RECURSIVE M-TESTS FOR THE CONSTANCY OF MULTIVARIATE REGRESSION RELATIONSHIPS OVER TIME

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ABSTRACT

Based on a general class of recursive M-estimators of regression parameters in a general multivariate linear model and allied M-statistics, some testing procedures for a possible change in the regression relationships occurring at an unknown time point are considered. The (asymptotic) theory of the proposed tests rests on some invariance principles for recursive M-estimators and related residual M-statistics, and these are studied too.

1. INTRODUCTION

Let $\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_n$ be independent random $p(\geq 1)$-vectors, taken at ordered time points $t_1,\ldots,t_n$, respectively. At time point t_i, we also have a $q(\geq 1)$-vector $\underset{\sim}{c}_i$ of known regression constants, and the following linear model is assumed to hold :

$$\underset{\sim}{X}_i = \underset{\sim}{\beta}^{(i)}\underset{\sim}{c}_i + \underset{\sim}{e}_i \ , \ i=1,\ldots,n \ , \tag{1.1}$$

where the $\underset{\sim}{\beta}^{(i)}$ ($= ((\beta_{jj'}^{(i)}))_{j=1,\ldots,p;j'=1,\ldots,q}$) are unknown ($p\times q$) matrices of regression parameters and the $\underset{\sim}{e}_i$ are independent and identically distributed (i.i.d.) random vectors (r.v.) with a distribution function (d.f.) F defined on the p-dimensional Euclidean space E^p. A special case of this model is the usual shift model where $q=1$ and $c_i = 1$ for every $i \geq 1$; in this case, the $\underset{\sim}{\beta}^{(i)}$ relate to location paramters (vectors). In the context of continuous inspection plans [viz., Page (1954)], one may want to test for the identity of these location vectors against a composite alternative that at an unknown time point (t_m for some m $:1<m\leq n$), a change occurs. In a somewhat more general situation, one may like to test for the null hypothesis

$$H_0: \underset{\sim}{\beta}^{(1)} = \ldots = \underset{\sim}{\beta}^{(n)} = \underset{\sim}{\beta} \text{ (unknown)}, \tag{1.2}$$

against the composite alternative that for some m ,

$$\underset{\sim}{\beta}^{(1)} = \ldots = \underset{\sim}{\beta}^{(m)} \neq \underset{\sim}{\beta}^{(m+1)} = \ldots = \underset{\sim}{\beta}^{(n)} \tag{1.3}$$

where m ($1 \leq m < n$) is unknown. Thus, if H_m stands for the alternative hypothesis in (1.3), we may formulate the composite alternative hypothesis as

$$\mathcal{H} = \bigcup_{m=1}^{n-1} H_m . \tag{1.4}$$

For the univariate case (i.e., $p=1$) and normal F, this testing problem has been treated in detail in Brown, Durbin and Evans (1975) and detailed references to earlier works were also cited there. A very similar problem may also arise in the context of a (generalized) sequential detection problem [viz., Shirayayev (1963, 1978) for the specific location model], where a series of independent random vectors is observed sequentially, such that at each time point the model in (1.1) is assumed to hold, and for some unknown m (possibly $+\infty$) (1.3) holds : The problem is to raise an alarm if $m < +\infty$. In this sequential scheme, one needs to choose a positive integer valued r.v. (i.e., a stopping rule) N , such that if $m < +\infty$, $E(N-m)^+$ [or some other characteristic of $(N-m)^+ = \max(N-m,0)$] should be as small as possible, while, if $m = +\infty$, then $E(N \mid m = \infty)$

should be as large as possible [or the probability of a false alarm i.e., $P\{ N < \infty \mid m = +\infty\}$ should be as small as possible]. It may be noted that in this setup of a sequential detection problem, one has a genuine sequential procedure based on a well defined stopping rule N, while, in the setup of (1.1)-(1.4) [which is usually referred to as the change-point model], the testing procedure may or may not be (quasi-) sequential in nature. If one adapts a quasi-sequential procedure, i.e., (1.2) tested against (1.3) recursively, though n is specified in advance, one may be able to have an early stopping , while, in a non-recursive testing scheme, the test for the change point is made only when all the n observations have been obtained. Methodologically, the testing procedures in the sequential detection problem and the quasi-sequential ones in the change-point problem are very similar, and, we intend to pursue these here.

For the quasi-sequential procedure in the change-point problem, recursive residuals play a very important role. Along with a general formulation of such recursive residuals, some invariance principles for them were considered in Sen (1982*a*). These enable one to test for (1.2) against (1.3) recursively, when F need not be multi-normal (when the sample size is not small). Recursive rank tests for the change-point problem were considered by Bhattacharya and Frierson (1981) for the particular model where under the null hypothesis , the random variables are i.i.d. ; these tests are adaptable for the particular case of the shift model, but, in general, for the regression model the null hypothesis fails to ensure the identity of the d.f.'s of the X_i , and hence, their recursive ranking scheme may run into difficiculties for the general model in (1.1). However, in such a case, one may use aligned (signed) rank statistics based on recursive residuals, and the general asymptotic theory of such recursive residual rank tests was developed in Sen (1983 a). Earlier, Sen (1982 b) considered general recursive U-statistics and studied the asymptotic theory of recursive tests for the change-point model based on such recursive U-statistics (and their jackknifed estimators of variances). For the univariate shift model, recursive M-tests for the change-point problem were studied in Sen(1983b). We intend to extend the methodology of this paper to general multivariate linear models.

Along with the preliminary notions, some non-recursive M-tests for the constancy of regression relationships over time are considered briefly in Section 2. Recursive M-tests are then introduced in Section 3. Along with the basic regularity conditions , some invariance principles for such recursive M-statistics are presented in Section 4. These results are then incorporated in Section 5 in the general formulation of the asymptotic theory of recursive M-tests. The concluding section deals with some general remarks.

2. NON-RECURSIVE M-TESTS

Our proposed tests (both recursive and non-recursive ones) are based on some M-estimators of regression parameters and some related (aligned) M-statistics. First, we introduce these estimators and statistics. Let $\underset{\sim}{\psi} = \{ \underset{\sim}{\psi}(t) = (\psi_1(t),\dots, \psi_p(t))', \ t \in R =(-\infty,\infty)\}$ be a suitable vector of score-functions , and, for every $\underset{\sim}{B} = ((b_{j\ell})) \in R^{pq}$, we write $\underset{\sim}{B}' = (\underset{\sim}{b}_1,\dots,\underset{\sim}{b}_p)$ and let

$$M_{nj\ell}(\underset{\sim}{b}_j) = \Sigma_{i=1}^{n} c_{i\ell}\, \psi_j(X_{ij} - \underset{\sim}{b}_j'\underset{\sim}{c}_i) \ , \ 1 \le j \le p; \ 1 \le \ell \le q \ , \quad \dots (2.1)$$

where $\underset{\sim}{c}_i = (c_{i1},\dots,c_{iq})'$ and $\underset{\sim}{X}_i = (X_{i1},\dots,X_{ip})'$, $i=1,\dots,n$. We assume that for each j (=1,...,p), ψ_j is nondecreasing, continuous and skew-symmetric (i.e., $\psi_j(u) + \psi_j(-u) = 0, \ \forall \ u \in R$), ψ_j has a bounded derivative inside $(-k_j,k_j)$ (where $0 < k_j < \infty$) and $\psi_j(x) = \psi_j(k_j)\text{sign}x$ for $|x| \ge k_j$. In passing, we may remark that this boundedness condition on the ψ_j is not unnatural for the M-procedures. In fact, for the particular Huber-estimator, one has $\psi_j(x) = x$ for $|x| \le k_j$ and $k_j\text{sign}x$ for $|x| > k_j$. Such a boundedness condition induces robustness against outliers, and, can mostly be justified on the ground of (local) minimax properties . Also, we assume that corresponding to the d.f. F in (1.1), the jth marginal d.f. , denoted by $F_{[j]}$, is symmetric about the origin , so that for every j(=1,...,p) , $\int_R \psi_j(x)dF_{[j]}(x) = 0$. Concerning the regressors $\underset{\sim}{c}_i$, we assume that there exists a positive definite (p.d.) and finite matrix $\underset{\sim}{C}_o$, such that for every (fixed) $t \in (0,1)$, as $n \to \infty$,

$$n^{-1}\Sigma_{i=1}^{[nt]} \underset{\sim}{c}_i \underset{\sim}{c}_i' = n^{-1}\underset{\sim}{C}_{[nt]} \to t\underset{\sim}{C}_o . \tag{2.2}$$

