RANDOM HYPERPLANE INTERSECTING A FIXED LINE IN R^n AND THE CAUCHY DISTRIBUTION

by

Ibrahim A. Salama

Department of Biostatistics
University of North Carolina at Chapel Hill

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Ibrahim A. Salama

Department of Biostatistics
University of North Carolina
Chapel Hill, NC 27514

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ABSTRACT

Let L be a fixed line in R^n given by the equation $L(t) = B + tV$, $B,V \in R^n$ and $t \in R$. Let $P(c)$ be a hyperplane in R^n given by the equation $c.P = 0$, where $c \in S^{n-1}$ and $c.V \neq 0$. Let $L(t_c)$ be the point of intersection of L and $P(c)$, and define $X_n(P(c)) = \mathrm{Sgn}(t_c)||L(t_c) - L(0)||$. Consider X_n as a map from $S_V^{n-1} = \{c \in S^{n-1} | c.V \neq 0\}$ to R, and assume a uniform distribution over S_V^{n-1}. We show that for every $n \geq 2$, X_n has the same distribution as $aX + b$, where X is standard Cauchy and a,b are constants. Some consequences of this result are discussed.

1. INTRODUCTION

Let L be a fixed line in R^n given by the equation $L(t) = B + tV$, $B,V \in R^n$ and $t \in R$. For simplicity of exposition we assume for the moment that $||B|| = ||V|| = 1$ and $B.V = 0$. Let $P(c)$ be a hyperplane in R^n given by the equation $c.P = 0$, where

$c \in S^{n-1}$ (the unit sphere in R^n) and $c.V \neq 0$. Let $L(t_c)$ be the point of intersection of L and P(c), and define $X_n(P(c)) = \mathrm{Sgn}(t_c)||L(t_c) - L(0)||$.

Let $P_v = \{P(c)|c \in S^{n-1}, c.V \neq 0\}$, then X_n may be viewed as a map from P_v to R, or equivalently as a map from $S_v^{n-1} = \{c \in S^{n-1}| c.V \neq 0\}$ to R. Assuming that P_v is given a uniform probability structure, which induces a uniform distribution over S_v^{n-1}, the question is: what is the probability distribution of the random variable $X_n(c)$, $c \in S_v^{n-1}$.

2. THE PROBABILITY DISTRIBUTION OF X_n AND RELATED RESULTS

First, note that the probability distribution of X_n is invariant under the group of transformations representing rotations in R^n. Hence, in answering our question, there is no loss in generality by assuming that the line L is given by $L(t) = e_1 + te_n$, $e_1, \ldots, e_n$ are the standard basis for R^n. Second: $X_n(c)$ is given by $X_n(c) = - c^tB/c^tV$. Finally: for n = 2, we have the well known result that $X_2(c)$ is Cauchy. In fact, this will be the case for every n as given by the following.

Theorem 1:

For $n \geq 2$, we have

$$P\{X_n \leq t\} = \frac{1}{2} + \frac{1}{\pi} \mathrm{Tan}^{-1}(t), \quad -\infty < t < \infty,$$

that is

$$f_{X_n}(t) = \frac{1}{\pi(1 + t^2)}, \quad -\infty < t < \infty.$$

Proof:

Assume $L(t) = e_1 + te_n$. Let S_u^{n-1} = the upper half of (S^{n-1} - the equator). For given $t \in R$, consider the corresponding point

$L(t) \varepsilon R^n$. Then we have $X_n(c) = t$ for all $c \varepsilon S_u^{n-1} \cap P(L(t))$. Let $S_u^{n-1}(t)$ be that cross section of S_u^{n-1} bounded by $P(e_n)$ and $P(L(t))$ and intersects R_+^n (the positive orthant of R^n). Then we have

$$P\{X_n \leq t\} = P\{c \ \varepsilon \ S_u^{n-1}(t)\} = \frac{A(S_u^{n-1}(t))}{A(S_u^{n-1})} ,$$

where $A(\cdot)$ is the area function. Let θ be the angle between $P(e_n)$ and $P(L(t))$, $0 < \theta < \pi$. Then it is easy to see that $A(S_u^{n-1}(t)) = \pi^{n/2 - 1}/\Gamma(n/2)\ \theta$. Let $\phi = \theta - \pi/2$, then since $\mathrm{Tan}(\phi) = t$, we get

$$A(S_u^{n-1}(t)) = \frac{\pi^{n/2}}{2\ \Gamma(n/2)} + \frac{\pi^{n/2 - 1}}{\Gamma(n/2)}\phi = \frac{\pi^{n/2}}{2\ \Gamma(n/2)} + \frac{\pi^{n/2 - 1}}{\Gamma(n/2)} \mathrm{Tan}^{-1}(t) .$$

