January 25, 1983

MIMEOGRAPH SERIES #1623

Mean Estimation Bias in Least Squares Estimation of Autoregressive Processes

by

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Abstract

Estimation of the parameters of an autoregressive process with a mean that is a function of time is considered. Approximate expressions for the bias of the least squares estimator of the autoregressive parameters that is due to estimating the unknown mean function are derived. For the case of a mean function that is a polynomial in time, a reparameterization that isolates the bias is given. Using the approximate expressions, a method of modifying the least squares estimator is proposed. A Monte Carlo study of the second-order autoregressive process is presented. The Monte Carlo results agree well with the approximate theory and, generally speaking, the modified least squares estimators performed better than the least squares estimator.

1. Introduction

Let the time series Y_t satisfy

$$Y_t = \sum_{i=1}^{r} X_{ti}\beta_i + P_t \ , \quad t = 1,2,\ldots, \tag{1}$$

where X_{ti} is a fixed function of time for $i = 1,2,\ldots, r$, P_t is a stationary p-th order normal autoregressive process satisfying

$$P_t = \sum_{j=1}^{p} \alpha_j P_{t-j} + e_t \ , \tag{2}$$

and $\{e_t\}$ is a sequence of normal independent $(0, \sigma^2)$ random variables. Assume that the roots of the characteristic equation

$$m^p - \alpha_1 m^{p-1} - \dots - \alpha_p = 0 \tag{3}$$

lie inside the unit circle. Assume that

$$\underset{\sim}{D}_n(\underset{\sim}{X}'\underset{\sim}{X})^{-1} \underset{\sim}{D}_n = O(1) \tag{4}$$

and

$$\underset{\sim}{D}_n^{-1} \underset{\sim}{X}_t' = O(n^{-1/2}) , \quad t = 1,2,\dots, n$$

where

$$\underset{\sim}{D}_n = \text{diag}[\{\Sigma_{t=1}^n X_{t1}^2\}^{1/2},\dots, \{\Sigma_{t=1}^n X_{tr}^2\}^{1/2}] ,$$

$$\underset{\sim}{X} = (\underset{\sim}{X}_1', \underset{\sim}{X}_2',\dots, \underset{\sim}{X}_n')' ,$$

and

$$\underset{\sim}{X}_t = (X_{t1}, X_{t2},\dots, X_{tr}) .$$

Also assume that there exists a nonsingular $r \times r$ matrix $\underset{\sim}{C}$ such that

$$\underset{\sim}{X}_t = \underset{\sim}{X}_{t-m}\underset{\sim}{C}^m , \text{ for all } t , \tag{5}$$

where $\underset{\sim}{C}^0 = \underset{\sim}{I}_r$. For example, these assumptions on X_{ti} are satisfied by polynomial and trignometric polynomial functions of time.

For the case $r = 1$, $p = 1$ and $\underset{\sim}{X}_t \equiv 1$, Mariott and Pope (1954), Orcutt and Winokur (1969), Salem (1971), Bora-Senta and Kounias (1980) and Lee (1981) proposed several estimators of α_1 . These authors also used Monte Carlo studies to compare the small sample properties of the various estimators. See Lee (1981). The small sample properties of the least squares estimator of $\underset{\sim}{\alpha}$ have received little attention for higher order processes. Salem (1971) extended the method of Mariott and Pope (1954) to obtain expressions for the approximate biases of the least squares estimator of $\underset{\sim}{\alpha}$ for the case $r = 1$, $p = 2$ and $X_{t1} \equiv 1$. The method proposed by Salem has no immediate extension for processes with $p > 2$. Bora-Senta and Kounias (1980) considered an iterative method of moments procedure as an alternative to the least squares estimation procedure for higher order processes with $r = 1$ and $X_{t1} \equiv 1$. Ansley and Newbold (1980) compared the properties of the maximum likelihood predictor, the exact least squares predictor and the conditional least squares predictor of Y_{n+1}, Y_{n+2} and Y_{n+10} by means of simulation.

We shall consider two procedures for estimating $\underset{\sim}{\alpha}$. In the first procedure Y_t is regressed on $(\underset{\sim}{X}_t, Y_{t-1}, Y_{t-2}, \ldots, Y_{t-p})$ and the coefficient of Y_{t-i} estimates α_i . There are a total of $n-p$ observations in the regression. Because the $\underset{\sim}{X}_t$ satisfy (5), this procedure is equivalent to computing residuals $\tilde{P}_{t-i}$, $i = 0,1,\ldots, p$, by regressing Y_{t-i} on $\underset{\sim}{X}_{t-i}$ and then estimating $\underset{\sim}{\alpha}$ by regressing $\tilde{P}_t$ on $\tilde{P}_{t-1}, \tilde{P}_{t-2}, \ldots, \tilde{P}_{t-p}$. We denote the estimator of $\underset{\sim}{\alpha}$ obtained in this way by $\tilde{\alpha}$. See Rao (1967) for a description of this procedure for the polynomial mean function.

The second procedure consists of two steps. First, the ordinary least squares estimator $\hat{\beta}$ of β is constructed, where

$$\hat{\beta} = (X'X)^{-1} X'Y , \tag{6}$$

and

$$Y = (Y_1, Y_2, \ldots, Y_n)' .$$

Let the least squares residuals be denoted by $\hat{P}_t$, where

$$(\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_n)' = Y - X\hat{\beta} = M Y , \tag{7}$$

and

$$M = I - X(X'X)^{-1} X' . \tag{8}$$

The second step of procedure two consists of estimating α by regressing $\hat{P}_t$ on $(\hat{P}_{t-1}, \hat{P}_{t-2}, \ldots, \hat{P}_{t-p})$. We call this estimator $\hat{\alpha}$.

In this paper we obtain an approximate expression for the bias in the estimators of α that arises from estimating β . We propose an adjustment in the estimators based upon the expression for the approximate bias. Using Monte Carlo simulation, the proposed estimator is compared with the original estimator.