We also assume that

$$\max_{1\leq j\leq q} (n^{-1}\Sigma_{i=1}^{n} |c_{ij}|^r) = O(1) , \text{ for } r=3,4. \tag{2.3}$$

Note that (2.2) and (2.3) ensure that

$$\max_{1\leq j\leq q} \max_{1\leq i\leq n} n^{-\frac{1}{2}} |c_{ij}| \to 0 , \text{ as } n \to \infty . \tag{2.4}$$

Based on the terminal M-estimator $\hat{\underset{\sim}{\beta}}_n' = (\hat{\underset{\sim}{\beta}}_{n1},\ldots,\hat{\underset{\sim}{\beta}}_{np})$ of $\underset{\sim}{\beta}'$, we consider the residuals

$$\hat{\underset{\sim}{X}}_{ni} = \underset{\sim}{X}_i - \hat{\underset{\sim}{\beta}}_n \underset{\sim}{c}_i , \; i= 1,\ldots,n. \tag{2.5}$$

We define the residual CUSUM M-scores by

$$\hat{\underset{\sim}{M}}_{nk} = \underset{\sim}{M}_k(\hat{\underset{\sim}{\beta}}_n) = ((M_{kj\ell}(\hat{\underset{\sim}{\beta}}_{nj}))) , \; k=1,\ldots,n. \tag{2.6}$$

Also, we define a pxp matrix $\underset{\sim}{S}_n = ((S_{njj'}))$ by letting

$$S_{njj'} = n^{-1}\Sigma_{i=1}^{n}\psi_j(\hat{X}_{nij})\psi_{j'}(\hat{X}_{nij'}) , \; j,j'=1,\ldots,p. \tag{2.7}$$

Finally, let $\hat{\underset{\sim}{M}}_{nk}^o$ be the pq-vector obtained from rolling out $\hat{\underset{\sim}{M}}_{nk}$, for k=1,...,n. Conventionally, we let $\hat{\underset{\sim}{M}}_{nk}^o = \underset{\sim}{0}$ for k=0. Note that for k=n, $\hat{\underset{\sim}{M}}_{nn}^o = \underset{\sim}{0}$, by the definition in (2.1). Consider then the set of partial scores statistics

$$\mathcal{L}_{nk} = (\hat{\underset{\sim}{M}}_{nk}^o)' (\underset{\sim}{S}_n \otimes \underset{\sim}{C}_n)^{-1} (\hat{\underset{\sim}{M}}_{nk}^o) , \; k = 0,\ldots,n. \tag{2.8}$$

The test for the change-point model is then based on the partial sequence in (2.8). If,for some k : $1\leq k\leq n$, $\mathcal{L}_{nk}$ exceeds a critical value $\ell_{n\alpha}$, we reject the null hypothesis of constancy of the regression relationships over time. Otherwise, we accept the null hypothesis. The problem is therefore to determine the critical level $\ell_{n\alpha}$, so that the level of significance of this test is equal to the preassigned value α :$0 < \alpha < 1$.

Let $W_i^o = \{W_i^o(t),\ t \in (0,1)\}$, i=1,...,m be m independent copies of a standard Brownian bridge , and let $B_m^o = \{B_m^o(t),\ t \in (0,1)\}$ be defined by letting $(B_m^o(t))^2 = \Sigma_{i=1}^{m}(W_i^o(t))^2$, $t \in (0,1)$. Thus, B_m^o is an m-dimensional tied-down Bessel Process. Let then $\lambda_\alpha^{(m)}$ be defined by

$$P\{\sup_{0\le t\le 1} B_m^o(t) > \lambda_\alpha^{(m)}\} = \alpha . \tag{2.9}$$

Then, we shall show that under the assumed regularity conditions,

$$\ell_{n\alpha} \to \lambda_\alpha^{(m)}, \text{ as } n \to \infty . \tag{2.10}$$

For the critical levels $\lambda_\alpha^{(m)}$, we may refer to Kiefer(1959) for $m \le 5$. For higher values of m, a recent programme worked out in DeLong (1981) may be used with advantage.

To establish (2.10), we need to prove the stochastic convergence of $\underset{\sim}{S}_n$ to $\underset{\sim}{\Sigma}^*$, where $\underset{\sim}{\Sigma}^* = ((\sigma^*_{jj'}))$ and $\sigma^*_{jj'} = E\psi_j(X_{ij}-\underset{\sim}{\beta}_j'\underset{\sim}{c}_i)\psi_{j'}(X_{ij'}-\underset{\sim}{\beta}_{j'}'\underset{\sim}{c}_i)$, for $j,j'=1,\ldots,p$, and also an invariance principle for the aligned M-statistics $\{\hat{\underset{\sim}{M}}{}^o_{nk} ; k \le n\}$. First, we note that by virtue of Theorem 3.1 of Jurečková and Sen (1984), as extended to the multivariate case in a coordinatewise manner, we have under the assumed regularity conditions,

$$n^{\frac{1}{2}}(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\beta}) = n^{-\frac{1}{2}}\, \underset{\sim}{\Gamma}^{-1}\underset{\sim}{M}_n(\underset{\sim}{\beta})\underset{\sim}{C}_o^{-1} + \underset{\sim}{O}_p(n^{-\frac{1}{2}}), \text{ as } n \to \infty , \tag{2.11}$$

where $\underset{\sim}{\Gamma} = \text{Diag.}(\gamma_1, \ldots, \gamma_p)$ with

$$\gamma_j = \int_{-\infty}^{\infty} \psi_j'(x)dF_{[j]}(x) = \int_{-k_j}^{+k_j} \psi_j'(x)dF_{[j]}(x) , \tag{2.12}$$

for $j=1,\ldots,p$. Let us define $\underset{\sim}{S}{}_n^o = ((S^o_{njj'}))$ by letting

$$S^o_{njj'} = n^{-1}\Sigma_{i=1}^n \psi_j(X_{ij}-\underset{\sim}{\beta}_j'\underset{\sim}{c}_i)\psi_{j'}(X_{ij'}-\underset{\sim}{\beta}_{j'}'\underset{\sim}{c}_i),\ j,j'=1,\ldots,p. \tag{2.13}$$

Also, we make use of the fact that the ψ_j are all bounded and they satisfy the Lipschitz condition that for every $x, y \in E$ and $j=1,\ldots,p$, $|\psi_j(x) - \psi_j(y)| \le K|x-y|$, where $K(<\infty)$ does not depend on x,y,j. Then by (2.5), (2.7), (2.11) and (2.13), we conclude that

$$\underset{\sim}{S}_n - \underset{\sim}{S}{}_n^o = \underset{\sim}{O}_p(n^{-\frac{1}{2}}), \text{ as } n \to \infty . \tag{2.14}$$

On the other hand, under H_0, the $\underset{\sim}{X}_i - \underset{\sim}{\beta}'\underset{\sim}{c}_i$ are i.i.d.r.v., so that by the Kintchine strong law of large numbers, we have

$$\underset{\sim}{S}{}_n^o \to E\underset{\sim}{S}{}_n^o = \underset{\sim}{\Sigma}^*, \text{ a.s., as } n \to \infty . \tag{2.15}$$

This establishes the stochastic convergence of $\underset{\sim}{S}_n$ to $\underset{\sim}{\Sigma}^*$. Next, we we consider the following asymptotic linearity result on M-statistics and this extends Lemma 3.1 of Jurečková and Sen (1984) to the multivariate case in the setup of an invariance principle.