Hence

$$P\{X_n \leq t\} = 1/2 + 1/\pi\ \mathrm{Tan}^{-1}(t) \ , \ t \ \varepsilon \ R$$

(since $A(S_u^{n-1}) = \dfrac{\pi^{n/2}}{\Gamma(n/2)}$) .

In the general case, L is given by $L(t) = B + tV$, such that $||V|| = 1$ and the angle θ between B and V satisfies $0 < \theta < \pi$. Define $Y_n(c) = \mathrm{Sgn}(t_c)||L(t_c) - L(0)||$. Let V_B be such that $||V_B|| = 1$, $V.V_B = 0$, and $B = ||B||\ \mathrm{Sin}\ \theta\ V_B + ||B||\ \mathrm{Cos}\ \theta\ V$. Since

$$Y_n(c) = -\frac{c^tB}{c^tV} = -\frac{1}{c^tV}\ [||B||\,\mathrm{Sin}\ \theta\ c^tV_B + ||B||\,\mathrm{Cos}\ \theta\ c^tV]$$

$$= ||B||\,\mathrm{Sin}\ \theta\ X_n(c) - ||B||\ \mathrm{Cos}\ \theta\ ,$$

we have

Theorem 2:

For $n \geq 2$, we have

$$P\{Y_n \le t = P\left\{X_n \le \frac{t + ||B|| \text{ Cos } \theta}{||B|| \text{ Sin } \theta}\right\}$$

$$= \frac{1}{2} + \frac{1}{\pi} \text{Tan}^{-1} \left(\frac{t + ||B|| \text{ Cos } \theta}{||B|| \text{ Sin } \theta}\right) , \quad t \in R$$

that is

$$f_{Y_n}(t) = \frac{||B|| \text{ Sin } \theta}{\pi[||B||^2 \text{ Sin}^2 \theta + (t + ||B|| \text{ Cos } \theta)^2]} , \quad t \in R$$

which is cauchy with location parameter $- ||B|| \text{ Cos } \theta$, and scale parameter $||B|| \text{ Sin } \theta$.

Corollary 3:

If A_1 and A_2 are any two non-colinear vectors in S^{n-1}, then the random variable $Y = A_1^t C / A_2^t C$, where C is chosen uniformly in S^{n-1}, has the same distribution as $aX + b$, where X is standard Cauchy and a,b are constants.

Corollary 4:

Let A_1, A_2 be any two orthogonal vectors in S^{n-1}. If $Y = C_1^t A_1 / C_1^t A_2 \cdot C_2^t A_1 / C_2^t A_2 = C_1^t (A_1 A_1^t) C_2 / C_1^t (A_2 A_2^t) C_2$, where C_1 and C_2 are chosen independently and uniformly in S^{n-1}, then $f_Y(y)$ is the density function of the product of two independent standard Cauchy. That is

$$f_Y(y) = \frac{1}{\pi^2 (y^2 - 1)} \log (y^2) , \quad -\infty < y < \infty .$$

Proof:

The probability density function of the product of two independent standard Cauchy is given by Springer and Thompson (1966).

Corollary 5:

Under the same notations and assumptions of Corollary 4, the random variable $Y = C_1^t\,(A_1A_2^t)\,C_2 / C_1^t\,(A_2A_1^t)\,C_2$ has the probability density function $f_Y(y)$ given above.

Let $X_1, \ldots, X_n$ be n independent standard Cauchy. Set $Y_n = \prod_{i=1}^{n} X_i$. Springer and Thompson (1966) gave the probability density function of Y_n, $n \leq 10$, explicitly. This result may be applied in our situation by choosing $X_i = C_i^t\,A_1/C_i^t\,A_2$ or $X_i = C_i^t\,A_{1i}/C_i^t\,A_{2i}$, where A_{1i}, A_{2i} are orthogonal unit vectors. Let

$$Z_{n,k} = \prod_{i=1}^{k} X_i \prod_{j=k+1}^{n} X_i^{-1} ,$$

and noting that $Z_{n,k}$ has the same distribution as Y_n, thus we may choose some of the X_i's as $X_i = C_i^t\,A_2/C_i^t\,A_1$.