2. Bias in estimators of $\underset{\sim}{\alpha}$.

The estimator of $\underset{\sim}{\alpha}$ obtained by method one can be written

$$\tilde{\underset{\sim}{\alpha}} = \tilde{\underset{\sim}{H}}^{-1} \tilde{\underset{\sim}{N}} , \tag{9}$$

where

$$\tilde{\underset{\sim}{H}} = (n-p)^{-1} \sum_{t=p+1}^{n} \tilde{\underset{\sim}{F}}'_{t-1}\tilde{\underset{\sim}{F}}_{t-1} ,$$

$$\tilde{\underset{\sim}{N}} = (n-p)^{-1} \sum_{t=p+1}^{n} \tilde{\underset{\sim}{F}}'_{t-1}\tilde{P}_{t} ,$$

$$\tilde{\underset{\sim}{F}}_{t-1} = (\tilde{P}_{t-1}, \tilde{P}_{t-2}, \ldots, \tilde{P}_{t-p}) .$$

From the arguments of Fuller and Hasza (1981) and Lee (1981), it follows that $\tilde{\underset{\sim}{\alpha}}$ is integrable and

$$E[\tilde{\underset{\sim}{\alpha}}] = \underset{\sim}{\alpha} + E[\underset{\sim}{H}^{-1} \underset{\sim}{A} \underset{\sim}{H}^{-1}(\underset{\sim}{A} \underset{\sim}{\alpha} - \underset{\sim}{d}) - \underset{\sim}{H}^{-1}(\underset{\sim}{A} \underset{\sim}{\alpha} - \underset{\sim}{d})]$$

$$+ O(n^{-2}) , \tag{10}$$

where

$$\underset{\sim}{A} = \tilde{\underset{\sim}{H}} - \underset{\sim}{H} ,$$

$$\underset{\sim}{d} = \tilde{\underset{\sim}{N}} - \underset{\sim}{N} ,$$

$$\underset{\sim}{H} = E[(n-p)^{-1} \Sigma_{t=p+1}^{n} \underset{\sim}{F}'_{t-1}\underset{\sim}{F}_{t-1}] ,$$

$$\tilde{N} = E[(n-p)^{-1} \Sigma_{t=p+1}^{n} \tilde{F}'_{t-1} P_t] ,$$

and

$$\tilde{F}_{t-1} = (P_{t-1}, P_{t-2}, \ldots, P_{t-p}) .$$

The bias in $\tilde{\alpha}$ arises from two sources. The first source of bias is inherent in estimating the product of the inverse of $\tilde{H}$ and the vector $\tilde{N}$. The second source of bias results from estimating the unknown function $\tilde{X}_t \tilde{\beta}$. The approximate bias in $\tilde{\alpha}$ arising from estimating the mean function is given by $E[-\tilde{H}^{-1}(\tilde{A}\,\tilde{\alpha} - \tilde{d})]$. An expression for $E[\tilde{A}\,\tilde{\alpha} - \tilde{d}]$ is derived in Theorem 1.

Theorem 1. Let Y_t satisfy model (1) and let $\tilde{\alpha}$ be given by (9). Assume that the roots of equation (3) lie inside the unit circle and that $\tilde{X}_t$ satisfies conditions (4) and (5). Then the i-th coordinate of $E\{\tilde{A}\,\tilde{\alpha} - \tilde{d}\}$ is

$$(n-p)^{-1} \operatorname{tr}[(\tilde{X}'_{(-p)}\tilde{X}_{(-p)})^{-1} \tilde{X}'_{(-p)}\tilde{B}_i\tilde{X}_{(-p)}] , \qquad (11)$$

where

$$\tilde{B}_i = E\{\tilde{e}_{(0)}\tilde{P}'_{(-i)}\} ,$$

$$\tilde{P}'_{(-i)} = (P_{p-i+1}, P_{p-i+2}, \ldots, P_{n-i}) ,$$

$$\tilde{e}'_{(0)} = (e_{p+1}, e_{p+2}, \ldots, e_n) ,$$

$$\underset{\sim}{X}'_{(-p)} = (\underset{\sim}{X}'_{p+1}, \underset{\sim}{X}'_{p+2}, \ldots, \underset{\sim}{X}'_{n}) .$$

Proof. Under procedure 1

$$\hat{\underset{\sim}{P}}_{(-i)} = \underset{\sim}{M}_{(-p)} \underset{\sim}{P}_{(-i)} , \qquad (12)$$

where

$$\underset{\sim}{M}_{(-p)} = \underset{\sim}{I} - \underset{\sim}{X}_{(-p)} (\underset{\sim}{X}'_{(-p)} \underset{\sim}{X}_{(-p)})^{-1} \underset{\sim}{X}'_{(-p)} .$$

The single matrix $\underset{\sim}{M}_{(-p)}$ can be used to construct all residuals because of assumption (5). It follows that

$$\hat{\underset{\sim}{\alpha}} - \underset{\sim}{\alpha} = (\hat{\underset{\sim}{F}}' \hat{\underset{\sim}{F}})^{-1} \hat{\underset{\sim}{F}}' \underset{\sim}{e}_{(0)}$$

$$= (\underset{\sim}{F}' \underset{\sim}{M}_{(-p)} \underset{\sim}{F})^{-1} \underset{\sim}{F}' \underset{\sim}{M}_{(-p)} \underset{\sim}{e}_{(0)} ,$$

where

$$\hat{\underset{\sim}{F}} = \underset{\sim}{M}_{(-p)} (\underset{\sim}{P}_{(-p)}, \underset{\sim}{P}_{(-p+1)}, \ldots, \underset{\sim}{P}_{(-1)}) ,$$

$$\underset{\sim}{F} = (\underset{\sim}{P}_{(-p)}, \underset{\sim}{P}_{(-p+1)}, \ldots, \underset{\sim}{P}_{(-1)}) .$$

We have

$$E\{\underset{\sim}{A}\, \underset{\sim}{\alpha} - \underset{\sim}{d}\} = (n-p)^{-1} E\{\underset{\sim}{F}' \underset{\sim}{X}_{(-p)} (\underset{\sim}{X}'_{(-p)} \underset{\sim}{X}_{(-p)})^{-1} \underset{\sim}{X}'_{(-p)} \underset{\sim}{e}_{(0)}\}$$

and the i-th coordinate of $E\{\underset{\sim}{A}\,\underset{\sim}{\alpha} - \underset{\sim}{d}\}$ is

$$(n-p)^{-1}\ E\{\underset{\sim}{P}'_{(-i)}\underset{\sim}{X}_{(-p)}(\underset{\sim}{X}'_{(-p)}\underset{\sim}{X}_{(-p)})^{-1}\ \underset{\sim}{X}'_{(-p)}\underset{\sim}{e}_{(0)}\}$$

$$= (n-p)^{-1}\ \mathrm{tr}[(\underset{\sim}{X}'_{(-p)}\underset{\sim}{X}_{(-p)})^{-1}\ \underset{\sim}{X}'_{(-p)}\underset{\sim}{B}_i\underset{\sim}{X}_{(-p)}]\ . \qquad \square$$

The contribution to the bias arising from the estimation of a polynomial mean function is evaluated in Theorem 2.