Let $K^* = [0,1] \times [-K, K]^q$ ($\subset E^{q+1}$), and for every $n(\geq 1)$, s $[0,1]$, $\underset{\sim}{t} \in [-K,K]^q$ and j $(=1,\ldots,p)$, define (a q-vector)

$$\underset{\sim}{N}_{nj}(s,\underset{\sim}{t}) = \Sigma_{i\leq[ns]} \underset{\sim}{c}_i[\psi_j(Y_{ij}+ n^{-\frac{1}{2}}\underset{\sim}{t}'\underset{\sim}{c}_i) - \psi_j(Y_{ij}) - n^{-\frac{1}{2}}\underset{\sim}{t}'\underset{\sim}{c}_i\gamma_j] , \qquad (2.16)$$

where $Y_{ij} = X_{ij} - \underset{\sim}{\beta}_j'\underset{\sim}{c}_i$ and the γ_j are deined by (2.12). Then, for each j, $\{\underset{\sim}{N}_{nj}(s,\underset{\sim}{t}), (s,\underset{\sim}{t}) \in K^*\}$ is a q-variate process defined on the space $D^q[K^*]$.

Lemma 2.1. Under the assumed regularity conditions, as $n \to \infty$,

$$\max_{1\leq j\leq p} \sup_{(s,\underset{\sim}{t})\in K^*} ||\underset{\sim}{N}_{nj}(s,\underset{\sim}{t})|| = O_p(1) \quad . \qquad (2.17)$$

[Note that here we assume that H_0 holds.]

The proof follows precisely on the same lines as in the proof of Lemma 3.1 of Jurečková and Sen (1984); the only extra step needed here is the reconstruction (for $s_1 \geq s_2$ and $\underset{\sim}{u}, \underset{\sim}{t} \in [-K,K]^q$) :

$$\underset{\sim}{N}_{nj}(s_1,\underset{\sim}{t}) - \underset{\sim}{N}_{nj}(s_2,\underset{\sim}{t}) - \underset{\sim}{N}_{nj}(s_1,\underset{\sim}{u}) + N_{nj}(s_2,\underset{\sim}{u})$$
$$= \Sigma_{[ns_2]<i\leq[ns_1]} \underset{\sim}{c}_i[\psi_j(Y_{ij}+n^{-\frac{1}{2}}\underset{\sim}{t}'\underset{\sim}{c}_i) - \psi_j(Y_{ij}+n^{-\frac{1}{2}}\underset{\sim}{u}'\underset{\sim}{c}_i) - n^{-\frac{1}{2}}(\underset{\sim}{t}-\underset{\sim}{u})'\underset{\sim}{c}_i\gamma_j] ; \qquad (2.18)$$

the fourth central moment of the right hand side of (2.18) is bounded by $K(s_1-s_2)^2||\underset{\sim}{t}-\underset{\sim}{u}||^4$, for some $K(< \infty)$, uniformly in s_1, s_2, $\underset{\sim}{t}$ and $\underset{\sim}{u}$, so that the rest of the proof follows as in Jurečková and Sen (1984).

Now, by virtue of (2.11) and Lemma 2.1, we may conclude, following some routine steps, that under H_0 and the assumed regularity conditions,

$$\max_{0\leq k\leq n} ||\hat{\underset{\sim}{M}}_{nk} - [\underset{\sim}{M}_k(\underset{\sim}{\beta}) - \underset{\sim}{M}_n(\underset{\sim}{\beta})\underset{\sim}{C}_n^{-1}\underset{\sim}{C}_k] || = O_p(1) \quad . \qquad (2.19)$$

On the other hand, the $\underset{\sim}{M}_k(\underset{\sim}{\beta})$ involve independent matrix-valued summands, so that the classical invariance principle holds under H_0 and the given regularity conditions. As such, if we define $\underset{\sim}{W}_n^* = \{\underset{\sim}{W}_n^*(t), t \in [0,1]\}$ by

$$\underset{\sim}{W}_n^*(t) = (\underset{\sim}{\Sigma}^* \otimes \underset{\sim}{C}_n)^{-\frac{1}{2}} \hat{\underset{\sim}{M}}^o_{n[nt]} \quad , t \in [0,1], \qquad (2.20)$$

and if $\underset{\sim}{W}^* = (W_1^o ,\ldots,W_{pq}^o)'$ be a vector of pq independent copies of a standard Brownian bridge on [0,1], then, we conclude from the above that under H_0 and the assumed regularity conditions, as $n \to \infty$,

$$\underset{\sim}{W}_n^* \xrightarrow{\mathfrak{D}} \underset{\sim}{W}^* \text{, in the } J_1\text{-topology on } D^{pq}[0,1]. \quad (2.21)$$

Since $\underset{\sim}{S}_n$ converges in probability to $\underset{\sim}{\Sigma}^*$, by virtue of (2.8) and (2.21), we conclude that under H_0 and the assumed regularity conditions, as $n \to \infty$,

$$\max_{0 \le k \le n} \mathcal{L}_{nk} \xrightarrow{\mathfrak{D}} \sup_{0 \le t \le 1} B_{pq}^{o\,2}(t) , \quad (2.22)$$

and this ensures that (2.10) holds.

The weak convergence in (2.21) may also be incorporated along with 'contiguity of probability measures' in the study of the asymptotic power properties of the test for local (contiguous) alternative hypotheses. Towards this, we assume that in (1.3), we have a sequence $\{m(n)\}$ of positive integers, such that as $n \to \infty$,

$$n^{-1}m(n) \to \pi : 0 < \pi < 1 , \quad (2.23)$$

and there exists a non-null pxq matrix $\underset{\sim}{\Theta}$, such that in (1.3)

$$\underset{\sim}{\beta}_{m(n)} = \underset{\sim}{\beta}_{m(n)+1} - n^{-\frac{1}{2}}\underset{\sim}{\Theta} . \quad (2.24)$$

We denote by $\{K_n\}$, the sequence of alternative hypotheses for which (2.23) and (2.24) hold. Also, we assume that for each j(=1, ...,p), the d.f. $F_{[j]}$ admits an absolutely continuous probability density function (p.d.f.) $f_{[j]}$ with a finite Fisher information $I(f_{[j]})$. This along with the assumed regularity conditions insure the contiguity of the probability measure under $\{K_n\}$ to that under the null hypothesis. As such the stochastic convergence of $\underset{\sim}{S}_n$ to $\underset{\sim}{\Sigma}^*$ under H_0 extends to that under $\{K_n\}$ as well. Further, (2.21) also extends to that under $\{K_n\}$, where we need to adjust $\underset{\sim}{W}^*$ by a suitable drift function. We denote by $\underset{\sim}{\Theta}^o$ the rolled out pq-vector form of $\underset{\sim}{\Theta}$, and define $\underset{\sim}{T} = \underset{\sim}{\Gamma}\underset{\sim}{\Sigma}^*\underset{\sim}{\Gamma}$ and let

$$\underset{\sim}{\omega}^o(t) = \begin{cases} -t(1-\pi)(\underset{\sim}{T} \otimes \underset{\sim}{C}_o)^{-\frac{1}{2}}\underset{\sim}{\Theta}^o, & 0 \le t \le \pi , \\ -\pi(1-t)(\underset{\sim}{T} \otimes \underset{\sim}{C}_o)^{-\frac{1}{2}}\underset{\sim}{\Theta}^o, & \pi \le t \le 1; \end{cases} \quad (2.25)$$

