ON ESTIMATORS OF BUNDLE-STRENGTH IN LENGTH-BIASED SAMPLING

by

Pranab Kumar Sen

Department of Biostatistics
University of North Carolina at Chapel Hill

Institute of Statistics Mimeo Series No. 1426

January 1983

ON ESTIMATORS OF BUNDLE-STRENGTH IN LENGTH-BIASED SAMPLING *

BY PRANAB KUMAR SEN

University of North Carolina, Chapel Hill.

Under a length-biased sampling scheme, an estimator of the strength of a bundle of parallel filaments and its jackknife version are considered. Various properties of these estimators are studied. Jackknife estimator of the variance function is also considered and its convergence property is studied.

AMS Subject Classification : 62E20, 60K99

Key Words and Phrases : Almost sure convergence, asymptotic normality, bias, bundle-strength of filaments, jackknife estimator, law of iterated logarithm, length-biased sampling, moment convergence.

* Work supported by the Office of Naval Research, Contract No. NR 681-001.

1. Introduction.

Daniels (1945) in the context of the distribution theory of the strength of a bundle of parallel filaments considered the statistic

$$(1.1) \qquad D_n = \max\{ (n-i+1)X_{n:i} : 1 \leq i \leq n \},$$

where $X_{n:1} <\dots< X_{n:n}$ stand for the order statistics corresponding to n independent and identically distributed (i.i.d.) nonnegative random variables (r.v.) $X_1,\dots,X_n$ from a continuous distribution function (d.f.) F, defined on $R^+ = (0,\infty)$. It may be noted [viz., Suh et. al (1970) and Sen(1973)] that $Z_n = n^{-1}D_n$ actually estimates the parameter

$$(1.2) \qquad \theta = \theta(F) = \sup\{ x[1-F(x)] : x \varepsilon R^+ \}.$$

It the context of some sampling problems in technology, Cox(1969) has stressed on the role of length-biased (l.b.) sampling with some emphasis on the estimation problems with the bundle-strength of fibres. A d.f. G($= G_F$) is called a <u>length-biased distribution</u> corresponding to a d.f. F, if

$$(1.3) \qquad G_F(x) = \mu^{-1} \int_0^x ydF(y) , \qquad y \varepsilon R^+$$

where

$$(1.4) \qquad \mu = \int_0^x xdF(x) \text{ is assumed to be finite and positive.}$$

For some other applications of l.b. sampling, we may also refer to Patil and Rao(1977,1978) and Coleman(1979), among others.

We conceive of a set $\{Y_1,\dots,Y_n\}$ of i.i.d.nonnegative r.v.'s from the l.b. distribution G, and our problem is to provide a suitable estimator of $\theta(F)$ in (1.2) (based on the Y_i) and to study its various properties. Along with the preliminary notions, the proposed estimator is introduced in Section 2. Asymptotic properties of the estimators are studied in Section 3. In this context, weak convergence results on the empirical l.b. distribution play a vital role and some of these are also presented along with. Some strong convergence results are also presented side by side. A jackknife variance estimator is presented in the last section.

2. **The estimator.** Note that by (1.3), $dG(x) = \mu^{-1}x dF(x)$, $x \in R^+$, so that $1-F(x) = \int_x^\infty dF(x) = \mu \int_x^\infty y^{-1} dG(y) = \mu K(x)$, say. As such, by (1.2) and (1.3),

$$\theta = \sup\{\mu x K(x): x \in R^+\} = (\sup\{xK(x): x \in R^+\})/K(0). \tag{2.1}$$

Note that $xK(x) \leq 1$ for all $x \in R^+$ and $K(0) = \mu^{-1}$, so that $\theta \leq \mu$.

Let now $G_n(y) = n^{-1}\Sigma_{i=1}^n I(Y_i \leq y)$, $y \in R^+$ be the empirical d.f. (based on the Y_i) and let

$$K_n(x) = \int_x^\infty y^{-1} dG_n(y) = n^{-1}\Sigma_{i=1}^n Y_i^{-1} I(Y_i \geq x), \quad x \in R^+. \tag{2.2}$$

Then, by (2.1) and (2.2), we consider the following estimator of θ :

$$\hat{\theta}_n = (\sup\{xK_n(x) : x \in R^+\})/K_n(0). \tag{2.3}$$

If $Y_{n:1} \leq \ldots \leq Y_{n:n}$ be the ordered r.v.'s corresponding to $Y_1,\ldots,Y_n$, then we may write (2.3) explicitly as

$$\hat{\theta}_n = \max_{1 \leq i \leq n} \{ (\Sigma_{j=i}^n Y_{n:i}/Y_{n:j})/ \Sigma_{k=1}^n Y_{n:k}^{-1} \}. \tag{2.4}$$