Theorem 2. Assume that the conditions of Theorem 1 are satisfied and assume that

$$\underset{\sim}{X}_t = (1,\ t, \ldots,\ t^{r-1})\ , \quad r > 1\ . \tag{13}$$

Then

$$E\{\underset{\sim}{A}\,\underset{\sim}{\alpha} - \underset{\sim}{d}\} = (n-p)^{-1}\ r\ \sigma^2(\sum_{j=0}^{\infty} \omega_j)\underset{\sim}{J} + O(n^{-2})$$

$$= (n-p)^{-1}\ r\ \sigma^2(1 - \sum_{j=1}^{p} \alpha_j)^{-1}\ \underset{\sim}{J} + O(n^{-2})\ , \tag{14}$$

where $\underset{\sim}{J} = (1,\ 1, \ldots,\ 1)'$ and the ω_j satisfy the set of homogeneous difference equations

$$\omega_j = \alpha_1\omega_{j-1} + \alpha_2\omega_{j-2} + \ldots + \alpha_p\omega_{j-p}\ , \quad j = 1,2,\ldots$$

with $\omega_0 = 1$ and $\omega_j = 0$ for $j < 0$.

Proof. It is clear that $\underset{\sim}{X}_t$ satisfies the conditions (4) and (5). The $\underset{\sim}{B}_i$ of Theorem 1 is

$$\underset{\sim}{B}_i = \begin{bmatrix} 0 & 0 & \dots & 0 & \omega_0 & \omega_1 & \omega_2 & \dots & \omega_{n-i-p-1} \\ 0 & 0 & \dots & 0 & 0 & \omega_0 & \omega_1 & \dots & \omega_{n-i-p-2} \\ \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & \omega_0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix} \sigma^2$$

Therefore,

$$\begin{aligned} \underset{\sim}{X}'_{(-p)}\underset{\sim}{B}_i\underset{\sim}{X}_{(-p)} &= \sum_{k=p+1}^{n-i-p} \underset{\sim}{X}'_k \sum_{j=0}^{n-i-p-k} \omega_j \underset{\sim}{X}_{k+i+j}\,\sigma^2 \\ &= \sum_{j=0}^{n-i-p-1} \omega_j \Big(\sum_{k=p+1}^{n-i-j} \underset{\sim}{X}'_k \underset{\sim}{X}_{k+i+j} \Big)\sigma^2 \\ &= \sum_{j=0}^{n-i-p-1} \omega_j \Big(\sum_{k=p+1}^{n} \underset{\sim}{X}'_k \underset{\sim}{X}_k \Big)\sigma^2 \\ &\quad + \sum_{j=0}^{n-i-p-1} \omega_j \Big[\sum_{k=p+1}^{n} \underset{\sim}{X}'_k (\underset{\sim}{X}_{k+i+j} - \underset{\sim}{X}_k) \Big]\sigma^2 \\ &\quad - \sum_{j=0}^{n-i-p-1} \omega_j \Big(\sum_{k=n-i-j+1}^{n} \underset{\sim}{X}'_k \underset{\sim}{X}_{k+i+j} \Big)\sigma^2 . \end{aligned}$$

Note that

$$(\underset{\sim}{X}'_{(-p)}\underset{\sim}{X}_{(-p)})^{\ell s} = C_{\ell s} n^{-(\ell+s+1)} + O(n^{-(\ell+s+2)})$$

for some constants $C_{\ell s}$, where $(\underset{\sim}{X}'\underset{\sim}{X})^{\ell s}$ is the (ℓ,s)-th element of $(\underset{\sim}{X}'_{(-p)}\underset{\sim}{X}_{(-p)})^{-1}$. Since $\Sigma_{j=0}^{\infty}|\omega_j|$ and $\Sigma_{j=0}^{\infty} j|\omega_j|$ are finite, we have

$$\operatorname{tr}[(\underset{\sim}{X}_{(-p)}\underset{\sim}{X}_{(-p)})^{-1}\ \underset{\sim}{X}'_{(-p)}\underset{\sim}{B}_i\underset{\sim}{X}_{(-p)}] = r\ \sigma^2(\sum_{j=0}^{\infty} \omega_j) + O(n^{-1})$$

and the i-th coordinate of $E[\underset{\sim}{A}\ \underset{\sim}{\alpha} - \underset{\sim}{d}]$ is

$$(n-p)^{-1}\ r\ \sigma^2(\sum_{j=0}^{\infty} \omega_j) + O(n^{-2})\ .$$

Because the roots of the equation (2) are less than one in absolute value, we have $\Sigma_{j=0}^{\infty}\ \omega_j = (1 - \Sigma_{j=1}^{p}\ \alpha_j)^{-1}$. □

We now consider the bias in the second method of constructing an estimator of $\underset{\sim}{\alpha}$.

Theorem 3. Let Y_t satisfy model (1) and let $\hat{\underset{\sim}{\alpha}}$ be given by

$$\hat{\underset{\sim}{\alpha}} = (\sum_{t=p+1}^{n} \hat{\underset{\sim}{F}}'_{t-1}\hat{\underset{\sim}{F}}_{t-1})^{-1} \sum_{t=p+1}^{n} \hat{\underset{\sim}{F}}'_{t-1}\hat{P}_t\ , \tag{15}$$

$$= (\hat{\underset{\sim}{F}}'\hat{\underset{\sim}{F}})^{-1}\ \hat{\underset{\sim}{F}}'\hat{\underset{\sim}{P}}_{(0)}\ ,$$

where

$$\hat{\underset{\sim}{F}}' = (\hat{\underset{\sim}{F}}'_{p-1}, \hat{\underset{\sim}{F}}'_p, \ldots, \hat{F}'_{n-1})\ ,$$

$$\hat{\underset{\sim}{F}}_{t-1} = (\hat{P}_{t-1}, \hat{P}_{t-2}, \ldots, \hat{P}_{t-p})\ ,$$

$$\hat{P}_t = Y_t - \underset{\sim}{X}_t\hat{\underset{\sim}{\beta}}\ ,$$

$$\hat{\underset{\sim}{P}}_{(0)} = (P_{p+1}, P_{p+2}, \ldots, P_n)'\ ,$$

and $\hat{\beta}$ is defined in (6). Assume that the roots of equation (3) lie inside the unit circle and that X_t satisfies conditions (4) and (5). Then

$$E\{\hat{\alpha} - \tilde{\alpha}\} = O(n^{-2}) .$$

Proof. We can write

$$\tilde{P}_{t(-i)} = \hat{P}_t + X_t(\hat{\beta} - \tilde{\beta}_{(i)}) ,$$

where $\tilde{P}_{t(-i)}$ is the t-th element of the vector $\tilde{P}_{(-i)}$ defined in (12) and

$$\tilde{\beta}_{(i)} = (X'_{(-i)}X_{(-i)})^{-1} X'_{(-i)}Y_{(-i)} ,$$

$$X'_{(-i)} = (X'_{p-i+1}, X'_{p-i+2}, \ldots, X'_{n-i}) .$$

It follows that

$$\begin{aligned} D_n(\hat{\beta} - \tilde{\beta}_{(i)}) &= [D_n(X'X)^{-1} D_n]D_n^{-1} X'P \\ &\quad - [D_n(X'_{(-i)}X_{(-i)})^{-1} D_n]D_n^{-1} X'_{(-i)}P_{(-i)} \\ &= [D_n(X'X)^{-1} D_n][D_n^{-1} X'P - D_n^{-1} X'_{(-i)}P_{(-i)}] \\ &\quad + [D_n\{(X'X)^{-1} - (X'_{(-i)}X_{(-i)})^{-1}\}D_n]D_n^{-1} X'_{(-i)}P_{(-i)} . \end{aligned}$$