$\underset{\sim}{\omega}^o = \{\underset{\sim}{\omega}^o(t),\ 0 \le t \le 1\}$. Then, we have under $\{K_n\}$,

$$\underset{\sim}{W}_n^* \xrightarrow{\mathfrak{D}} \underset{\sim}{W}^* + \underset{\sim}{\omega}^o \text{, in the } J_1\text{-topology on } D^{pq}[0,1]. \quad (2.26)$$

Thus, if we define $\omega_Q^* = \{\omega^*(t),\ 0 \le t \le 1\}$ by letting

$$\omega_Q^*(t) = [\underset{\sim}{\omega}^o(t)]'[\underset{\sim}{\omega}^o(t)] ,\ 0 \le t \le 1 , \quad (2.27)$$

then under $\{K_n\}$ and the assumed regularity conditions, we have

$$\max_{0\le k\le n} \mathcal{L}_{nk} \quad \xrightarrow{\mathfrak{D}} \quad \sup_{0\le t\le 1}\{ B_{pq}^{o2}(t) + \omega_o^*(t)\} \text{ , as } n \to \infty. \tag{2.28}$$

Thus, the asymptotic power function of the test is given by

$$P\{ B_{pq}^{o2}(t) + \omega_o^*(t) \ge \lambda_\alpha^{(pq)} \text{ , for some } t \in [0,1] \}. \tag{2.29}$$

We shall make some further comments on this test in Section 3 .

3. RECURSIVE M-TESTS

Note that the use of the terminal estimator $\hat{\underset{\sim}{\beta}}_n$ to define the residuals in (2.5) has rendered a non-recursive character of the test based on the $\mathcal{L}_{nk}$ in (2.8), though the $\mathcal{L}_{nk}$ are computed at the successive observations. To construct some recursive tests, one needs to use some recursive estimates of the regression parameters and to incorporate them in the definition of so called recursive residuals. Also, one needs to estimate $\underset{\sim}{\Sigma}^*$ recursively. With these in mind, we proceed as follows.

Based on $\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_k$, the M-statistics $M_{kj\ell}(\underset{\sim}{b}_j)$, $j=1,\ldots,p$, $\ell = 1,\ldots,q$ are defined as in (2.1), for $k=1,\ldots,n$. With the same set of regularity conditions as in Section 2, let $\hat{\underset{\sim}{\beta}}_k' = (\hat{\beta}_{k1},\ldots,\hat{\beta}_{kp})$ be the M-estimator of $\underset{\sim}{\beta}$, based on the $M_{kj\ell}$, for $k=1,\ldots,n$. At the kth stage, we may consider then the residuals

$$\hat{\underset{\sim}{X}}_{ki} = \underset{\sim}{X}_i - \hat{\underset{\sim}{\beta}}_{k-1}\underset{\sim}{c}_i \text{ , } i=1,\ldots,k \text{ (} k=1,\ldots,n). \tag{3.1}$$

For $k \le q$, conventionally, we may let $\hat{\underset{\sim}{X}}_{ki} = \underset{\sim}{0}$, $i=1,\ldots,k$. We let $\hat{\underset{\sim}{X}}_{ki} = (\hat{X}_{ki,1},\ldots,\hat{X}_{ki,p})'$ and define the recursive M-statistics by

$$\hat{\underset{\sim}{M}}_k^* = (\hat{M}_{k1}^*,\ldots,\hat{M}_{kp}^*)' \text{ , } \hat{M}_{kj}^* = \Sigma_{i=1}^k \psi_j(\hat{X}_{ii,j}), \ 1\le j\le p. \tag{3.2}$$

Conventionally, we let $\hat{\underset{\sim}{M}}_k^* = \underset{\sim}{0}$, for $k=0,\ldots,q$. Also, analogous to (2.7), we define $\underset{\sim}{S}_k$, replacing n and the $\underset{\sim}{X}_{ni}$ by k and $\underset{\sim}{X}_{ki}$, respectively, for $k=1,\ldots,n$. Consider then the partial scores statistics

$$\mathcal{L}_{nk}^* = n^{-1}\{(\hat{\underset{\sim}{M}}_k^*)'\tilde{\underset{\sim}{S}}_k^{-1}(\hat{\underset{\sim}{M}}_k^*)\} \text{ , } k = q+1,\ldots,n, \tag{3.3}$$

where $\tilde{\underset{\sim}{S}}_k = \underset{\sim}{S}_k$ if $ch_p(\underset{\sim}{S}_k) \ge \varepsilon$, (for some $\varepsilon > 0$) and $\varepsilon \underset{\sim}{I}_p$, otherwise; $ch_p(\underset{\sim}{A})$ stands for the smallest characteristic root of $\underset{\sim}{A}$. Unlike in

(2.8), here the estimate S_k is used at the kth stage, and S_k may be singular for $k \leq q$. While, we may use a generalized inverse in (3.3) to eliminate this problem, there is little loss of generality in letting, conventionally, $\mathcal{L}^*_{nk} = 0$, for $k \leq q$. Further, in (2.8), we have used for the $\hat{M}^o_{kn}$ matrices of order pxq, whereas in (3.3), we have used p-vectors for the $\hat{M}^*_k$. This is mainly in the spirit of using robust recursive residuals, and the use of these lower dimensional vectors may enhance the power of the procedure. We may also note that since the observations X_i are taken sequentially at the ordered time points $\{t_i\}$ and the recursive statistics $\mathcal{L}^*_{nk}$ are based only on the set $X_1, \dots, X_k$, $k \geq 1$, so that a test for the null hypothesis H_0 can be based in a quasi-sequential manner. Operationally, the test procedure consists in computing the $\mathcal{L}^*_{nk}$ at the successive time-points : If, for some k(= K), for the first time, $\mathcal{L}^*_{nK}$ exceeds some critical level $\ell^*_{n\alpha}$, one stops at that time point t_K along with the rejection of H_0; if no such $K(\leq n)$ exists, one proceeds on to the last stage and accepts H_0. The basic problem is to determine the critical level,such that

$$P\{\mathcal{L}^*_{nk} > \ell^*_{n\alpha} \text{ for some } k: k \leq n \mid H_0\} = \alpha, \tag{3.4}$$

where α is the desired level of significance of this quasi-sequential procedure. We shall provide suitable asymptotic approximations for the $\ell^*_{n\alpha}$ under the same regularity conditions as in Section 2. In this quasi-sequential procedure, one allows an early stopping based on the cumulative evidence at the successive stages. The stopping number K is a positive integer valued random variable and $K \leq n$, with probability one.

To study the asymptotic properties of the proposed M-test (and to determine $\ell^*_{n\alpha}$ in (3.4)), we need to study first some invariance principles relating to the recursive estimators M^*_k and S_k, for $k = 1, \dots, n$. We present these basic results in the next section. These results are then incorporated in Section 5 for the study of the general properties of the proposed tests.