Corollary 6:

Let A_1, A_2 be any two orthogonal unit vectors. If $Y = C^t\,(A_1A_1^t)\,C/C^t\,(A_2A_2^t)\,C$, where C is chosen uniformly in S^{n-1}, then $f_Y(y) = 1/\pi(1 + y)\sqrt{y}$, $0 \leq y < \infty$. That is, Y is $\beta_2(1/2, 1/2)$.

3. CAUCHY AS PRODUCT OF INDEPENDENT RANDOM VARIABLES

Let $I_0 = (-\pi, \pi)$, $I_1 = (-\pi/2, \pi/2)$ and $\Omega = I_0 \times I_1^{n-2}$. For S^{n-1} consider the parametrization $T: \Omega \to S^{n-1}$, where for $\Theta \in \Omega$ we have

$$T(\Theta) = T(\theta_0, \ldots, \theta_{n-2}) = U_n = (U_{n1}, \ldots, U_{nn}) \in S^{n-1}, \text{ and}$$

$$U_{n1} = \text{Sin } \theta_{n-2}$$
$$U_{n2} = \text{Cos } \theta_{n-2} \text{ Sin } \theta_{n-3}$$
$$\vdots$$
$$U_{n,\, n-2} = \text{Cos } \theta_{n-2} \cdots \text{Cos } \theta_2 \text{ Sin } \theta_1$$
$$U_{n,\, n-1} = \text{Cos } \theta_{n-2} \cdots \text{Cos } \theta_1 \text{ Sin } \theta_0$$
$$U_{nn} = \text{Cos } \theta_{n-2} \cdots \text{Cos } \theta_1 \text{ Cos } \theta_0$$

Regard $\theta \varepsilon \Omega$ as a random vector, and let g be the probability density function (p.d.f.) over Ω which is carried by T to the uniform distribution over S^{n-1}. Noting that the area element associated with T is given by $\lambda(\Theta) = \prod_{i=1}^{n-2} \text{Cos}^i \theta_i \, d\theta$, thus $g(\theta) = \prod_{i=1}^{n-2} \text{Cos}^i \theta_i / A(S^{n-1})$. Hence, the marginal p.d.f.'s $g_0, \ldots, g_{n-2}$ of $\theta_0, \ldots, \theta_{n-2}$ are given by

$$g_0(\theta) = \frac{1}{2\pi} \quad , \quad -\pi < \theta < \pi \ ,$$

$$g_i(\theta) = a_i^{-1} \text{Cos}^i \theta \quad , \quad \pi/2 < \theta < \pi/2 \quad , \quad i = 1, \ldots, n-2 \ ,$$

where $a_i = 2^i \beta\left(\frac{i+1}{2}, \frac{i+1}{2}\right)$, and $\theta_0, \ldots, \theta_{n-2}$ are independent.

Assume that $L(t) = e_1 + te_n$, and for $U_n \varepsilon S^{n-1}$ let $X = U_n^t e_n / U_n^t e_1 = U_{nn}/U_{n1}$. By theorem 1, X is standard Cauchy. In general for $i \neq j$ U_{ni}/U_{nj} is standard Cauchy. Thus we have

Theorem 7:

For every $m \geq 1$, there exists m independent (nondegenerate) random variables $X_1, \ldots, X_m$ such that $\prod_{i=1}^{m} X_i$ is standard Cauchy.

Corollary 8:

Let X be standard cauchy, then:

(i) for every $m \geq 1$ and every integer $k \neq 0$, there exists m independent (nondegenerate) random variables $X_1, \ldots, X_m$ such that $\prod_{i=1}^{m} X_i$ has the same distribution as X^k.

(ii) for every $m \geq 1$ and every real $\alpha \neq 0$, there exists m independent (nondegenerate) random variables $X_1, \ldots, X_m$ such that $\prod_{i=1}^{m} X_i$ has the same distribution as $|X|^{\alpha}$.