Observe that

$$\mathbf{D}_n^{-1} \mathbf{X}'\mathbf{P} - \mathbf{D}_n^{-1} \mathbf{X}'_{(-i)}\mathbf{P}_{(-i)} = \mathbf{D}_n^{-1}\left[\sum_{t=1}^{p-i} \mathbf{X}'_t P_t + \sum_{t=n-i+1}^{n} \mathbf{X}'_t P_t \right]$$

$$+ O_p(n^{-1/2})$$

and

$$\mathbf{D}_n\{(\mathbf{X}'\mathbf{X})^{-1} - (\mathbf{X}'_{(-i)}\mathbf{X}_{(-i)})^{-1}\}\mathbf{D}_n = O(n^{-1}) \ .$$

Therefore,

$$E\{\hat{\boldsymbol{\beta}} - \widehat{\boldsymbol{\beta}}_{(i)}\} = 0$$

and

$$E\{\mathbf{D}_n(\hat{\boldsymbol{\beta}} - \widehat{\boldsymbol{\beta}}_{(i)})(\hat{\boldsymbol{\beta}} - \widehat{\boldsymbol{\beta}}_{(i)})'\mathbf{D}_n\} = O(n^{-1}) \ .$$

Now

$$\hat{\boldsymbol{\alpha}} - \widehat{\boldsymbol{\alpha}} = [n^{-1} \hat{\mathbf{F}}'\hat{\mathbf{F}}]^{-1} n^{-1} \hat{\mathbf{F}}'\hat{\mathbf{P}}_{(0)}$$

$$- [n^{-1} \widehat{\mathbf{F}}'\widehat{\mathbf{F}}]^{-1} n^{-1} \widehat{\mathbf{F}}'\widehat{\mathbf{P}}_{(0)}$$

$$= [n^{-1} \hat{\mathbf{F}}'\hat{\mathbf{F}}]^{-1}[n^{-1} \hat{\mathbf{F}}'\hat{\mathbf{P}}_{(0)} - n^{-1} \widehat{\mathbf{F}}'\widehat{\mathbf{P}}_{(0)}]$$

$$+ \{[n^{-1} \hat{\mathbf{F}}'\hat{\mathbf{F}}]^{-1} - [n^{-1} \widehat{\mathbf{F}}'\widehat{\mathbf{F}}]^{-1}\}[n^{-1} \widehat{\mathbf{F}}'\widehat{\mathbf{P}}_{(0)}]$$

where

$$\tilde{P}_{(0)} = (\tilde{P}_{p+1}, \tilde{P}_{p+2}, \ldots, \tilde{P}_n)$$

$$= \hat{P}_{(0)} .$$

For fixed i and j ,

$$n^{-1} \sum_{t=p+1}^{n} \tilde{P}_{t-i}\tilde{P}_{t-j} = n^{-1} \sum_{t=p+1}^{n} \{[\hat{P}_{t-i} + X_{t-i}(\hat{\beta} - \tilde{\beta}_{(i)})][\hat{P}_{t-j} + X_{t-j}(\hat{\beta} - \tilde{\beta}_{(j)})]\}$$

$$= n^{-1} \sum_{t=p+1}^{n} \hat{P}_{t-i}\hat{P}_{t-j} + n^{-1} \sum_{t=p+1}^{n} \hat{P}_{t-i}X_{t-j}(\hat{\beta} - \tilde{\beta}_{(i)}) + n^{-1} \sum_{t=p+1}^{n} \hat{P}_{t-j}X_{t-i}(\hat{\beta} - \tilde{\beta}_{(j)}) + n^{-1} \sum_{t=p+1}^{n} X_{t-i}(\hat{\beta} - \tilde{\beta}_{(i)})X_{t-j}(\hat{\beta} - \tilde{\beta}_{(j)}) .$$

Note that

$$\sum_{t=p+1}^{n} \hat{P}_{t-i}X_{t-j}D_n^{-1} = \sum_{t=p+1}^{n} \hat{P}_{t-i}X_{t-i}C_{|j-i|}D_n^{-1}$$

$$= \sum_{t=p+1}^{n} \hat{P}_t X_t C_{|j-i|}D_n^{-1}$$

$$+ \sum_{t=p+1-i}^{p} \hat{P}_t \underset{\sim}{X}_t \underset{\sim}{C}_{|j-i|} \underset{\sim}{D}_n^{-1}$$

$$- \sum_{t=n-i+1}^{n} \hat{P}_t \underset{\sim}{X}_t \underset{\sim}{C}_{|j-i|} \underset{\sim}{D}_n^{-1}$$

$$= \sum_{t=p+1-i}^{p} \hat{P}_t \underset{\sim}{X}_{t+|j-i|} \underset{\sim}{D}_n^{-1}$$

$$- \sum_{t=n-i+1}^{n} \hat{P}_t \underset{\sim}{X}_{t+|j-i|} \underset{\sim}{D}_n^{-1}$$

$$= O_p(n^{-1/2}) \, ,$$

where $\underset{\sim}{C}_m = \underset{\sim}{C}^m$. Therefore,

$$n^{-1} \sum_{t=p+1}^{n} \tilde{P}_{t-i} \tilde{P}_{t-j} = n^{-1} \sum_{t=p+1}^{n} \hat{P}_{t-i} \hat{P}_{t-j} + O_p(n^{-2}) \, .$$

It can be shown that the expectation of powers of the elements of

$$[n^{-1} \sum_{t=p+1}^{n} \hat{\underset{\sim}{F}}_t' \hat{\underset{\sim}{F}}_t]^{-1}$$

and

$$[n^{-1} \sum_{t=p+1}^{n} \tilde{\underset{\sim}{F}}_t' \tilde{\underset{\sim}{F}}_t]^{-1}$$

are bounded. See Fuller and Hasza (1981). Therefore, since $\hat{\underset{\sim}{\alpha}}$ and $\tilde{\underset{\sim}{\alpha}}$ are square integrable we have

$$E[\hat{\underset{\sim}{\alpha}} - \tilde{\underset{\sim}{\alpha}}] = O(n^{-2}) \, . \qquad \square$$

