \tilde{P}\{G(t) \le x\} - \tilde{P}\{G(t) \le x-\varepsilon\} + \tilde{P}\{G(t) \le x-\varepsilon, |G(t+T)-G(t)| \ge \varepsilon\}$$

$$\le \tilde{P}\{G(t) \le x\} - \tilde{P}\{G(t) \le x-\varepsilon\} + \tilde{P}\{|G(t+T)-G(t)| \ge \varepsilon\} \qquad (3.21)$$

$$\to J(x) - J(x-\varepsilon) \text{ by assumption.}$$

But x is a continuity point of J and ε is arbitrary. Thus $\tilde{P}\{G(t) \le x < G(t+T)\} \to 0$ as $t \to \infty$; the argument for $\tilde{P}\{G(t) > x \ge G(t+T)\}$ is exactly the same. □

Theorem 3.4 Suppose $\underset{\sim}{G}(t)$ is a sluggish C process of dimension m and that $\Phi(x)$ is a function of bounded variation, $0 \le x < \infty$. If, for every $\varepsilon > 0$ there is a $\Delta(\varepsilon)$ making

$$\tilde{E}|G_j(t)|\chi(|G_j(t)| \ge \Delta) < \varepsilon \quad , \quad 1 \le j \le m ,$$

for all sufficiently large t and if $\tilde{\mu}_1 < \infty$, then

$$\tilde{E}\Phi(\zeta_t)\underset{\sim}{G}(t) - \tilde{E}\underset{\sim}{G}(t) \int_{0^-}^{\infty} \Phi(x)\tilde{K}(dx) \to \underset{\sim}{0} \text{ as } t \to \infty \qquad (3.22)$$

Proof: We will assume with no loss of generality that $\Phi(x) \ge 0$. Let A(t) be a sluggish type C event and define

$$Z_A(x,t) \equiv \tilde{P}\{\zeta_t \le x, A(t)\} - \tilde{K}(x)\ \tilde{P}\{A(t)\} .$$

Note that $Z_A(x,t) \to 0$ as $t \to \infty$ by Theorem 3.2. Also, $Z_A(0-,t) = 0$; $Z_A(\infty,t) = 0$; $|Z_A(x,t)| \le 1$ for all (x,t); and, for fixed t, $Z_A(x,t)$ is a function of bounded variation in x, $0 \le x < \infty$.

Because $\Phi(x)$ is of bounded variation and $Z_A(x,t)$ is uniformly bounded,

$$\int_{0^-}^{\infty} Z_A(x,t)\Phi(dx) \to 0 \text{ as } t \to \infty .$$

Integrating by parts one easily finds that

$$\int_{0^-}^{\infty} \Phi(x)\ Z_A(dx,t) \to 0 \qquad \text{as } t \to \infty .$$

But by definition of $Z_A(x,t)$, this simply says

$$\tilde{E}\Phi(\zeta_t)\chi(A(t)) - \tilde{P}\{A(t)\} \int_{0^-}^{\infty} \Phi(x)\tilde{K}(dx) \to 0 \text{ as } t \to \infty \quad .$$

Hence if $\underset{\sim}{H}(t)$ is an m dimensional process with j^{th} component $H_j(t) = \sum_{k=1}^{N_j} \alpha_{jk}\chi(A_{jk}(t))$, where each event $A_{jk}(t)$ is a sluggish type C event and the $\{\alpha_{jk}\}$ are arbitrary reals, we can conclude immediately that

$$\tilde{E}\Phi(\zeta_t)\underset{\sim}{H}(t) - \tilde{E}\underset{\sim}{H}(t)\int_{0^-}^{\infty} \Phi(x)\tilde{K}(dx) \to \underset{\sim}{0} \text{ as } t \to \infty . \tag{3.23}$$

We shall call a process like $\underset{\sim}{H}(t)$ a <u>simple</u> process.