4. SOME RELATED RESULTS

Consider the random variables $X_i \equiv \sin\theta$, $Y_i \equiv \cos\theta$, and $Z_i \equiv \cot\theta$, where $\theta \sim g_i$ (~ means has the p.d.f.) and i = 0, 1, 2, ... Let $f_i(h_i, q_i)$ be the p.d.f. of $X_i(Y_i, Z_i)$ respectively. Thus we have

$$f_i(x) = a_i^{-1}(1 - x^2)^{\frac{i-1}{2}} \quad , \quad -1 < x < 1 \quad , \; i = 0, 1, 2, \ldots,$$

$$h_0(x) = f_0(x) \; ,$$

$$h_i(x) = 2\, a_i^{-1}\, x^i/\sqrt{1 - x^2} \quad , \quad 0 < x < 1 \quad , \; i = 1, 2, \ldots,$$

and

$$q_i(x) = a_i^{-1}\, x^i/(1 + x^2)^{\frac{i}{2} + 1} \quad , \quad -\infty < x < \infty \quad , \quad i = 0, 1, 2, \ldots.$$

For $n \geq 2$, write (with the conversion $X_{-1} \equiv 1$)

$$U_{ni} = X_{n-i-1} \cdot \prod_{j=n-i}^{n-2} Y_i \quad , \quad i = 1, \ldots, n \; .$$

Defining $Y_{nk} = \prod_{i=k}^{n} Y_i$, $k = 0, 1, \ldots, n$, then

$$U_{n,n-k} = Y_{n-2,k} \cdot X_{k-1} \quad , \quad k = 0, 1, \ldots, n-1 \; .$$

Lemma 9:

For every $n \geq 2$, the U_{ni}'s are identically distributed. Hence their common p.d.f. is f_{n-2}.

Proof:

The fact that the U_{ni}'s are identically distributed follows at once from the symmetry of S^{n-1}. Hence the common p.d.f. is that of U_{n1}.

Notes:

(1) f_i is the beta p.d.f. over $(-1, 1)$ with parameters $\left(\frac{i+1}{2}, \frac{i+1}{2}\right)$ and will be denoted here by $\bar{\beta}\left(\frac{i+1}{2}, \frac{i+1}{2}\right)$.

(2) $Y_i^2 \sim \beta_1\left(\frac{i+1}{2}, \frac{1}{2}\right)$. Since the product of independent $\beta_1(a, b)$ and $\beta_1(a + b, c)$ is $\beta_1(a, b + c)$, then $\prod_{k=i}^{n} Y_i^2 = Y_{nk}^2 \sim \beta_1\left(\frac{k+1}{2}, \frac{n-k+1}{2}\right)$. Hence

$$Y_{nk} \sim 2\left[\beta\left(\frac{k+1}{2}, \frac{n-k+1}{2}\right)\right]^{-1} y^k(1 - y^2)^{\frac{n-k-1}{2}} \quad , \quad 0 < y < 1 .$$

(3) Let $D = \{W_1, \ldots, W_n\}$ be a set of random variables. Define the random variable $D^* = \prod_{i=1}^{n} W_i$. D may be partitioned into arbitrary disjoint sets $D_1, \ldots, D_m$. For every partition of D we have a corresponding representation of D^* (in D) given by $D^* = \prod_{i=1}^{m} D_i^*$. We say $D \sim f^*$ if $D^* \sim f^*$. If D_1 and D_2 are two sets of random variables, then we say $D_1 \overset{*}{=} D_2$ if D_1^* and D_2^* are identically distributed.

This notion provides us with an economical way to state our results. Statements concerning product of random variables may be considered through this notion. For example, lemma 9 is a statement about the equality ($\overset{*}{=}$) of n sets. Let P be the number of partitions of a set D, then the statement $D \sim f^*$ is in fact P statements concerning product of random variables. The following corollary is formulated in this notion.

Corollary 10:

Consider the following sets of independent random variables:

$$D_0 = \{Z_0\},\ D_1 = \{Z_1,\ Y_0\}, \dots,\ D_i = \{Z_i,\ Y_{i-1}, \dots,\ Y_0\},\ \dots$$

then

$$D_i \stackrel{*}{=} D_0 \quad , \quad i = 1, 2, \dots .$$

In particular, if X and Y are independent such that $X \sim \overline{\beta}\left(\frac{i+1}{2}, \frac{i+1}{2}\right)$ and $Y \sim q_{i+1}$, then for every integer $i \geq 0$, XY is standard Cauchy.

If $D = \{W_1, \dots, W_n\}$, we define $D^2 = \{W_1^2, \dots, W_n^2\}$.