Note that for $\underset{\sim}{X}_t$ given in (13), the coordinates of $E[\underset{\sim}{A}\,\underset{\sim}{\alpha} - \underset{\sim}{d}]$ are all equal to $(n-p)^{-1}\, r\sigma^2(1 - \Sigma_{j=1}^{p} \alpha_j)^{-1} + O(n^{-2})$. Pantula (1982) obtained an analogous expression for $\underset{\sim}{X}_t = (t^{i_1}, t^{i_2}, \ldots, t^{i_r})$, where $0 < i_1 < i_2 < \cdots < i_r$ are integers. Theorem 2 also suggests that it is possible to isolate the effect of estimating the mean function by transforming the model (1). For $p > 2$, consider the following reparameterization

$$P_t = \delta_1 P_{t-1} + \sum_{j=2}^{p} \delta_j (P_{t-j+1} - P_{t-j}) + e_t \ , \tag{16}$$

where

$$\alpha_1 = \delta_1 + \delta_2 \ ,$$

$$\alpha_2 = \delta_{j+1} - \delta_j \ , \quad j = 2,3,\ldots, p-1 \tag{17}$$

and $\alpha_p = -\delta_p$. Therefore, $\delta_1 = \Sigma_{j=1}^{p} \alpha_j$ and δ_1 is equal to one if and only if one of the roots of the equation (3) is one. The least squares estimator $\hat{\underset{\sim}{\delta}}$ of $\underset{\sim}{\delta}$ is obtained by regressing $\hat{P}_t$ on $\hat{P}_{t-1}$, $(\hat{P}_{t-1} - \hat{P}_{t-2})$, $(\hat{P}_{t-2} - \hat{P}_{t-3}), \ldots, (\hat{P}_{t-p+1} - \hat{P}_{t-p})$, where $\hat{P}_t = Y_t - \underset{\sim}{X}_t \hat{\underset{\sim}{\beta}}$ and $\hat{\beta}$ is given by (6). From Theorem 2, it follows that an approximate expression for the bias in $\hat{\underset{\sim}{\delta}}$ arising from estimating the mean function is

$$\underset{\sim}{G}^{-1}\, \underset{\sim}{f} + O(n^{-2}) \ , \tag{18}$$

where

$$\underset{\sim}{f} = r\sigma^2(n-p)^{-1}(1-\delta_1)^{-1}(1, 0, 0, \ldots, 0)' ,$$

$$\underset{\sim}{G} = E[(n-p)^{-1} \sum_{t=p+1}^{n} \underset{\sim}{G}_t'\underset{\sim}{G}_t] ,$$

and

$$\underset{\sim}{G}_t = (P_{t-1}, P_{t-1} - P_{t-2}, P_{t-2} - P_{t-3}, \ldots, P_{t-p+1} - P_{t-p}) .$$

The expression in (14) suggests the following method of correcting for the bias due to estimating the mean function:

(a) Obtain the least squares estimator of $\underset{\sim}{\delta}$,

$$\hat{\underset{\sim}{\delta}} = \hat{\underset{\sim}{G}}^{-1} \hat{\underset{\sim}{g}} ,$$

where

$$\hat{\underset{\sim}{G}} = (n-p)^{-1} \sum_{t=p+1}^{n} \hat{\underset{\sim}{G}}_t'\hat{\underset{\sim}{G}}_t ,$$

$$\hat{\underset{\sim}{G}}_t = (\hat{P}_{t-1}, \hat{P}_{t-1} - \hat{P}_{t-2}, \ldots, \hat{P}_{t-p+1} - \hat{P}_{t-p}) ,$$

and

$$\hat{\underset{\sim}{g}} = (n-p)^{-1} \sum_{t=p+1}^{n} \hat{\underset{\sim}{G}}_t'\hat{P}_t .$$

Modify the estimator to incorporate the condition that $\delta_1 < 1$, where the modified estimator is

$$\underset{\sim}{\delta}^* = \hat{\underset{\sim}{\delta}} + \hat{\underset{\sim}{G}}^{-1} \hat{\underset{\sim}{h}} ,$$

with

$$\hat{\underset{\sim}{h}} = (\hat{h}_1, 0, \ldots, 0)' \text{ , if } p > 2$$

$$= \hat{h}_1 \text{ , if } p = 1$$

$$\hat{h}_1 = (\hat{G}^{11})^{-1}(1 - \hat{\delta}_1) \text{ , if } \hat{\delta}_1 > 1$$

$$= 0 \text{ , if } \hat{\delta}_1 < 1$$

and $\hat{G}^{11}$ is the upper left element of $\hat{\underset{\sim}{G}}^{-1}$.

(b) Obtain the adjusted least squares estimator as

$$\underset{\sim}{\delta}^+ = \hat{\underset{\sim}{\delta}} + \hat{\underset{\sim}{G}}^{-1} \hat{\underset{\sim}{f}} ,$$

where

$$\hat{\underset{\sim}{f}} = (\hat{f}_1, 0, 0, \ldots, 0)' \text{ , if } p > 2$$

$$= \hat{f}_1 \text{ , if } p = 1$$

$$\hat{f}_1 = (\hat{G}^{11})^{-1}(1 - \hat{\delta}_1) \text{ , if } \hat{\delta}_1 > 1 \text{ or}$$

$$\text{if } \hat{\sigma}^2 \hat{G}^{11} > (1 - \hat{\delta}_1)^2(n-p)$$

$$= [(n-p)(1 - \hat{\delta}_1)]^{-1} r\hat{\sigma}^2 \text{ , otherwise ,}$$

and $\hat{\sigma}^2$ is the residual mean square error of the regression in (a).

(c) Obtain the corresponding estimators, $\hat{\alpha}^*$ and $\hat{\alpha}^+$ using the relations in (11).

The estimator $\hat{\alpha}^+$ of α is relatively easy to construct and, hence, is of practical importance. A Monte Carlo study of the estimators $\hat{\alpha}^*$ and $\hat{\alpha}^+$ is presented in the next section.