Let $\underset{\sim}{G}(t)$ be any bounded, sluggish C process. Then for any small $\delta > 0$ we can, by choosing the $\{\alpha_{jk}\}$ suitably and letting $A_{jk}(t) \equiv \{\alpha_{jk} < G_j(t) \le \alpha_{jk} + \delta\}$, construct a simple process $\underset{\sim}{H}(t)$ with j^{th} component $H_j(t) = \sum_{k=1}^{N_j} \alpha_{jk}\chi(A_{jk}(t))$ such that each $A_{jk}(t)$ is a sluggish type C event and

$$H_j(t) < G_j(t) \le H_j(t) + \delta \tag{3.24}$$

Hence if $D_j(t) \equiv \tilde{E}\Phi(\zeta_t)G_j(t) - \tilde{E}\ G_j(t) \int_{0^-}^{\infty} \Phi(x)\tilde{K}(dx)$,

we see

$$D_j(t) \le \tilde{E}\Phi(\zeta_t)[H_j(t)+\delta] - \tilde{E}H_j(t)\int_{0^-}^{\infty} \Phi(x)\tilde{K}(dx) . \tag{3.25}$$

Since $\Phi(\cdot)$ is bounded (and positive) we can find a finite C such that $\Phi(x) = |\Phi(x)| \le C$ for all $x \ge 0$. We then infer from (3.23) and (3.25) that

$$\overline{\lim_{t\to\infty}}\ D_j(t) \le \delta\ C.$$

Since δ is arbitrarily small, $\overline{\lim_{t\to\infty}}\ D_j(t) \le 0$, $1 \le j \le m$. A similar argument shows

$$\overline{\lim_{t\to\infty}}\ D_j(t) \ge 0,\ 1 \le j \le m.$$

Therefore equation (3.22) holds whenever $\underset{\sim}{G}$ is bounded.

The extension of this result to unbounded processes $\underset{\sim}{G}$ whose tails exhibit the integrability property hypothesized is now easy. Choose ε and a concomitant Δ. Let

$$\underset{\sim}{G}_1(t) \equiv \begin{cases} \underset{\sim}{G}(t) & \text{whenever } |G_j(t)| < \Delta \text{ , } 1 \le j \le m \\ \underset{\sim}{0} & \text{otherwise} \end{cases}$$

and

$$\underset{\sim}{G}_2(t) \equiv \underset{\sim}{G}(t) - \underset{\sim}{G}_1(t)$$

Then

$$\tilde{E}\Phi(\zeta_t)\underset{\sim}{G}(t) = E\Phi(\zeta_\epsilon)\underset{\sim}{G}_1(t) + E\Phi(\zeta_\epsilon)\underset{\sim}{G}_2(t) \text{ .}$$

A simple approximation argument using $|\Phi(x)| \le C$ will obviously complete the proof. □

The following lemma helps one verify that the uniform integrability condition of the previous theorem is satisfied. The J_t appearing in the lemma is intended to be the distribution function of a real valued process $G(t)$ or a single component of a vector valued process.

<u>Lemma 3.5</u> For each $t \ge 0$, let $J_t(\cdot)$ be a distribution function. Assume that

$$J_t \overset{W}{\to} J \quad \text{as } t \to \infty$$

and

$$\int |x| \; J_t(dx) \to \int |x| \; J(dx) < \infty \tag{3.26}$$

Then, given $\varepsilon > 0$, there is a $\Delta(\varepsilon)$ such that

$$\int_{\{|x| \ge \Delta\}} |x| J_t(dx) < \varepsilon$$

for all sufficiently large t.

Proof: Choose Δ so that

$$I_\Delta \equiv \int_{|x| > \frac{1}{2}\Delta} |x| J(dx) < \varepsilon \text{ .}$$

Let

$$g(x) = \begin{cases} |x| & \text{for } 0 \le |x| < \tfrac{1}{2}\Delta \\ \Delta - |x| & \text{for } \tfrac{1}{2}\Delta \le |x| < \Delta \\ 0 & \text{for } |x| \ge \Delta \end{cases}$$

Then by the Helly-Bray theorem, as $t \to \infty$,

$$\int g(x)\, J_t(dx) \to \int g(x)\, J(dx). \tag{3.27}$$

If we subtract this result from (3.26), it follows immediately that

$$\overline{\lim_{t\to\infty}} \int_{\{|x| \ge \Delta\}} |x|\, J_t(dx) \le I_\Delta < \varepsilon$$

The lemma follows easily.