4. INVARIANCE PRINCIPLES FOR RECURSIVE M-STATISTICS

To fix notations, we define for every $\underset{\sim}{B} = (\underset{\sim}{b}_1,\ldots,\underset{\sim}{b}_p)' \in E^{pq}$,

$$\underset{\sim}{\psi}(\underset{\sim}{X}_i - \underset{\sim}{B}\underset{\sim}{c}_i) = (\psi_1(X_{i1} - \underset{\sim}{b}_1'\underset{\sim}{c}_i),\ldots,\ \psi_p(X_{ip} - \underset{\sim}{b}_p'\underset{\sim}{c}_i))' \tag{4.1}$$

for $i=1,\ldots,n$. Note that by definition, the $\underset{\sim}{\psi}(\underset{\sim}{X}_i - \underset{\sim}{B}\underset{\sim}{c}_i)$ are bounded random vectors for every i and $\underset{\sim}{B}$. Hence, if we choose any sequence $\{k_n\}$ of positive integers, such that

$$k_n \to \infty \quad \text{but} \quad n^{-\frac{1}{2}}k_n \to 0 \quad \text{as } n \to \infty\,, \tag{4.2}$$

then, we have

$$\max_{k \le k_n} \|\, n^{-\frac{1}{2}} \Sigma_{i \le k}\ \underset{\sim}{\psi}(\hat{\underset{\sim}{X}}_{ii})\,\| \to 0\ , \text{ in probability}, \tag{4.3}$$

where the $\hat{\underset{\sim}{X}}_{ii}$ are defined by (3.1). We also define the $S_{kjj'}$ as in (2.7) , but for a sample of size k, for $k=1,\ldots,n$. Note that

$$\max_{k \le n} \|\underset{\sim}{S}_k\| \text{ is bounded with probability 1 .} \tag{4.4}$$

Consequently, by (3.2), (3.3),(4.2), (4.3) and (4.4), we have

$$\max_{k \le k_n} \mathcal{L}^*_{nk} \to 0\ , \text{ in probability, as } n \to \infty\ ; \tag{4.5}$$

in the sequel, we choose $k_n = [n^{\eta}]$ where $\eta = 625/1296$ ($< \frac{1}{2}$).

Next, we note that under H_0, $\{\underset{\sim}{M}_k(\underset{\sim}{\beta});\ k \ge 0\}$ forms a (pxq matrix valued) martingale sequence (actually with independent zero mean increments), where the scores are all bounded (ensuring the existence of moments of all finite orders). Hence, we may use the law of iterated logarithm and conclude that under H_0 and the assumed regularity conditions, as $n \to \infty$,

$$\max_{k_n < k \le n} \{(k \text{ loglog } k)^{-\frac{1}{2}}\ \|\underset{\sim}{M}_k(\underset{\sim}{\beta})\|\} = O_p(1)\ . \tag{4.6}$$

Further, we define $n = n_o$ and let

$$n_1 = [n^{5/6}],\ n_2 = [n^{25/36}],\ n_3 = [n^{125/216}] \text{ and } n_4 = k_n. \tag{4.7}$$

Defining the $\underset{\sim}{N}_{nj}(s,\underset{\sim}{t})$ as in (2.16) and letting $\underset{\sim}{N}^*_{nj}(s,\underset{\sim}{t})$ be equal to $n^{-1/12}(\text{loglog } n)^{-\frac{1}{2}}\underset{\sim}{N}_{nj}(s,\ n^{1/12}(\text{loglog } n)^{\frac{1}{2}}\underset{\sim}{t})$, for $s,\underset{\sim}{t} \in K^*$, we obtain as in Lemma 2.1 that under H_0 and the assumed regularity conditions,

$$n^{-1} \max_{n_1 \le s \le 1}\ \sup_{\underset{\sim}{t} \in K^*} \|\,\underset{\sim}{N}^*_{nj}(s,\underset{\sim}{t})\| = O_p(1), \tag{4.8}$$

for every j(=1,...,p). In a similar manner, we have

$$\max_{1\le j\le p} \max_{n_{r+1}/n_r \le s\le 1} \sup_{\underset{\sim}{t} \in K^*} \| \underset{\sim}{N}_{n_r j}(s,\underset{\sim}{t})\| = O_p(1), \qquad (4.9)$$

for every r (=0,1,2,3). Combining these and noting that $n_r^{1/12} \le k^{1/10}$ for every $k : n_{r+1} \le k \le n_r$, r=0,1,2,3, we obtain that under H_0 and the assumed regularity conditions, as $n \to \infty$,

$$\max_{1\le j\le p} \max_{k_n\le k\le n} \sup_{\underset{\sim}{t} \in (\text{loglog } k)^{\frac{1}{2}}[-K,K]^q} k^{-1/10} \|\underset{\sim}{N}_{kj}(1,\underset{\sim}{t})\|$$
$$= O_p((\text{loglog } n)^{\frac{1}{2}}). \qquad (4.10)$$

By virtue of (2.16), (3.6) and (4.10), we obtain that under H_0 and the assumed regularity conditions, as $n \to \infty$,

$$\max_{k_n\le k\le n} k^{9/10}\| (\hat{\underset{\sim}{\beta}}_k - \underset{\sim}{\beta}) - \underset{\sim}{\Gamma}^{-1}\underset{\sim}{M}_k(\underset{\sim}{\beta})\underset{\sim}{C}_k^{-1} \| = O_p((\text{loglog } n)^{\frac{1}{2}}). \quad (4.11)$$

Let E_k denote the conditional expectation given $\underset{\sim}{X}_i$, $i \le k$, for $k \ge 1$, Recall that the ψ_j are all bounded and satisfy the Lipschitz condition. Then, on letting $\underset{\sim}{Z}_i = \underset{\sim}{\psi}(\hat{\underset{\sim}{X}}_{ii})-\underset{\sim}{\psi}(\underset{\sim}{X}_i-\underset{\sim}{\beta}\underset{\sim}{c}_i)-E_{i-1}\underset{\sim}{\psi}(\hat{\underset{\sim}{X}}_{ii}), i\ge q$, we note that the $\|\underset{\sim}{Z}_i\|$ are bounded r.v., $E_{i-1}\underset{\sim}{Z}_i = 0$ and $E\|\underset{\sim}{Z}_i\|^2 = E[E_{i-1}(\|\underset{\sim}{Z}_i\|^2)] \to 0$ as i increases. Consequently, using the martingale property of the partial sums $\Sigma_{i\le k} \underset{\sim}{Z}_i$, $k \ge 1$, it follows by standard steps that under H_0 and the assumed conditions,

$$\max_{k\le n} \{n^{-\frac{1}{2}} \| \Sigma_{i\le k} \underset{\sim}{Z}_i\| \} \to 0, \text{ in probability, as } n\to\infty. \quad (4.12)$$

Also, note that

$$E_{i-1}\underset{\sim}{\psi}(\hat{\underset{\sim}{X}}_{ii}) = \int \underset{\sim}{\psi}(\underset{\sim}{x} -(\hat{\underset{\sim}{\beta}}_{i-1} - \underset{\sim}{\beta})\underset{\sim}{c}_i)dF(\underset{\sim}{x})$$
$$= \int[\underset{\sim}{\psi}(\underset{\sim}{x}- (\hat{\underset{\sim}{\beta}}_{i-1}- \underset{\sim}{\beta})\underset{\sim}{c}_i) - \underset{\sim}{\psi}(\underset{\sim}{x})]\, dF(\underset{\sim}{x})$$
$$= - \underset{\sim}{\Gamma}(\hat{\underset{\sim}{\beta}}_{i-1} - \underset{\sim}{\beta})\underset{\sim}{c}_i + o(\|(\hat{\underset{\sim}{\beta}}_{i-1} -\underset{\sim}{\beta})\underset{\sim}{c}_i\|), \qquad (4.13)$$

so that

$$\max_{k_n\le k\le n} \| n^{-\frac{1}{2}} \Sigma_{i\le k} \{E_{i-1}\underset{\sim}{\psi}(\hat{\underset{\sim}{X}}_{ii}) + \underset{\sim}{\Gamma}(\hat{\underset{\sim}{\beta}}_{i-1} - \underset{\sim}{\beta})\underset{\sim}{c}_i\} \|$$
$$\le n^{-\frac{1}{2}}\Sigma_{k\le n} \{o(\|(\hat{\underset{\sim}{\beta}}_{i-1} - \underset{\sim}{\beta})\underset{\sim}{c}_i \|)\}$$
$$= o_p(1) , \qquad (4.14)$$

where the last step follows from (4.11) along with the fact that by virtue of the martingale invariance principle under the d_q-norm [c.f. Theorem 2.4.8 of Sen (1981)], for every $\eta >0$,

$$\max_{k\le n}\{ n^{-\frac12}(k/n)^{-\frac12+\eta}\| \underset{\sim}{M}_k(\underset{\sim}{\beta})\| \} = O_p(1) . \tag{4.15}$$