It may be noted that $\{K_n(x), x \in R^+ ; n \geq 1\}$ is a reverse martingale (process), and hence, $\{\sup\{xK_n(x): x \in R^+\}, n \geq 1\}$ is a reverse sub-martingale, where we note that $\sup\{xK_n(x) : x \in R^+\} \leq 1$, with probability one, for all $n \geq 1$. Also, $\{K_n(0); n \geq 1\}$ is a reverse martingale with $EK_n(0) = K(0) = \mu^{-1} < \infty$, by (1.4). Hence, by the reverse (sub-)martingale convergence theorem, as $n \to \infty$,

$$\sup\{xK_n(x): x \in R^+\} \to \sup\{xK(x): x \in R^+\}, \text{ almost surely (a.s.)} \tag{2.5}$$

and

$$K_n(0) \to K(0) \text{ a.s.} \tag{2.6}$$

From, (2.1), (2.5) and (2.6), we conclude that whenever $E_G Y^{-1} = \int_0^\infty y^{-1} dG(y) < \infty$,

$$\hat{\theta}_n \text{ converges a.s. to } \theta \text{ as } n \to \infty. \tag{2.7}$$

Note that for the strong consistency of the estimator we do not need other regularity conditions, as will be introduced subsequently for more refined results. Towards this, we define the Kolmogorov-norm

$$U_n = n^{\frac{1}{2}} \sup\{|G_n(x) - G(x)| : x \in R^+\}, \tag{2.8}$$

and note that [viz., Dvoretzky, Kiefer and Wolfowitz(1956)] for every $n \geq 1$,

(2.9) $$P\{ U_n \geq u \} \leq 2 \exp\{ -2u^2 \} \quad , \forall\, u \in R^+ ,$$

so that U_n is uniformly (in n) exponentially integrable. Further, by partial integration, we obtain that for every $n \geq 1$ and $x \in R^+$,

(2.10) $$\begin{aligned} n^{\frac{1}{2}} x |K_n(x) - K(x)| &= n^{\frac{1}{2}} x \left| \int_x^\infty y^{-1} d(G_n(y) - G(y)) \right| \\ &\leq n^{\frac{1}{2}} |G_n(x) - G(x)| + \left| x \int_x^\infty n^{\frac{1}{2}} (G_n(y) - G(y))\, y^{-2} dy \right| \\ &\leq U_n + U_n\, x \int_x^\infty y^{-2} dy = 2\, U_n , \end{aligned}$$

so that $\sup\{ n^{\frac{1}{2}} x |K_n(x) - K(x)| : x \in R^+ \} \leq 2U_n$, with probability 1, for every n. Note that the law of iterated logarithm holds for $\{U_n\}$. If we assume that

(2.11) $$\nu_G = E_G Y^{-2} = \int_0^\infty y^{-2} dG(y) < \infty \quad ,$$

then, noting that $K_n(0)$ is an average of i.i.d.r.v. with finite second moment, we conclude that under (2.11), as $n \to \infty$, (i)

(2.12) $$n^{\frac{1}{2}} \{ K_n(0) - K(0) \} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \nu_G - \mu^{-2}),$$

so that $n^{\frac{1}{2}} |K_n(0) - K(0)| = O_p(1)$, and (ii)

(2.13) $$\{ n/(\log\log n) \}^{\frac{1}{2}} |K_n(0) - K(0)| = O(1) \text{ a.s.}$$

By (2.1) and (2.3), we have then

(2.14) $$\begin{aligned} | \hat{\theta}_n - \theta | &= \left| \sup_{x \in R^+} \{ x K_n(x)/K_n(0) \} - \sup_{x \in R^+} \{ x K(x)/K(0) \} \right| \\ &\leq \sup\{ x |K_n(x) - K(x)| \}/K_n(0) + |K_n(0) - K(0)| \theta / K_n(0) , \end{aligned}$$

so that by (2.10), (2.12), (2.13) and (2.14), we conclude that under (2.11),

(2.15) $$n^{\frac{1}{2}} |\hat{\theta}_n - \theta| = O_p(1) ,$$

(2.16) $$\{ n/(\log\log n) \}^{\frac{1}{2}} |\hat{\theta}_n - \theta| = O(1) \text{ a.s., as } n \to \infty .$$

For the study of other properties of the estimator, we need some extra regularity conditions which are introduced below. We assume that $\{xK(x), x \in R^+\}$ has a unique (global) maximum (θ/μ) at a point $x_o : 0 < x_o < \infty$, so that for every δ (> 0), sufficiently small, there exists a $\eta > 0$, such that $\sup\{ xK(x) : |x - x_o| > \eta \} < \theta/\mu - \delta$. Also, we assume that in some neighbourhood of x_o , $h(x) = xK(x)$ (= $\mu^{-1} x[1-F(x)]$) has a continuous first order derivative $h'(x)$

($= \mu^{-1}\{1 - F(x) - xf(x)\}$). Note that f is the density function of the d.f. F and by definition $h'(x_o) = 0$ and

(2.17) $\xi(x,x_o) = \text{sign}(x_o - x)h'(x)$ is ≥ 0 for all $x : |x-x_o| \leq \eta$.