§4 RESULTS AND EXAMPLES

We now use the technical lemmas of the last section to derive results about transient cumulative processes which parallel those results cited in Section 1. Although we state these results in the case $d = 1$, given appropriate assumptions about $\tilde{\mu}_{rs}(j)$, $1 \le j \le d$, multivariate analogs can be derived easily. We will appeal to Theorem 3.4 repeatedly; each time we will take $\Phi(x) = e^{-\sigma x}$.

Lemma 4.1 Let $W(t) \in \mathbb{R}'$ be a cumulative process such that $EY_1^* = \kappa_1^* < \infty$ and $\tilde{\mu}_1 < \infty$. Then for every $\varepsilon > 0$,

$$P\left\{ \left| \frac{W(t)}{t} - \frac{\tilde{\kappa}_1}{\tilde{\mu}_1} \right| > \varepsilon \,\middle|\, A(t) = 1 \right\} \to 0 \text{ as } t \to \infty \tag{4.1}$$

Proof: Let $A(t) \equiv \left\{ \left| \frac{W(t)}{t} - \frac{\tilde{\kappa}_1}{\tilde{\mu}_1} \right| > \varepsilon \right\}$. $A(t)$ is sluggish because

$$\tilde{P}\{\chi(A(t)) \ne \chi(A(t+T))\} \le \tilde{P}\{A(t)\} + \tilde{P}\{A(t+T)\} \to 0 \text{ as } t \to \infty$$

since $\frac{W(t)}{t} \to \frac{\tilde{\kappa}_1}{\tilde{\mu}_1}$ a.s. $(\tilde{P})$ when $\tilde{\mu}_1 < \infty$ and $\tilde{\kappa}_1^* < \infty$. Thus

$$P\left\{\left|\frac{W(t)}{t} - \frac{\tilde{\kappa}_1}{\tilde{\mu}_1}\right| > \varepsilon \;\middle|\; A(t) = 1\right\}$$

$$= \frac{\tilde{E}e^{-\sigma\zeta_t}\chi(A(t))}{\tilde{E}e^{-\sigma\zeta_t}} = \tilde{P}(A(t)) + o(1)$$

by Theorem 3.4. The lemma follows.

Lemma 4.2 Suppose W(t) is a real-valued cumulative process such that $\tilde{\mu}_1 < \infty$ and $\tilde{\kappa}_2^* < \infty$. Then

$$\eta_1(t) \equiv \frac{W(t) - \tilde{\kappa}_1 N(t)}{\tilde{\sigma}_y\sqrt{\frac{t}{\tilde{\mu}_1}}}$$

is a sluggish process.

Proof:

From (1.7), we know that $\tilde{P}\{\eta_1(t) \le \alpha\} \to \Phi(\alpha)$ and hence by Lemma 3.3 it is sufficient to prove that $\eta_1(t+T) - \eta_1(t) \overset{\tilde{P}}{\to} 0$.

$$\tilde{P}\left\{\left|\frac{W(t+T) - \tilde{\kappa}_1 N(t+T)}{\tilde{\sigma}_y\sqrt{\frac{t+T}{\tilde{\mu}_1}}} - \frac{W(t) - \tilde{\kappa}_1 N(t)}{\tilde{\sigma}_y\sqrt{\frac{t}{\tilde{\mu}_1}}}\right| > \varepsilon\right\}$$

$$= \tilde{P}\left\{\left|\frac{W(t+T)-W(t)}{\tilde{\sigma}_y\sqrt{\frac{t+T}{\tilde{\mu}_1}}} - \frac{\tilde{\kappa}_1[N(t+T)-N(t)]}{\tilde{\sigma}_y\sqrt{\frac{t+T}{\tilde{\mu}_1}}}\right.\right.$$

$$\left.\left.\frac{W(t) - \tilde{\kappa}_1 N(t)}{\sigma_y\sqrt{\frac{t}{\tilde{\mu}_1}}}\left(1 - \sqrt{\frac{t}{t+T}}\right)\right| > \varepsilon\right\}. \qquad (4.2)$$