Therefore, from (4.12) and (4.14), we obtain that under H_0 and the assumed regularity conditions, as $n \to \infty$,

$$\max_{k_n\le k\le n} \|n^{-\frac12} \Sigma_{i\le k} \{\underset{\sim}{\psi}(\hat{\underset{\sim}{X}}_{ii}) - \underset{\sim}{\psi}(\underset{\sim}{X}_i - \underset{\sim}{\beta}\underset{\sim}{c}_i) + \underset{\sim}{\Gamma}(\hat{\underset{\sim}{\beta}}_{i-1} - \underset{\sim}{\beta})\underset{\sim}{c}_i\}\| \overset{P}{\to} 0. \tag{4.16}$$

Finally, recall that by (4.11), simultaneously for all $k: k_n \le k \le n$,

$$\begin{aligned}
&n^{-\frac12}\Sigma_{i\le k}\{ \underset{\sim}{\psi}(\underset{\sim}{X}_i - \underset{\sim}{\beta}\underset{\sim}{c}_i) - \underset{\sim}{\Gamma}(\hat{\underset{\sim}{\beta}}_{i-1} - \underset{\sim}{\beta})\underset{\sim}{c}_i \} \\
&= n^{-\frac12}\Sigma_{i\le k}\{ \underset{\sim}{\psi}(\underset{\sim}{X}_i - \underset{\sim}{\beta}\underset{\sim}{c}_i) - \underset{\sim}{M}_{i-1}(\underset{\sim}{\beta})\underset{\sim}{C}_{i-1}^{-1}\underset{\sim}{c}_i\} + O_p(n^{-2/5}\log\log n) \\
&= n^{-\frac12}\Sigma_{i\le k} [\underset{\sim}{\psi}(\underset{\sim}{X}_i - \underset{\sim}{\beta}\underset{\sim}{c}_i) - \Sigma_{j=1}^{i-1}\underset{\sim}{\psi}(\underset{\sim}{X}_j - \underset{\sim}{\beta}\underset{\sim}{c}_j)\underset{\sim}{c}_j'\underset{\sim}{C}_{i-1}^{-1}\underset{\sim}{c}_i] + O_p(n^{-2/5}\log\log n) \\
&= n^{-\frac12}\Sigma_{i\le k} (\Sigma_{j\le i} a_{ij}\, \underset{\sim}{\psi}(\underset{\sim}{X}_j - \underset{\sim}{\beta}\underset{\sim}{c}_j)) + O_p(n^{-2/5}\log\log n) ,
\end{aligned} \tag{4.17}$$

where $a_{ij} = -\underset{\sim}{c}_j'\underset{\sim}{C}_{i-1}^{-1}\underset{\sim}{c}_i$ for $j<i$, $a_{ii} = 1$ and, conventionally, we let $a_{ij} = 0$ for $j > i$. Note that

$$\begin{aligned}
\Sigma_{j\le i} a_{ij}^2 &= 1 + \Sigma_{j\le i-1} \underset{\sim}{c}_i'\underset{\sim}{C}_{i-1}^{-1}\underset{\sim}{c}_j\underset{\sim}{c}_j'\underset{\sim}{C}_{i-1}^{-1}\underset{\sim}{c}_i \\
&= 1 + \underset{\sim}{c}_i'\underset{\sim}{C}_{i-1}^{-1}(\Sigma_{j\le i-1} \underset{\sim}{c}_j\underset{\sim}{c}_j')\underset{\sim}{C}_{i-1}^{-1}\underset{\sim}{c}_i \\
&= 1 + \underset{\sim}{c}_i'\underset{\sim}{C}_{i-1}^{-1}\underset{\sim}{c}_i \\
&= 1 + \mathrm{Tr.}(\underset{\sim}{C}_{i-1}^{-1}\underset{\sim}{c}_i\underset{\sim}{c}_i') \\
&= 1 + \mathrm{Tr.}(\underset{\sim}{C}_{i-1}^{-1}\underset{\sim}{C}_i - \underset{\sim}{I}_q) \\
&= 1 + O(i^{-1}), \text{ by (2.2);}
\end{aligned} \tag{4.18}$$

$$\Sigma_j a_{ij}a_{i'j} = 0 , \text{ for every } i \ne i' = 1,\dots,n. \tag{4.19}$$

Thus, if we define $\underset{\sim}{W}_n = \{ \underset{\sim}{W}_n(t),\ t \in [0,1]\}$ by letting

$$\underset{\sim}{W}_n(t) = n^{-\frac12}\underset{\sim}{\Sigma}^{*-\frac12} \Sigma_{i\le[nt]} \underset{\sim}{\psi}(\hat{\underset{\sim}{X}}_{ii}) , \quad t \in [0,1], \tag{4.20}$$

then, by an appeal to (4.16), (4.17), (4.18), (4.19) and Theorem 1 of Sen (1982a), it follows that under H_0 and the assumed regularity conditions, as $n \to \infty$,

$$\underset{\sim}{W}_n \overset{\mathfrak{D}}{\to} \underset{\sim}{W} , \text{ in the Skorokhod } J_1\text{-topology on } D^p[0,1], \tag{4.21}$$

where $\underset{\sim}{W} = \{ \underset{\sim}{W}(t),\ t \in [0,1]\}$ consists of p independent copies of a standard Brownian motion on [0,1]. Also, using (2.14), (2.15) along with (4.11), it follows that under H_0 and the assumed regularity conditions, as $n \to \infty$,

$$\max_{k_n \le k \le n} || \underset{\sim}{S}_k - \underset{\sim}{\Sigma}^* || \to 0 \text{ , in probability.} \qquad (4.22)$$

The invariance principles in (4.21) and (4.22) provide the basic key to the study of the main results pertaining to the recursive M-statistics. Defining $\underset{\sim}{W}$ as in after (4.21), we consider a p-parameter Bessel process $B_p = \{B_p(t),\ t \in [0,1]\}$, where

$$B_p^2(t) = [\underset{\sim}{W}(t)]'[\underset{\sim}{W}(t)] \text{ , } t \in [0,1] \text{ .} \qquad (4.23)$$

Then, from (3.3), (4.5), (4.20), (4.21) and (4.22), we conclude that under H_0 and the assumed regularity conditions, as $n \to \infty$,

$$\max_{k \le n} \mathcal{L}^*_{nk} \xrightarrow{\mathcal{D}} \sup_{0 \le t \le 1} B_p^2(t) \text{ .} \qquad (4.24)$$

We shall find this result very useful in the next section.