3. Monte Carlo Study

Estimators for a second-order autoregressive process with constant mean, and a second-order autoregressive process with mean function linear in time are studied. Samples of size 25 and 50 were generated for 18 sets of parameter values (α_1, α_2). The (α_1, α_2) and the roots of the characteristic equation are given in Table 1. Independent normal (0, 1) random variables were generated for the e_t. For the processes with one of the roots equal to one, the initial observations Y_1, Y_2 are set equal to zero. For the other parameter values, Y_1 and Y_2 are defined by

$$Y_1 = \{\gamma(0)\}^{1/2} e_1 ,$$

$$Y_2 = \{\gamma(0)\}^{1/2} [\alpha_1(1 - \alpha_2)^{-1} e_1 + \{1 - \alpha_1^2(1 - \alpha_2)^{-2}\} e_2] ,$$

where $\gamma(0) = [(1 + \alpha_2)(1 - \alpha_1 - \alpha_2)(1 + \alpha_1 - \alpha_2)]^{-1}(1 - \alpha_2)$. The remaining observations are given by

$$Y_t = \alpha_1 Y_{t-1} + \alpha_2 Y_{t-2} + e_t .$$

For each (α_1, α_2, n) combination, various point estimates are computed using the same set of observations. This is repeated for 1,000 sets of observations. Method two is used to estimate $\underset{\sim}{\beta}$ throughout the study. Sample biases, variances and mean square errors for each estimator are obtained by averaging over the 1,000 replications. For a second-order autoregressive process,

$$\underset{\sim}{H}^{-1} E[\underset{\sim}{A}\, \underset{\sim}{\alpha} - \underset{\sim}{d}] = r(1 + \alpha_2)(n - 2)^{-1}(1, 1)' + O(n^{-2}) . \quad (21)$$

For the sample mean model $r = 1$ and for the time trend model $r = 2$. For the mean model, Salem (1971) and Lee (1981) used (21) to adjust for the bias arising from estimating the unknown mean. Table 2 contains the empirical bias of the estimators of α_2 and δ_1 for the constant mean model. For positive values of δ_1, the least squares estimator δ_1^* has larger absolute bias than the adjusted estimator δ_1^+. Unless both roots are negative, δ_1^+ underestimates δ_1. The difference between the empirical biases of δ_1^+ of δ_1^* is close to the theoretical difference $2(n-2)^{-1}(1+\alpha_2)$. This is illustrated in Figure 1. The adjusted estimator δ_1^+ also has smaller absolute bias than the least squares estimator δ_1^*, except when both roots are negative. For most of the parameter values considered, α_2^+ has smaller absolute bias than α_2^*. If α_2 is close to negative one, then the bias in the least squares estimator of α_2 arising from estimating the unknown mean is close to zero and the other source of bias is positive.

Table 3 contains the empirical mean square errors of the two estimators of α_2 and δ_1 . The mean square errors of δ_1^+ are less than those of δ_1^* for positive values of δ_1 . The mean square errors of δ_1^+ is as much as 30 percent less than those of δ_1^* for positive values of δ_1 , even for sample size 50. When both roots are negative or complex, the mean square error of δ_1^+ is larger than that of δ_1^* . The ratio of the mean square errors of δ_1^+ to δ_1^* is largest for the δ_1 values that correspond to complex roots. For all positive α_2 , α_2^+ has smaller mean square error than α_2^* .

Table 4 contains the empirical bias of the estimators of α_2 and δ_1 for the model with an estimated time trend. Except when both roots are negative or complex, δ_1^+ has smaller absolute bias than δ_1^* . The difference in the biases of the estimators α_2^+ and α_2^* and δ_1^* are very close to $2(n-2)^{-1}(1+\alpha_2)$ and $4(n-2)^{-1}(1+\alpha_2)$, respectively. Therefore, for positive values of α_2 the reduction in the bias of the least squares estimator is large. When α_2 is negative, α_2^+ has larger absolute bias than α_2^* . When the roots are complex, δ_1^* underestimates the true value of δ_1 , whereas δ_1^+ overestimates the true value.

For positive values of δ_1 , δ_1^+ has smaller mean square error than δ_1^* . When α_2 is negative and the roots lie inside unit circle, the mean square error of δ_1^+ is larger than that of δ_1^* . The gain in mean square error of δ_1^+ over δ_1^* is as much as 50%. For values of δ_1 greater than 0.5, the ratio of mean square errors of δ_1^+ to δ_1^* is about 40%. Generally, α_2^+ has smaller mean square error than α_2^* . If

α_2 is negative, then α_2^* has smaller mean square error than α_2^+. Generally speaking, the adjustment produces large gains in mean square error for those parametric configurations that originally had large mean square errors and produces losses for those parameters that originally had small mean squares. Therefore, the mean square error function for the adjusted estimator is much flatter than the corresponding function for the least squares estimator.

Summary

For an autoregressive process with mean function a polynomial in time, the bias in the least squares estimator arising from estimating the mean function has a relatively simple form. This form can be used to correct the estimator for the bias arising from estimating the mean function. The Monte Carlo study demonstrates that the mean square error of the adjusted estimator is smaller than that of the least squares estimator for a wide range of parameter values. The mean square error is reduced about 40 percent by the adjustment for $n = 50$ when $\delta_1(= \alpha_1 + \alpha_2)$ is positive and α_2 is negative.

Table 1. The roots of the characteristic equation and the parameter values.

Set No.	Roots		Values of the parameters		
	m_1	m_2	α_1	$\alpha_2=-\delta_2$	$\delta_1=\alpha_1+\alpha_2$
1	1.0	0.8	1.8	-0.80	1.00
2	1.0	0.5	1.5	-0.50	1.00
3	1.0	0.0	1.0	0.00	1.00
4	1.0	-0.5	0.5	0.50	1.00
5	1.0	-0.8	0.2	0.80	1.00
6	0.8	0.5	1.3	-0.40	0.90
7	0.8	0.1	0.9	-0.08	0.82
8	0.8	-0.5	0.3	0.40	0.70
9	0.8	-0.8	0.0	0.64	0.64
10	0.5	0.0	0.5	0.00	0.50
11	0.5	-0.5	0.0	0.25	0.25
12	0.5	-0.8	-0.3	0.40	0.10
13	0.1	-0.5	-0.4	0.05	-0.35
14	0.1	-0.8	-0.7	0.08	-0.62
15	-0.5	-0.8	-1.3	-0.40	-1.70
16	0.7 ±	0.7i	1.4	-0.98	0.42
17	0.2 ±	0.8i	0.4	-0.68	-0.28
18	-0.7 ±	0.7i	-1.4	-0.98	-2.38

Table 2. Empirical bias of various estimators for the mean model.