Hence

$$\tilde{P}\{|\eta_1(t+T)-\eta_1(t)| > \varepsilon \}$$

$$\leq \tilde{P}\left\{\left| \frac{W(t+T)-W(S_{N(t+T)})+W(S_{N(t+T)})-W(S_{N(t)+1})+W(S_{N(t)+1})-W(t)}{\tilde{\sigma}_y\sqrt{\frac{t+T}{\tilde{\mu}_1}}} \right| > \frac{\varepsilon}{3} \right\}$$

$$+ \tilde{P}\left\{\left| \frac{\tilde{\kappa}_1[N(t+T)-N(t)]}{\tilde{\sigma}_y\sqrt{\frac{t+T}{\tilde{\mu}_1}}} \right| > \frac{\varepsilon}{3} \right\} + \tilde{P}\left\{\left| \frac{W(t)-\tilde{\kappa}_1 N(t)}{\tilde{\sigma}_y\sqrt{\frac{t}{\tilde{\mu}_1}}} \right| > \frac{\varepsilon\sqrt{t+T}}{3(\sqrt{t+T}-\sqrt{t})} \right\}$$

$$= A + B + C. \tag{4.3}$$

In his paper introducing the concept of cumulative processes, Smith (1955) proved that if $\tilde{\mu}_1 < \infty$ and $\tilde{\kappa}_2^* < \infty$,

(i) $\dfrac{W(t)-W(S_{N(t)})}{\sqrt{t}} \to 0$ a.s. $(\tilde{P})$ and

(ii) $\dfrac{W(S_{N(t)+1})-W(t)}{\sqrt{t}} \to 0$ a.s. $(\tilde{P})$.

These two results coupled with the fact that

$$\frac{W(S_{N(t+T)})-W(S_{N(t)+1})}{\sqrt{t+T}} = \frac{\sum_{j=N(t)+2}^{N(t+T)} Y_j}{N(t+T)-N(t)-1} \cdot \frac{N(t+T)-N(t)-1}{\sqrt{t+T}} \to 0$$

a.s. $(\tilde{P})$ by the Strong Law of Large Numbers ensure that $A \to 0$. $B \to 0$ by the Strong Law also. Finally, $C \to 0$ by the asymptotic normality of $\eta_1(t)$.

<u>Lemma 4.3</u> Suppose $W(t)$ is a cumulative process such that $\tilde{\mu}_2 < \infty$ and $\tilde{\kappa}_2^* < \infty$. Then

$$\eta_2(t) = \frac{W(t) - \frac{\tilde{\kappa}_1 t}{\tilde{\mu}_1}}{\sqrt{\frac{\tilde{\gamma} t}{\tilde{\mu}_1}}} \text{ is sluggish.} \tag{4.4}$$

Proof: (1.8) and obvious modifications to the argument in Lemma 4.2 will suffice.

Theorem 4.4 Let W(t) be a cumulative process such that $\tilde{\mu}_1 < \infty$ and $\tilde{\kappa}_2^* < \infty$. Then

$$\text{(i)} \quad P\left\{ \frac{W(t)-\tilde{\kappa}_1 N(t)}{\tilde{\sigma}_y \sqrt{\frac{t}{\tilde{\mu}_1}}} \le \alpha \mid A(t)=1 \right\} \to \Phi(\alpha) \tag{4.5}$$

(ii) If, in addition, $\tilde{\mu}_2 < \infty$,

$$P\left\{ \frac{W(t) - \frac{\tilde{\kappa}_1 t}{\tilde{\mu}_1}}{\sqrt{\frac{\tilde{\gamma} t}{\tilde{\mu}_1}}} \le \alpha | A(t) = 1 \right\} \to \Phi(\alpha) . \tag{4.6}$$

Proof: Using the notation $\eta_j(t)$ established in Lemmas 4.2 and 4.3, $\{\eta_j(t) \le \alpha\}$ is a sluggish event if condition j holds; j = 1, 2.

$$P\{\eta_j(t) \le \alpha | A(t)=1\} = \frac{\tilde{E} e^{-\sigma\zeta_t} \chi(\eta_j(t) \le \alpha)}{\tilde{E} e^{-\sigma\zeta_t}} \to \Phi(\alpha) \text{ by Theorem 3.4.} \quad \square$$