As in Section 2, we may consider some local (contiguous) alternatives [viz., (2.23)-(2.24)] and extend these invariance principles to such cases too. In this context, we assume that as k increases,

$$\bar{\underset{\sim}{c}}_k = k^{-1}\Sigma_{i=1}^k \underset{\sim}{c}_i \to \bar{\underset{\sim}{c}} \quad \text{where } ||\bar{\underset{\sim}{c}}|| < \infty \text{ .} \qquad (4.25)$$

First, we note that by virtue of the contiguity of probability measures under $\{K_n\}$ to that under H_0 , the stochastic equivalences in (4.16), (4.17) and (4.22), holding under H_0 , remain in tact under $\{K_n\}$ as well. Secondly,the tightness of $\{W_n\}$ (under H_0),ensured by (4.21), also holds under $\{K_n\}$ as well. Hence, to study the weak convergence of $\{W_n\}$ under $\{K_n\}$, all we need to study is the convergence of the finite dimensional distributions of W_n to those of some p-dimensional Wiener process with an apprpriate drift function. In view of (4.17) being validated under $\{K_n\}$, the asymptotic multinormality result will follow readily through the use of some standard central limit theorems on the $\underset{\sim}{\psi}(\underset{\sim}{X}_i - \underset{\sim}{\beta}\underset{\sim}{c}_i)$, and hence, we need to study the nature of the drift function only. Towards this, recall that by virtue of (2.23) (where we take $\underset{\sim}{\beta}_{m(n)} = \underset{\sim}{\beta}$ and $\underset{\sim}{\beta}_{m(n)+1} = \underset{\sim}{\beta} + n^{-\frac{1}{2}} \underset{\sim}{\Theta}$, for some (fixed) $\underset{\sim}{\Theta} \in E^{pq}$), for every $i \le m(n)$, $\underset{\sim}{\psi}(\underset{\sim}{X}_i - \underset{\sim}{\beta}\underset{\sim}{c}_i)$ has expectation $\underset{\sim}{0}$, while for $i > m(n)$, $\underset{\sim}{\psi}(\underset{\sim}{X}_i - \underset{\sim}{\beta}\underset{\sim}{c}_i)$ has expectation $n^{-\frac{1}{2}}\underset{\sim}{\Gamma}\underset{\sim}{\Theta}\underset{\sim}{c}_i + \underset{\sim}{o}(n^{-\frac{1}{2}})$. As such, for every $k \le m(n)$, under K_n ,

$n^{-1/2}\Sigma_{i\le k}\Sigma_{j\le i}\ a_{ij}\psi(X_j - \beta c_j)$ has expectation equal to 0 , while, for $k > m(n) = m$,

$$E\{n^{-1/2}\Sigma_{i\le k}\Sigma_{j\le i}\ a_{ij}\psi(X_j - \beta c_j)\ \mid\ K_n\}$$
$$= n^{-1}\Sigma_{m<i\le k}\Sigma_{m<j\le i}\ a_{ij}\Gamma\Theta c_j \quad + o(1)$$
$$= \Gamma\Theta\ \{(k/n)\bar{c}_k - n^{-1}\Sigma_{m<i\le k}\Sigma_{m<j<i}[-c_j' C_{i-1}^{-1} c_i]c_j\} + o(1)$$
$$= \Gamma\Theta\ \{(k/n)\bar{c}_k - n^{-1}\Sigma_{m<i\le k}\ [C_{i-1} - C_m]C_{i-1}^{-1}c_i\ \} + o(1)$$
$$= \Gamma\Theta\ \{(k/n)\bar{c}_k - n^{-1}\Sigma_{m<i\le k}\ [\ I - C_m C_{i-1}^{-1}]c_i\ \} \quad + o(1)$$
$$= \Gamma\Theta\ \{(m/n)\bar{c}_m + (m/n)\Sigma_{m<i\le k}\ (m^{-1}C_m)((i-1)C_{i-1}^{-1})[\tfrac{i}{i-1}\bar{c}_i - \bar{c}_{i-1}]\} + o(1)$$
$$= \Gamma\Theta\ \{(m/n)\Sigma_{m<i\le k}\ [m^{-1}C_m(i-1)C_{i-1}^{-1}](i-1)^{-1}\bar{c}_{i-1}\} + o(1)$$
$$= \Gamma\Theta\ \{(m/n)\Sigma_{m<i\le k}\ [\ I + O(\tfrac{1}{m})][\bar{c} + o(1)]/(i-1)\} + o(1)$$
$$= \Gamma\Theta\ \{(m/n)\bar{c}\ \log(k/m)\} + o(1), \text{ by (2.2) and (4.25).} \tag{4.26}$$

Hence, if we define $\omega = \{\omega(t),\ 0\le t\le 1\}$ by letting

$$\omega(t) = \begin{cases} 0\ , & 0\le t\le \pi\ , \\ \pi(\Sigma^*)^{-1/2}\Gamma\Theta\bar{c}\ \log(t/\pi)\ , & \pi < t \le 1, \end{cases} \tag{4.27}$$

then, we conclude that under $\{K_n\}$ and the assumed regularity conditions, as $n \to \infty$,

$$W_n \xrightarrow{\mathcal{D}} W + \omega, \text{ in the } J_1\text{-topology on } D^p[0,1]\ . \tag{4.28}$$

Since (4.22) holds under $\{K_n\}$ as well, the weak convergence result in (4.28) also provides an extension of (4.24) to that under $\{K_n\}$. Towards this, we define $\omega^* = \{\omega^*(t),\ 0\le t\le 1\}$ by letting

$$\omega^*(t) = [\omega(t)]'[\omega(t)]\ ,\ 0 \le t \le 1. \tag{4.29}$$

Then, under $\{K_n\}$ and the assumed regularity conditions, as $n \to \infty$,

$$\max_{k\le n} \mathcal{L}^*_{nk} \xrightarrow{\mathcal{D}} \sup_{0\le t\le 1} \{B_p^2(t) + \omega^*(t)\}\ . \tag{4.30}$$

The asymptotic properties of the proposed tests may now be studied with the aid of (4.24) and (4.30).

5. ASYMPTOTIC PROPERTIES OF RECURSIVE M-TESTS

For the Bessel process B_p, defined by (4.23), we define the critical level $\lambda_{p,\alpha}$ by

$$P\{ \sup_{0 \le t \le 1} B_p(t) > \lambda_{p,\alpha} \} = \alpha \ (0 < \alpha < 1) . \qquad (5.1)$$

For specific values of p and α, these critical levels have been computed by DeLong(1980). Let us now consider the critical value $\ell^*_{n\alpha}$ in (3.4). By virtue of (4.24) and (5.1), we conclude that under the assumed regularity conditions (of Section 2),

$$\ell^*_{n\alpha} \to \lambda^2_{p,\alpha} \quad \text{as } n \to \infty . \qquad (5.2)$$

The rapidity of the convergence in (5.2), of course, depends on F,p, q, the $\underset{\sim}{c}_i$ as well as the scores $\psi_1,\ldots,\psi_p$. Since, we are dealing here with boundedly continuous scores, the rate of convergence is expected to be faster than that of the case of maximum likelihood estimator based procedures (where the score functions are essentially unbounded in the majority of the cases). Moreover, the M-statistics considered here are robust against outliers and gross errors, so that the dependece of (5.2) on the underlying d.f. F is likely to be less stringent than in the case of the other procedure.

Let us next study the consistency of the proposed tests for any fixed alternative. Note that under (1.3), for some m: $m/n \to \pi : 0 < \pi < 1$ and $\underset{\sim}{\beta}_{m+1} - \underset{\sim}{\beta}_m = \underset{\sim}{\Delta} \ (\neq \underset{\sim}{0})$, (4.4) remains valid, and for $k \le m$, the $\underset{\sim}{S}_k$ still converge stochastically to $\underset{\sim}{\Sigma}^*$, while for $k > m$, the stochastic limit of $\underset{\sim}{S}_k$ is given by $\underset{\sim}{\Sigma}^* + \underset{\sim}{\Sigma}^{**}_k$, where $\underset{\sim}{\Sigma}^{**}_k$ is a positive semidefinite matrix whose elements are all bounded. On the other hand, for $k > m$, the $\underset{\sim}{\psi}(\hat{\underset{\sim}{X}}_{kk})$ would have centering vectors progressing away from $\underset{\sim}{0}$ (as k moves away from m), and hence, by (3.3), for k greater than m, the $\mathcal{L}^*_{nk}$ will be stochastically larger. Simple computations show that for every k: $m/n < k/n \le 1$, $n^{-1}\mathcal{L}^*_{nk}$ converges stochatically to a positive constant, so that $\mathcal{L}^*_{nk}$ exceeds the null critical level $\ell^*_{n\alpha}$ (converging to a fixed $\lambda^2_{p,\alpha}$) with probability converging to 1, as $n \to \infty$. This establishes the consistency of the proposed test for any fixed alternative.