Finally, for the moment convergence properties of the estimator, we may also need to assume that for some $r(\geq 2)$,

(2.18) $E_G Y^r = \int_0^\infty y^r dG(y) = \mu^{-1} E_F X^{r+1} = \nu^*_{r+1} < \infty$.

We conclude this section with the remark that by definition

(2.19) $\hat{\theta}_n \geq x_o K_n(x_o)/K(0) = \theta^*_n$, say,

and, we shall show that under the regularity conditions mentioned above, for the asymptotic theory , $\hat{\theta}_n$ may be replaced by θ^*_n [on which standard theory holds under general conditions].

3. Asymptotic properties of $\hat{\theta}_n$. Note that by (2.10), (2.12) and (2.19),

(3.1) $\hat{\theta}_n \geq \theta^*_n \geq \theta - Cn^{-\frac{1}{2}}$, in probability,

where $C(0 < C < \infty)$ is a suitable constant. If we let

(3.2) $I^*_n = \{ x : xK(x)/K(0) < \theta - 2Cn^{-\frac{1}{2}} \}$,

then, proceeding as in (2.14), we conclude that

(3.3) $|\sup\{xK_n(x)/K_n(0): x \in I^*_n\} - \sup\{xK(x)/K(0): x \in I^*_n\}| = O_p(n^{-\frac{1}{2}})$,

and hence, by (3.1), (3.2) and (3.3), we obtain that as $n \to \infty$,

(3.4) $\sup\{ xK_n(x)/K_n(0) : x \in I^*_n \} < \theta^*_n$, in probability.

For n adequately large, $I_n = R^+ \setminus I^*_n = \{x : xK(x)/K(0) \geq \theta - 2Cn^{-\frac{1}{2}}\}$ reduces to a shrinking neighbourhood of x_o , so that by the assumptions made after (2.16), for $x \in I_n$, we have

(3.5) $$xK_n(x) - x_oK_n(x_o) = x[K_n(x)-K(x)]-x_o[K_n(x_o)-K(x_o)] + xK(x) - x_oK(x_o)$$
$$= x[K_n(x)-K(x)-K_n(x_o)+K(x_o)] + (x-x_o)[K_n(x_o)-K(x_o)] - \xi(x',x_o)|x-x_o| ,$$

where, by (2.17), $\xi(x',x_o)$ is nonnegative, and $x' = ax + (1-a)x_o$, $0<a<1$.

Further, as in (2.10), we have

(3.6) $n^{\frac{1}{2}}x|\ K_n(x)-K(x) - K_n(x_o) + K(x_o)| \ \leq 2U_n|\ x - x_o|/x_o\ ,\ \forall x \in I_n$,

so that by (3.5), (3.6) and (2.9), we conclude that for every $x \in I_n$,

(3.7) $xK_n(x)/K_n(0) = x_o K_n(x_o)/K_n(0) - \xi(x',x_o)|x - x_o| + o_p(n^{-\frac{1}{2}})$.

Therefore, by (3.7), $\sup\{\ xK_n(x)/K_n(0) : x \in I_n\} = x_o K_n(x_o)/K_n(0) + o_p(n^{-\frac{1}{2}})$, and hence, using (3.4), we conclude that

(3.8) $n^{\frac{1}{2}}|\hat{\theta}_n - \theta_n^*| \to 0$, in probability, as $n \to \infty$.

[In fact, in (3.1) and (3.2), we may replace $Cn^{-\frac{1}{2}}$ by $C(n/\{\text{loglog } n\})^{-\frac{1}{2}}$ and conclude that (3.4) holds a.s. with C being replaced by $C(\text{loglog } n)^{\frac{1}{2}}$. As such, (3.8) can be strengthened to $o((\text{loglog } n)^{\frac{1}{2}})$ a.s., as $n \to \infty$. If further, we assume that h(x) has a second order derivative h'' in some neighbourhood of x_o (where $h''(x_o) < 0$, by definition), then the diameter of I_n is $O(n^{-\frac{1}{4}}(\text{loglog})^{\frac{1}{4}})$, so that by the same steps, it follows that

(3.9) $n^{\frac{1}{2}}|\hat{\theta}_n - \theta_n^*| = O(n^{-\frac{1}{4}}(\text{loglog } n)^{\frac{1}{2}})$ a.s., as $n \to \infty$.