Let us re-examine the "time until ruin" problem. We found that

$$P\{\tau^* \le t\} = P\{\sum_{k=1}^{M(t)} L_k > u\}$$

where M(t) is the renewal count for the transient renewal process $\{Z_i\}$. Note that $L_k^* \equiv L_k$. Let $J(z,\ell)$ be the joint distribution of Z_1 and L_1 and suppose there is a σ making

$$\int_0^\infty \int_0^\infty e^{\sigma z} J(dz,d\ell) = 1.$$

Define $J(dz,d\ell) = e^{\sigma z} J(dz,d\ell)$. If $\tilde{E} Z_1^2 < \infty$ and $\tilde{E} L_1^2 < \infty$, then Theorem 4.4 implies

$$P\left\{ \frac{\sum_{k=1}^{M(t)} L_k - \frac{\tilde{E}L_1 t}{\tilde{E}Z_1}}{\sqrt{\frac{\tilde{\gamma}t}{\tilde{E}Z_1}}} > \frac{u - \frac{\tilde{E}L_1 t}{\tilde{E}Z_1}}{\sqrt{\frac{\tilde{\gamma}t}{\tilde{E}Z_1}}} \,\Big|\, A(t) = 1 \right\}$$

$$\sim\Phi\left(\frac{\frac{\tilde{E}L_1 t}{\tilde{E}Z_1} - u}{\sqrt{\frac{\tilde{\gamma}t}{\tilde{E}Z_1}}} \right) \to 1 \text{ as } t \to \infty \; .$$

where $\tilde{\gamma} = E(L_1 - \frac{\tilde{E}L_1}{\tilde{E}Z_1} Z_1)^2$. Therefore

$$P\{\tau^* \le t\} \sim \frac{(1-\omega)e^{-\sigma t}}{\sigma \tilde{E}Z_1} \tag{4.7}$$

where

$$1-\omega = P\{Z_1 = \infty\} \; .$$

Equation (4.7) agrees with Cramer's estimate for the time until ruin (1955).

<u>Theorem 4.5</u> Suppose $W(t)$ is a cumulative process such that $\tilde{\mu}_2 < \infty$ and $\tilde{\kappa}_2{}^* < \infty$. Then

$$E[(W(t) - \frac{\tilde{\kappa}_1 t}{\tilde{\mu}_1})^2 \,|\, A(t) = 1] = \frac{\tilde{\gamma}t}{\tilde{\mu}_1} + o(t) \; . \tag{4.8}$$

Proof: Let

$$\eta_2(t) = \frac{W(t) - \frac{\tilde{\kappa}_1 t}{\tilde{\mu}_1}}{\sqrt{\frac{\tilde{\gamma}t}{\tilde{\mu}_1}}} \; , \text{ as before.}$$

$\tilde{P}\{\eta_2(t)^2 \le \alpha\} \to 2\Phi(\sqrt{\alpha}) - 1$, $\alpha \ge 0$ since

$\tilde{P}\{\eta_2(t) \le \alpha\} \to \Phi(\alpha)$.

Smith (1955) has shown that $\tilde{E}\eta_2(t)^2 \to 1$ which is, of course, the first moment of the limiting distribution of $\eta_2(t)^2$. Note that as $\eta_2(t)$ is a sluggish process, $\eta_2(t)^2$ is too. Therefore, by Theorem 3.4 and Lemma 3.5,

$$\frac{\tilde{E}e^{-\sigma\zeta_t}\eta_2(t)^2}{\tilde{E}e^{-\sigma\zeta_t}} - \tilde{E}\eta_2(t)^2 \to 0 \text{ as } t \to \infty \,. \tag{4.9}$$

Hence

$$E[\eta_2(t)^2 | A(t) = 1] = \frac{Ee^{-\sigma\zeta_t}\eta_2(t)^2}{\tilde{E}e^{-\sigma\zeta_t}} \to 1. \qquad \square$$

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A cumulative process based on a transient renewal process is defined. Various applications are described. The transient cumulative process is then studied conditionally upon "surviving". It is shown that in this conditional sense various theorems about ordinary cumulative processes (particularly asymptotic normality theorems) carry over to the present case.

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