We have noticed in Section 3 that the recursive M-tests allow the possibility of an earlier stopping. Define the stopping variable K as in before (3.4). Note that by definition, we have for every k: $1 \leq k \leq n-1$,

$$P\{ K > k\} = P\{ \mathcal{L}^*_{nr} \leq \ell^*_{n\alpha} \text{ , for every } r \leq k\}, \tag{5.3}$$

while, $P\{ K > n \} = 0$. Hence, by (4.24) and (5.2), under H_0 and the assumed regularity conditions, for every $t \in (0,1)$, as $n \to \infty$,

$$\begin{aligned} P\{ K > [nt] \mid H_0 \} &\to P\{\sup_{0 \leq u \leq t} B^2_p(t) \leq \lambda^2_{p,\alpha} \} \\ &= P\{\sup_{0 \leq u \leq 1} B^2_p(u) \leq t^{-1}\lambda^2_{p,\alpha} \}. \end{aligned} \tag{5.4}$$

As a result, we conclude that

$$E\{ n^{-1}K \mid H_0 \} \to \int_0^1 P\{ \sup_{0 \leq u \leq 1} B^2_p(u) \leq t^{-1}\lambda^2_{p,\alpha} \} \, dt . \tag{5.5}$$

We may again refer to DeLong(1980) for some numerical studies relating to the right hand side of (5.5) (made in a different context). In a similar manner, it follows that for any fixed change in the regression parameter (i.e., for a given $\pi : 0 < \pi < 1$ and $\underset{\sim}{\Delta} \neq \underset{\sim}{0}$), for every $\eta > 0$, as $n \to \infty$,

$$P\{ K > [n(\pi + \eta)] \mid \underset{\sim}{\Delta} \neq \underset{\sim}{0} , 0 < \pi < 1 \} \to 1 , \tag{5.6}$$

so that with a probability converging to 1, the procedure terminates soon after a change-point occurs. Notice that there is always a probability of stopping earlier even if no change in the $\underset{\sim}{\beta}_i$ occur ; of course, such a probability is usually quite small.

Let us next consider the case of local (contiguous) alternative hypotheses, treated in Sections 2 and 4. Here, by virtue of (4.30), we conclude that under $\{K_n\}$ and the assumed regularity conditions, the asymptotic power of the recursive M-tests is given by

$$P\{ B^2_p(t) + \omega^*(t) > \lambda^2_{p,\alpha} \text{ , for some } t \in [0,1] \} , \tag{5.7}$$

where ω^* is defined by (4.29). Here also, the results of DeLong (1980) may be used to provide some numerical results. There is , however, a difference in the nature of the two drift functions in the two cases, and hence, the results may not be strictly comparable.

Parallel to (5.4) and (5.5), we have here

$$P\{ K > [nt] \mid K_n \} \rightarrow P\{ \sup_{0 \le u \le t} (B_p^2(u) + \omega^*(u)) \le \lambda^2_{p,\alpha} \}, \quad (5.8)$$

$$E(n^{-1}K \mid K_n) \rightarrow \int_0^1 P\{ \sup_{0 \le u \le t} (B_p^2(u) + \omega^*(u)) \le \lambda^2_{p,\alpha} \} \, dt. \quad (5.9)$$

Now, looking at (4.27) and (4.29), we gather that for $\pi < t \le 1$,

$$\omega^*(t) = \pi^2 (\bar{\underset{\sim}{c}}' \underset{\sim}{\Theta}' \underset{\sim}{\Gamma} (\underset{\sim}{\Sigma}^*)^{-1} \underset{\sim}{\Gamma} \underset{\sim}{\Theta} \bar{\underset{\sim}{c}})(\log(t/\pi))^2 , \quad (5.10)$$

and $\omega^*(t) = 0$ for $t \le \pi$. Whereas π , $\bar{\underset{\sim}{c}}$, $\underset{\sim}{\Theta}$ and $\log(t/\pi)$ do not depend on the particular set of score functions (i.e., $\underset{\sim}{\psi}$), the matrix $\underset{\sim}{\Gamma} (\underset{\sim}{\Sigma}^*)^{-1} \underset{\sim}{\Gamma}$ depends on the choice of $\underset{\sim}{\psi}$ and on the underlying d.f. F. The same matrix appears in the non-centrality parameter of the usual M-test statistic for the classical linear model , and hence, the usual asymptotically optimal choice of $\underset{\sim}{\psi}$ for such models remain in tact in the current context too.

6. SOME GENERAL REMARKS

We may observe that the right hand side of (5.4) is bounded from below by $1 - \alpha$, uniformly in $t \in (0,1]$, and as t becomes small, this converges to 1. Thus, the right hand side of (5.5) is bounded from below by $1 - \alpha$, and, in reality, it is quite close to 1. This shows that in the null case, there is not much chance of an early termination. On the other hand, in (5.8) and (5.9), for $t > \pi$, the drift function $\omega^*(t)$ is positive and it reduces the corresponding probabilities. Hence, the expected stopping time becomes smaller. The study of the forms of the right hand sides of (5.8) and (5.9) demands the distribution theory of the first passage time for a Bessel process with a segmented square log-function, as in (5.10). Such results are not that precisely known. However, numerical studies (as in DeLong (1980,1981)) or simulation studies can be made in specific cases to gather more ideas. Whereas the recursive M-tests enjoy this scope for an early termination, the non-recursive ones in Section 2 do not. One may, however, attempt to compare (2.29) and (5.7) for the study of the asymptotic relative efficiency (A.R.E.) of the recursive tests with respect to the other one (keeping aside the issue of early stopping). There are some technical problems in this context. First, in

(2.29), we have a pq-parameter Bessel process, while in (5.7), we have a p-parameter Bessel process, where $q \geq 1$; even, for $q = 1$, these two processes are not the same. In (2.29), we have a tied-down process vanishing at both the ends, while in (5.7), the Bessel process vanishes at the lower extremity only. Hence, **comparing** solely the two drift functions ω^* and ω_o^* may not provide any real picture of their A.R.E. Secondly, $\omega_o(t)$ in (2.25) is segmented linear in t, while $\omega(t)$ in (4.27) is segmented logarithmic in t. Even if we express (2.29) in terms of a (drifted) Bessel process (on the entire line $[0,\infty)$), the range of that process will be different from that in (5.07) and the two drift functions will also be of different forms. As such, we are not in a position to study the Pitman A.R.E. of the non-recursive tests with respect to the recursive ones (based on the same set of scores). Intuitively, however, for $q > 1$, the recursive procedures may perform better than the non-recursive ones for a broad class of situations. Numerical studies in specific cases are suggested to cast more light on the relative performances of these procedures. We conclude this section with the remark that the recursive procedures in Section 3 may easily be adapted in the context of the sequential detection problem. Actually, (4.11), (4.15), (4.16) and (4.17) all extend naturally to a.s. results (as $n \to \infty$), and hence, in a sequential setup, one may use (4.17) and a.s. invariance principles for the triangular scheme$\{ \Sigma_{i<k} \Sigma_{j<i} a_{ij} \underset{\sim}{\psi}(\underset{\sim}{X}_j - \underset{\sim}{\beta} \underset{\sim}{c}_j) , k \geq k_n \}$ and construct sequential procedures based on the recursive M-statistics. These invariance principles provide asymptotic expressions for the probability of a false alarm as well as for the quickest detection when there is a genuine change-point.

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