This stronger result (under more stringent conditions) is, however, not needed in the sequel.]

Let us denote by

(3.10) $W_n(x) = n^{\frac{1}{2}}\{K_n(x) - K(x)\}\ ,\ x \in R^+$.

Then, by definition, we have

(3.11)
$$\begin{aligned} n^{\frac{1}{2}}(\theta_n^* - \theta) &= n^{\frac{1}{2}}x_o\{\ K_n(x_o)/K_n(0) - K(x_o)/K(0)\ \} \\ &= n^{\frac{1}{2}}x_o\ \{\ (K(x_o) + n^{-\frac{1}{2}}W_n(x_o))/(K(0)+n^{-\frac{1}{2}}W_n(0)) - K(x_o)/K(0)\ \} \\ &= \theta\ \{\ W_n(x_o)/K(x_o) - W_n(0)/K(0) + O_p(n^{-\frac{1}{2}})\}. \end{aligned}$$

Now, parallel to (2.11), we let

(3.12) $\nu_G(x) = \int_x^\infty y^{-2}dG(y),\ x \in R^+$, so that $\nu_G = \nu_G(0)$.

The joint asymptotic normality of $(W_n(0), W_n(x_o))$ follows by a direct appeal to the central limit theorem [using the Cramér-Wold characterization], and hence, using (3.11) and (3.12), we obtain by some routine steps that as $n \to \infty$,

(3.13) $n^{\frac{1}{2}}(\theta_n^* - \theta) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^{*2})$,

where

(3.14) $\sigma^{*2} = \theta^2\{ \nu_G(x_o)(1/K(x_o) - 1/K(0))^2 + (K(0))^{-1}(\nu_G(0) - \nu_G(x_o)) \}$.

Combining (3.08), (3.13) and (3.14), we arrive at the following:

Theorem 1. Under (1.4), (2.11), (2.17) and the uniqueness of x_0, as $n \to \infty$,

(3.15) $$n^{\frac{1}{2}}(\hat{\theta}_n - \theta) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^{*2}).$$

It may be noted that for both the sequence $\{K_n(x_o)\}$ and $\{K_n(0)\}$, the law of iterated logarithm holds [under (2.11)], so that in (3.11), we have actually

(3.16) $$n^{\frac{1}{2}}(\theta_n^* - \theta) = \theta\{W_n(x_o)/K(x_o) - W_n(0)/K(0)\} + O(n^{-\frac{1}{2}}(\log\log n)^{\frac{1}{2}}) \text{ a.s.},$$

as $n \to \infty$. The first term on the right hand side of (3.16) is expressible as

(3.17) $n^{-\frac{1}{2}}\Sigma_{i=1}^n Z_i$ where $Z_i = \theta\{I(Y_i \geq x_o)/\{Y_i K(x_o)\} - \mu/Y_i\}$, $i \geq 1$.

The Z_i are i.i.d.r.v. with mean 0 and variance σ^{*2}, given by (3.14). Hence, the law of iterated logarithm applies to $\{n^{\frac{1}{2}}(\theta_n^* - \theta)\}$ as well. On the other hand, by the remarks made after (3.8), under (2.11) and (2.17), $n^{\frac{1}{2}}|\theta_n - \theta|$ $= o((\log\log n)$) a.s., as $n \to \infty$, so that the law of iterated logarithm holds for $\{n^{\frac{1}{2}}(\hat{\theta}_n - \theta)\}$. We may also note that in view of the backward invariance principles for $\{n^{\frac{1}{2}}[K_m(x) - K(x)], x \in R^+ ; m \geq n\}$ [along the lines of Theorem 2.6.3 of Sen (1981)], (3.8) may be strenghtened to $\sup\{ n^{\frac{1}{2}}|\hat{\theta}_m - \theta| : m \geq n \} \to 0$, in probability, as $n \to \infty$, while, for θ_n^*, being the ratio of two means, a similar backward invariance principle may be worked out as in Theorem 3.3.4 of Sen(1981). Thus, for the tail sequence $\{n^{\frac{1}{2}}(\hat{\theta}_m - \theta) : m \geq n\}$, a backward invariance principle (relating to the weak convergence to a Wiener process) holds under the same regularity conditions as in Theorem 1. This result is particularly useful in studying the asymptotic normality result for $\hat{\theta}_n$ for random sample sizes.