Set No.	True α_2	Bias of the estimators $\hat{\alpha}_2$	$\tilde{\alpha}_2$	$\hat{\delta}_1$	$\tilde{\delta}_1$
n=25					
1	-0.80	0.153	0.169	-0.070	-0.048
2	-0.50	0.069	0.092	-0.127	-0.086
3	0.00	-0.061	-0.024	-0.223	-0.151
4	0.50	-0.173	-0.119	-0.321	-0.216
5	0.80	-0.230	-0.169	-0.363	-0.243
6	-0.40	-0.002	0.027	-0.120	-0.064
7	-0.08	-0.055	-0.012	-0.164	-0.079
8	0.40	-0.147	-0.084	-0.243	-0.117
9	0.64	-0.192	-0.119	-0.278	-0.133
10	0.00	-0.083	-0.035	-0.149	-0.054
11	0.25	-0.120	-0.063	-0.173	-0.058
12	0.40	-0.139	-0.075	-0.197	-0.069
13	0.05	-0.088	-0.040	-0.115	-0.018
14	0.08	-0.084	-0.034	-0.095	0.005
15	-0.40	-0.009	0.021	0.021	0.081
16	-0.98	0.023	0.026	0.000	0.005
17	-0.68	-0.025	0.043	-0.008	0.027
18	-0.98	0.023	0.025	0.042	0.047
n=50					
1	-0.80	0.078	0.083	-0.027	-0.018
2	-0.50	0.037	0.046	-0.061	-0.042
3	0.00	-0.025	-0.007	-0.109	-0.074
4	0.50	-0.081	-0.055	-0.157	-0.106
5	0.80	-0.119	-0.089	-0.185	-0.127
6	-0.40	-0.006	0.007	-0.052	-0.025
7	-0.08	-0.045	-0.025	-0.080	-0.041
8	0.40	-0.076	-0.046	-0.128	-0.068
9	0.64	-0.086	-0.051	-0.130	-0.061
10	0.00	-0.045	-0.024	-0.072	-0.030

Table 2. (Continued)

Set No.	True α_1	Bias of the estimators $\hat{\alpha}_2$	$\tilde{\alpha}_2$	$\hat{\delta}_1$	$\tilde{\delta}_1$
11	0.25	-0.067	-0.041	-0.092	-0.039
12	0.40	-0.065	-0.036	-0.086	-0.027
13	0.05	-0.045	-0.022	-0.054	-0.010
14	0.08	-0.052	-0.030	-0.066	-0.021
15	-0.40	-0.004	0.010	0.012	0.038
16	-0.98	0.015	0.016	0.002	0.004
17	-0.68	0.006	0.014	-0.006	0.009
18	-0.98	0.016	0.017	0.027	0.028

Table 3. Empirical mean square error multiplied by ten various estimators for the mean model.

Set No.	True α_2	Mean square error (multiplied by ten) $\hat{\alpha}_2$	$\tilde{\alpha}_2$	$\hat{\delta}_1$	$\tilde{\delta}_1$
n=25					
1	-0.80	0.642	0.671	0.105	0.069
2	-0.50	0.467	0.499	0.282	0.178
3	0.00	0.470	0.440	0.795	0.498
4	0.50	0.792	0.642	1.709	1.110
5	0.80	0.953	0.697	2.123	1.335
6	-0.40	0.391	0.419	0.314	0.216
7	-0.08	0.468	0.468	0.609	0.426
8	0.40	0.615	0.508	1.280	0.898
9	0.64	0.776	0.592	1.800	1.311
10	0.00	0.463	0.454	0.796	0.681
11	0.25	0.546	0.489	1.233	1.066
12	0.40	0.660	0.576	1.736	1.554
13	0.05	0.474	0.452	1.369	1.363
14	0.08	0.489	0.475	1.588	1.652
15	-0.40	0.372	0.412	1.468	1.670
16	-0.98	0.061	0.065	0.025	0.027
17	-0.68	0.235	0.269	0.381	0.424
18	-0.98	0.060	0.065	0.212	0.229
n=50					
1	-0.80	0.204	0.205	0.013	0.009
2	-0.50	0.192	0.199	0.063	0.041
3	0.00	0.220	0.214	0.199	0.130
4	0.50	0.254	0.220	0.410	0.264
5	0.80	0.280	0.216	0.573	0.367
6	-0.40	0.173	0.180	0.076	0.059
7	-0.08	0.210	0.206	0.187	0.147
8	0.40	0.255	0.229	0.470	0.372
9	0.64	0.232	0.193	0.544	0.436

Table 3. (Continued)

Set No.	True α_2	Mean square error (multiplied by ten) $\hat{\alpha}_2$	$\tilde{\alpha}_2$	$\hat{\delta}_1$	$\tilde{\delta}_1$
10	0.00	0.206	0.200	0.298	0.264
11	0.25	0.247	0.228	0.542	0.494
12	0.40	0.239	0.218	0.638	0.598
13	0.05	0.217	0.211	0.603	0.599
14	0.08	0.217	0.207	0.719	0.709
15	-0.40	0.174	0.182	0.675	0.718
16	-0.98	0.019	0.020	0.008	0.008
17	-0.68	0.115	0.122	0.170	0.178
18	-0.98	0.023	0.023	0.074	0.074

Table 4. Empirical bias of various estimators for time trend model.

Set No.	True α_2	Bias of the estimators $\hat{\alpha}_2$	$\tilde{\alpha}_2$	$\hat{\delta}_1$	$\tilde{\delta}_1$
n=25					
1	-0.80	0.082	0.115	-0.169	-0.096
2	-0.50	-0.015	0.033	-0.266	-0.159
3	0.00	-0.158	-0.077	-0.438	-0.263
4	0.50	-0.316	-0.196	-0.594	-0.353
5	0.80	-0.390	-0.248	-0.672	-0.388
6	-0.40	-0.062	-0.006	-0.237	-0.119
7	-0.08	-0.149	-0.068	-0.313	-0.148
8	0.40	-0.251	-0.131	-0.427	-0.186
9	0.64	-0.283	-0.141	-0.483	-0.200
10	0.00	-0.143	-0.052	-0.264	-0.081
11	0.25	-0.193	-0.081	-0.315	-0.091
12	0.40	-0.201	-0.074	-0.312	-0.060
13	0.05	-0.130	-0.033	-0.197	-0.003
14	0.08	-0.112	-0.010	-0.158	0.045
15	-0.40	0.068	0.138	0.190	0.330
16	-0.98	0.027	0.032	-0.003	0.008
17	-0.68	0.015	0.051	-0.034	0.037
18	-0.98	0.055	0.063	0.109	0.125
n=50					
1	-0.80	0.046	0.058	-0.063	-0.038
2	-0.50	-0.000	0.021	-0.120	-0.075
3	0.00	-0.075	-0.034	-0.210	-0.128
4	0.50	-0.155	-0.096	-0.311	-0.192
5	0.80	-0.206	-0.135	-0.361	-0.220
6	-0.40	-0.020	0.007	-0.092	-0.037
7	-0.08	-0.058	-0.019	-0.126	-0.047
8	0.40	-0.119	-0.061	-0.188	-0.072
9	0.64	-0.135	-0.066	-0.222	-0.084

Table 6. (Continued)

Set No.	True α_2	Bias of the estimators $\hat{\alpha}_2$	$\tilde{\alpha}_2$	$\hat{\delta}_1$	$\tilde{\delta}_1$
10	0.00	-0.069	-0.026	-0.118	-0.033
11	0.25	-0.095	-0.042	-0.152	-0.047
12	0.40	-0.098	-0.038	-0.153	-0.034
13	0.05	-0.066	-0.020	-0.098	-0.008
14	0.08	-0.062	-0.015	-0.089	0.005
15	-0.40	-0.001	0.026	0.022	0.076
16	-0.98	0.018	0.020	0.001	0.005
17	-0.68	0.007	0.022	-0.014	0.016
18	-0.98	0.028	0.031	0.055	0.060

Table 5. Empirical mean square error multiplied by ten of various estimators for time trend model.