In the rest of this section, we study the bias and mean square results for the estimator $\hat{\theta}_n$. Note that by (2.10) and (2.14),

(3.18) $$n^{\frac{1}{2}}|\hat{\theta}_n - \theta| \leq 2U_n/K_n(0) + \theta|W_n(0)|/K_n(0),$$

where by the elementary inequality between the harmonic and arithmatic means, (3.19) $\{K_n(0)\}^{-1} = (n^{-1}\Sigma_{i=1}^{n} Y_i^{-1})^{-1} \leq n^{-1}\Sigma_{i=1}^{n} Y_i = \bar{Y}_n$,

so that we have

(3.20) $$n^{\frac{1}{2}}| \hat{\theta}_n - \theta | \leq \bar{Y}_n\{2U_n + \theta|W_n(0)|\} \leq \tfrac{1}{2} \bar{Y}_n^2 + \tfrac{1}{2}\{2U_n + \theta |W_n(0)|\}^2$$
$$\leq \tfrac{1}{2} \bar{Y}_n^2 + 4U_n^2 + \theta^2 W_n^2(0).$$

Now, under (2.18), for r=2, $\bar{Y}_n^2$ is uniformly (in n) integrable, while, under (1.4) and (2.11), U_n^2 and $W_n^2(0)$ are uniformly integrable. Further, by (3.8), (3.16) and (3.18), we have

(3.21) $$n^{\frac{1}{2}} (\hat{\theta}_n - \theta) = \bar{Z}_n^* + o_p(1) \ ; \ \bar{Z}_n^* = n^{-\frac{1}{2}}\Sigma_{i=1}^{n} Z_i ,$$

where the Z_i have 0 mean and finite variance σ^{*2} , and $\bar{Z}_n^{*2}$ is uniformly integrable. Hence, by (3.20), (3.21) and a version of the Dominated Convergence Theorem, we conclude that

(3.22) $$E\{ n^{\frac{1}{2}} (\hat{\theta}_n - \theta)\} \to 0 , \text{ as } n \to \infty ,$$

so that $E\hat{\theta}_n = \theta + o(n^{-\frac{1}{2}})$ which shows that the relative bias converges to 0 as $n \to \infty$. Further, by (3.18) and (3.19),

(3.23) $$n(\hat{\theta}_n - \theta)^2 \leq 8U_n^2\bar{Y}_n^2 + 2\theta^2\bar{Y}_n^2 W_n^2(0) , \text{ for every } n ,$$

where, by (2.9), for every finite k(> 0), U_n^k is uniformly (in n) integrable. Hence, using the Hölder inequality, it can be shown that $U_n^2\bar{Y}_n^2$ is uniformly integrable whenever (2.18) holds for some r > 2. Further, note that

(3.24) $$W_n^2(0)\bar{Y}_n^2 = n^{-3}\{ \Sigma_{i=1}^{n} (1-\mu^{-1}Y_i) + \Sigma_{1\leq i\neq j\leq n} Y_j (Y_i^{-1} - \mu^{-1}) \}^2 ,$$

and this is uniformly integrable whenever (2.11) and (2.18) hold. Hence, again using the Dominated Convergence Theorem along with (3.21) and (3.23), we have

(3.25) $$E\{n(\hat{\theta}_n - \theta)^2 \} \to \sigma^{*2} , \text{ as } n \to \infty .$$

Thus, in (3.15), one may also use the natural parameters and conclude that

(3.26) $$(\hat{\theta}_n - \theta)/\{\mathrm{Var}(\hat{\theta}_n)\}^{\frac{1}{2}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1) , \text{ as } n \to \infty .$$

It may be noted that (3.18), (3.19) and some simple inequalities may be used to establish the convergence of higher order moments of $n^{\frac{1}{2}}(\hat{\theta}_n - \theta)$. However, this will require higher order moment conditions on Y^{-1} as well as Y.

4. Jackknife estimators. If we write $\hat{\lambda}_n = \hat{\theta}_n K_n(0)$ and $\lambda = \theta/\mu$, then, we have

(4.1) $\hat{\theta}_n = \theta + \mu(\hat{\lambda}_n - \lambda) + (\theta/\mu)(1/K_n(0) - \mu) + (\hat{\lambda}_n - \lambda)(1/K_n(0) - \mu)$.

We have noticed earlier that $\{\hat{\lambda}_n\}$ is a reverse sub-martingale sequence converging a.s. to λ, as $n \to \infty$, and hence, it can be shown easily that $E\hat{\lambda}_n \geq \lambda$, for every finite n. Similarly, $K_n(0)$ unbiasedly estimates μ^{-1} and is nonnegative. Hence, $E\{K_n(0)\}^{-1} \geq \mu$, for every finite n, though $K_n(0)$ converges a.s. to μ^{-1} as $n \to \infty$. The last term on the right hand side of (4.1) represents a covariance term and is of lower order of magnitude . Hence, from (4.1) we conclude that upto the first order , $\hat{\theta}_n$ may not be unbiased and has a nonnegative bias, for every finite n. For this reason, it may be worthwhile to consider the jackknife estimator to reduce this bias and also to provide an estimator of the unknown variance σ^{*2}. Towards this goal, we define $\hat{\theta}_n = \hat{\theta}(\underset{\sim}{Y}_n)$, $\hat{\lambda}_n = \hat{\lambda}(\underset{\sim}{Y}_n)$, $\underset{\sim}{Y}_n = (Y_1, \ldots, Y_n)$ and $\hat{\theta}_{n-1}^{(i)} = \hat{\theta}(\underset{\sim}{Y}_{n-1}^{(i)})$, $\hat{\lambda}_{n-1}^{(i)} = \hat{\lambda}(\underset{\sim}{Y}_{n-1}^{(i)})$, where $\underset{\sim}{Y}_{n-1}^{(i)}$ is the subvector of $\underset{\sim}{Y}_n$ deleting Y_i , for $i=1,\ldots,n$. Let then

(4.2) $\hat{\theta}_{n,i} = n\hat{\theta}_n - (n-1)\hat{\theta}_{n-1}^{(i)}$, $i=1,\ldots,n$; $\hat{\hat{\theta}}_n = n^{-1}\Sigma_{i=1}^n \hat{\theta}_{n,i}$,

and

(4.3) $s_n^2 = (n-1)^{-1} \Sigma_{i=1}^n (\hat{\theta}_{n,i} - \hat{\hat{\theta}}_n)^2$.

Thus, $\hat{\hat{\theta}}_n$ is the usual jackknife estimator of θ and s_n^2 is the estimator of the variance σ^{*2}. Our basic goal is to study the properties of these estimators. For this, we define $G_{n-1}^{(i)}$, $K_{n-1}^{(i)}$ etc. as in Section 2 where $\underset{\sim}{Y}_n$ is replaced by $\underset{\sim}{Y}_{n-1}^{(i)}$, for $i=1,\ldots,n$. Also, we write

(4.4) $\hat{\theta}_n = x_n K_n(x_n)$ and $\hat{\theta}_{n-1}^{(i)} = x_{n-1}^{(i)} K_{n-1}^{(i)}(x_{n-1}^{(i)})$, $i=1,\ldots,n$;

where x_n (and the $x_{n-1}^{(i)}$) relate to the particular order statistics $(Y_{n:r})$ at which the maximum is attained. Note that under the assumed uniqueness of x_o ,

(4.5) $x_n \to x_o$ a.s., as $n \to \infty$.

Now, proceeding as in (2.10), we obtain that

(4.6) $\sup_{x \in R^+} |x\{K_{n-1}^{(i)}(x) - K_n(x)\}| \leq 2 \sup_{x \in R^+} |G_{n-1}^{(i)}(x) - G_n(x)|$

$= 2 \sup_x \{|(n-1)^{-1}\{I(Y_i \leq x) - G_n(x)\}|\} \leq 2/(n-1), \forall\, i=1,\ldots,n;$

$$(4.7)\quad \max_{1\le i\le n} |\hat{\lambda}_{n-1}^{(i)} - \hat{\lambda}_n| = \max_{1\le i\le n} \{|\sup_x xK_{n-1}^{(i)}(x) - \sup_x xK_n(x)|\}$$

$$\le \max_{1\le i\le n} \{\sup_x | x\{ K_{n-1}^{(i)}(x) - K_n(x)\}| \} \le 2/(n-1) ;$$

$$(4.8)\quad \max_{1\le i\le n} | K_{n-1}^{(i)}(0) - K_n(0)| \le (n-1)^{-1} \max_{1\le i\le n} |Y_i^{-1} - K_n(0)|$$

$$= o(n^{-\frac{1}{2}}) \text{ a.s., as } n \to \infty ,$$

where the last step is based on the well known result that for a distribution having finite second moment, the sample range is $o(n^{\frac{1}{2}})$ a.s., as $n \to \infty$ [and (2.11) insures this for the Y_i^{-1}]. Let us rewrite

$$(4.9)\quad \hat{\lambda}_{n-1}^{(i)} - \hat{\lambda}_n = x_{n-1}^{(i)}K_{n-1}^{(i)}(x_{n-1}^{(i)}) - x_nK_n(x_n)$$

$$= x_{n-1}^{(i)}[K_{n-1}^{(i)}(x_{n-1}^{(i)}) - K_n(x_{n-1}^{(i)})] + x_{n-1}^{(i)}K_n(x_{n-1}^{(i)}) - x_nK_n(x_n) ,$$

so that by (4.6) and (4.7),

$$(4.10)\quad \max_{1\le i\le n} |x_{n-1}^{(i)}K_n(x_{n-1}^{(i)}) - x_nK_n(x_n) | \le 4/(n-1).$$