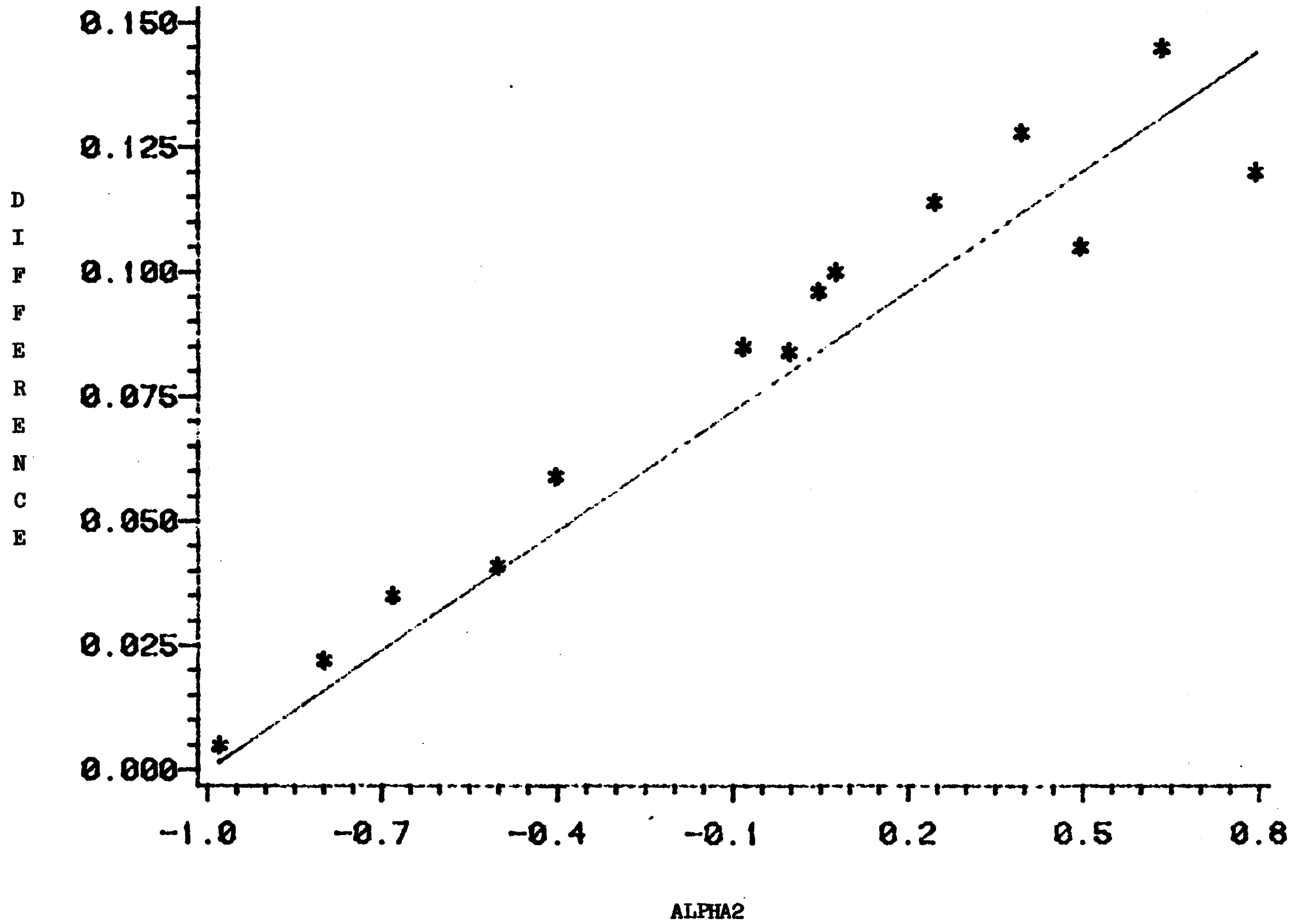
Set No.	True α_2	Mean square error (multiplied by ten) $\hat{\alpha}_2$	$\tilde{\alpha}_2$	$\hat{\delta}_1$	$\tilde{\delta}_1$
n=25					
1	-0.80	0.322	0.407	0.382	0.181
2	-0.50	0.277	0.328	0.916	0.464
3	0.00	0.596	0.469	2.371	1.188
4	0.50	1.423	0.890	4.446	2.277
5	0.80	1.892	1.046	5.513	2.633
6	-0.40	0.332	0.347	0.823	0.415
7	-0.08	0.547	0.434	1.488	0.783
8	0.40	1.033	0.658	2.718	1.399
9	0.64	1.220	0.712	3.492	1.803
10	0.00	0.589	0.499	1.404	0.914
11	0.25	0.776	0.563	2.048	1.368
12	0.40	0.804	0.552	2.302	1.666
13	0.05	0.598	0.537	1.655	1.548
14	0.08	0.538	0.508	1.692	1.784
15	-0.40	0.477	0.717	1.946	3.024
16	-0.98	0.052	0.061	0.025	0.028
17	-0.68	0.234	0.377	0.376	0.455
18	-0.98	0.087	0.105	0.318	0.386
n=50					
1	-0.80	0.128	0.141	0.055	0.028
2	-0.50	0.148	0.159	0.197	0.107
3	0.00	0.239	0.205	0.587	0.310
4	0.50	0.429	0.293	1.234	0.646
5	0.80	0.597	0.370	1.668	0.874
6	-0.40	0.152	0.162	0.151	0.085
7	-0.08	0.219	0.206	0.299	0.175
8	0.40	0.348	0.264	0.701	0.434
9	0.64	0.360	0.241	0.942	0.571

Table 5. (Continued)

Set No.	True α_2	Mean square error (multiplied by ten) $\hat{\alpha}_2$	$\tilde{\alpha}_2$	$\hat{\delta}_1$	$\tilde{\delta}_1$
10	0.00	0.241	0.218	0.405	0.299
11	0.25	0.290	0.237	0.682	0.517
12	0.40	0.295	0.234	0.866	0.703
13	0.05	0.257	0.238	0.720	0.683
14	0.08	0.246	0.230	0.818	0.809
15	-0.40	0.168	0.190	0.650	0.762
16	-0.98	0.024	0.026	0.009	0.010
17	-0.68	0.114	0.129	0.175	0.190
18	-0.98	0.032	0.035	0.112	0.122

DIFFERENCE IN BIASES

(N=25) MEAN MODEL : DELTA1



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