But

$$(4.11)\quad x_{n-1}^{(i)}K_n(x_{n-1}^{(i)}) - x_nK_n(x_n) = x_{n-1}^{(i)}[K_n(x_{n-1}^{(i)}) - K(x_{n-1}^{(i)})] -$$

$$x_n[K_n(x_n) - K(x_n)] + \{x_{n-1}^{(i)}K(x_{n-1}^{(i)}) - x_nK(x_n)\},$$

where, by (2.10), the first two terms on the right hand side are each $O(n^{-\frac{1}{2}}(\log\log n)^{\frac{1}{2}})$ a.s., as $n \to \infty$, so that by (4.10), the last term is also $O(n^{-\frac{1}{2}}(\log\log n)^{\frac{1}{2}})$ a.s., as $n \to \infty$. Combining this with (4.5), we obtain that

$$(4.12)\quad \max_{1\le i\le n} | x_{n-1}^{(i)} - x_n | \to 0 \text{ a.s., as } n \to \infty .$$

Also, we note that for each $i(=1,\ldots,n)$,

$$(4.13)\quad \hat{\theta}_{n,i} = \{K_n(0)\}^{-1}\{ (n\hat{\lambda}_n - (n-1)\hat{\lambda}_{n-1}^{(i)}) + \hat{\theta}_{n-1}^{(i)}\ n[K_{n-1}^{(i)}(0) - K_n(0)]\},$$

so that using (4.7), (4.8) and (4.13), we conclude that as $n \to \infty$,

$$(4.14)\quad \max_{1\le i\le n} | \hat{\theta}_{n,i} - \frac{1}{K_n(0)}\{(n\hat{\lambda}_n - (n-1)\hat{\lambda}_{n-1}^{(i)}) + \hat{\theta}_n n[K_n(0) - K_{n-1}^{(i)}(0)]\}| \to 0 \text{ a.s.}$$

Further,

$$(4.15)\quad n[K_n(0) - K_{n-1}^{(i)}(0)] = (n-1)^{-1}[Y_i^{-1} - K_n(0)] ,\ i=1,\ldots,n,$$

and, by definition,

$$(4.16)\quad n\hat{\lambda}_n - (n-1)\hat{\lambda}_{n-1}^{(i)} \le nx_nK_n(x_n) - (n-1)x_nK_{n-i}^{(i)}(x_n) = x_n[Y_i^{-1}I(Y_i \ge x_n)];$$

(4.17) $n\hat{\lambda}_n - (n-1)\hat{\lambda}^{(i)}_{n-1} \geq n x^{(i)}_{n-1} K_n(x^{(i)}_{n-1}) - (n-1) x^{(i)}_{n-1} K^{(i)}_{n-1}(x^{(i)}_{n-1}) = x^{(i)}_{n-1} Y_i^{-1} I(Y_i \geq x^{(i)}_{n-1})$,

so that by (4.12), (4.16) and (4.17), we obtain that

(4.18) $n\hat{\lambda}_n - (n-1)\hat{\lambda}^{(i)}_{n-1} = x_n Y_i^{-1} I(Y_i \geq x_n) + u_{ni}$, $i=1,\ldots,n$,

where

(4.19) $\max_{1\leq i\leq n} |u_{ni}| \leq 1$ and $n^{-1}\Sigma_{i=1}^n |u_{ni}| \to 0$ a.s., as $n \to \infty$.

Therefore, by (4.14), (4.18) and (4.19), we conclude that

(4.20) $\hat{\hat{\theta}}_n = \{K_n(0)\}^{-1} x_n\{n^{-1}\Sigma_{i=1}^n Y_i^{-1} I(Y_i \geq x_n)\} + \hat{\theta}_n . 0 + o(1)$ a.s.

$= x_n K_n(x_n)/K_n(0) + o(1)$ a.s.

$= \hat{\theta}_n + o(1)$ a.s., as $n \to \infty$.

Similarly, by (4.3), (4.14), (4.18), (4.19) and (4.20), we have

(4.21) $s_n^2 = (n-1)^{-1}\Sigma_{i=1}^n (\hat{\theta}_{n,i} - \hat{\hat{\theta}}_n)^2$

$= \{K_n(0)\}^{-2}\{ x_n^2 (n-1)^{-1}\Sigma_{i=1}^n [Y_i^{-1} I(Y_i \geq x_n) - K_n(x_n)]^2 +$

$\hat{\theta}_n^2\, n(n-1)^{-2}\Sigma_{i=1}^n (Y_i^{-1} - K_n(0))^2 -$

$2n(n-1)^{-1}\hat{\theta}_n x_n (n-1)^{-1}\Sigma_{i=1}^n [Y_i^{-1} - K_n(0)][Y_i^{-1} I(Y_i \geq x_n) - K_n(x_n)]\} + o(1)$

$= \hat{\theta}_n^2 \{(n-1)^{-1}\Sigma_{i=1}^n Y_i^{-2} I(Y_i \geq x_n)/K_n^2(x_n) + (n-1)^{-1}\Sigma_{i=1}^n Y_i^{-2}/K_n^2(0)$

$- 2(n-1)^{-1}\Sigma_{i=1}^n Y_i^{-2} I(Y_i \geq x_n)/K_n(0)K_n(x_n) \} + o(1)$ a.s., as $n \to \infty$.

If parallel to (3.12), we define the sample counterpart

(4.22) $V_n(x) = \int_x^\infty y^{-2} dG_n(y) = n^{-1}\Sigma_{i=1}^n Y_i^{-2} I(Y_i \geq x)$, $x \in R^+$,

then, under (2.11), it is easy to show that

(4.23) $\sup\{ |V_n(x) - \nu_G(x)| : x \in R^+ \} \to 0$ a.s., as $n \to \infty$.

Also, by the continuity and boundedness of $\nu_G(x)$, for every $\varepsilon > 0$, there exists a δ (> 0), such that

(4.24) $|\nu_G(x) - \nu_G(x_o)| < \varepsilon$ for every $x: |x - x_o| < \delta$.

Now, by (4.21) and (4.22),

(4.25) $s_n^2 = [n/(n-1)]\, \hat{\theta}_n^2 \{ V_n(x_n)/K_n^2(x_n) + V_n(0)/K_n^2(0) - 2V_n(x_n)/K_n(0)K_n(x_n)\}$

$+ o(1)$ a.s., as $n \to \infty$.

Since $\sup\{|K_n(x) - K(x)| : x \in R^+\}$ converges a.s. to 0 as $n \to \infty$, and $K(x)$ is a continuous and nonnegative function of x, by using (4.5), (4.23), (4.24) and (4.25), we obtain that

$$s_n^2 = \sigma^{*2} + o(1) \text{ a.s., as } n \to \infty , \tag{4.26}$$

where σ^{*2} is defined by (3.14). This gives the almost sure convergence of the jackknife estimator s_n^2 .

REFERENCES

COLEMAN, R. (1979). *An Introduction to Mathematical Stereology*. Memoirs No.3, Dept. of Theoretical Statistics, Univ.of Aarhus, Denmark.

COX, D.R. (1969). Some sampling problems in technology. In *New Developments in Survey Sampling* (eds: N.L.Johnson and H. Smith), Wiley, New York.

DANIELS, H.E. (1945). The statistical theory of the strength of bundle of threads. *Proc. Roy. Soc. Ser.A* 183, 405-435.

DVORETZKY, A., KIEFER, J. and Wolfowitz, J. (1956). Asymptotic minimax character of the sample distribution function and the classical multinomial estimator. *Ann. Math. Statist.* 27, 642-669.

PATIL, G.P. and RAO, C.R. (1977). The weighted distributions: A survey of their applications. In *Applications of Statistics* (ed:P.R. Krishnaiah). North Holland, Amsterdam, pp.383-405.

PATIL, G.P. and RAO, C.R. (1978). Weighted distributions and size-biased sampling with applications to wildlife populations and human families. *Biometrics* 34, 179-189.

SEN, P.K. (1973). On fixed-size confidence bands for the bundle strength of filaments. *Ann. Statist.*1, 526-537.

SEN, P.K. (1981). *Sequential Nonparametrics: Invariance Principles and Statistical Inference*. Wiley, New York.

SEN, P. K., BHATTACHARYYA, B.B. and SUH, M.W. (1973). Limiting behavior of the extrema of certain sample functions. *Ann.Statist.* 1, 297-311.

SUH, M.W., BHATTACHARYYA, B.B. and GRANDAGE, A.H.E. (1970). On the distribution and moments of the strength of a bundle of filaments. *Jour. Appl. Probability* 7, 712-720.