Corollary 11:

With the same notations and assumptions of Corollary 10, we have

$$D_i^2 \stackrel{*}{=} D_0^2 \quad , \quad i = 1, 2, \dots .$$

In particular, if X and Y are independent such that $X \sim \beta_1\left(\frac{1}{2}, \frac{i+1}{2}\right)$ and $Y \sim \beta_2\left(\frac{i+2}{2}, \frac{1}{2}\right)$, then for every integer $i \geq 0$, $XY \sim \beta_2\left(\frac{1}{2}, \frac{1}{2}\right)$.

Theorem 12:

The content of Theorem 7 and Corollary 8 applies to $\overline{\beta}\left(\frac{i+1}{2}, \frac{i+1}{2}\right)$ and $\beta_2\left(\frac{1}{2}, \frac{1}{2}\right)$ (with minor modifications).

The following statements are stated here because they are easily obtained from our results.

Corollary 13:

Let C_1 and C_2 be two points chosen independently and uniformly in S^{n-1}. Let d_n be the distance between C_1 and C_2. Then $\frac{1}{4} d_n^2 \sim \beta_1\left(\frac{n-1}{2}, \frac{n-1}{2}\right)$. Hence

$$d_n \sim a_{n-2}^{-1}\, x^{n-2}\left(1 - \frac{x^2}{4}\right)^{\frac{n-3}{2}} , \quad 0 < x < 2 .$$

Proof:

Without loss of generality fix $C_1 = e_1$ and set $C_2 = U_n$ to be chosen uniformly in S^{n-1}. Thus

$$d_n^2 = \|e_1 - U_n\|^2 = 2 - 2 \sin \theta, \ \theta \sim g_{n-2} .$$

Now, we generalize lemma 9 as follows:

Corollary 14:

Let P_k be a fixed subspace of R^n of dimension k, and $\pi_k: R^n \to P_k$ be the orthogonal projection onto P_k. Let $C(U_n)$ be a point (vector) chosen uniformly in S^{n-1}, and d_{nk} be the length of the orthogonal projection of $C(U_n)$ onto P_k. Then $d_{nk}^2 \sim \beta_1\left(\frac{k}{2}, \frac{n-k-1}{2}\right)$.

Proof:

We may assume that P_k is spanned by $\{e_{n-k+1}, \ldots, e_n\}$. Thus

$$\|\pi_k(U_n)\|^2 = \sum_{i=n-k+1}^{n} U_{ni}^2 = \sum_{i=k-1}^{n-2} Y_i^2 .$$

From Corollary 14 we have,

Theorem 15:

Let A be an $n \times n$ symmetric idempotant matrix such that Rank $(A) = k \leq n - 1$. If C is chosen uniformly in S^{n-1} then $C^t AC \sim \beta_1\left(\frac{k}{2}, \frac{n-k-1}{2}\right)$.

Corollary 16:

For every integer $n,m \geq 0$ we have

(i) $$\int_{-\pi/2}^{\pi/2} \cos^n\theta \, d\theta = 2^n \beta\left(\frac{n+1}{2}, \frac{n+1}{2}\right) = \beta\left(\frac{1}{2}, \frac{n+1}{2}\right) .$$

(ii) $$\int_0^{\pi/2} \operatorname{Sin}^m\theta \operatorname{Cos}^n\theta \, d\theta = \frac{1}{2}\,\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right) .$$

Proof:

(i) For $n = 0, 1, 2, \ldots$, let $a_n = \int_{-\pi/2}^{\pi/2} \operatorname{Cos}^n\theta \, d\theta$. If $X \sim f_n$, then X is $\bar{\beta}\left(\frac{n+1}{2}, \frac{n+1}{2}\right)$. Hence the first equality. But $X^2 \sim \beta_1\left(\frac{1}{2}, \frac{n+1}{2}\right)$, hence the second equality.

(ii) Consider the random vector $U_{n+2} = (U_{n+2,\,1}, \ldots, U_{n+2,\,n+2})$. now

$$E\left(|U_{n+2,\,1}|^m\right) = a_n^{-1} \int_{-\pi/2}^{\pi/2} |\operatorname{Sin}\theta|^m \operatorname{Cos}^n\theta \, d\theta .$$

$$= 2a_n^{-1} \int_0^{\pi} \operatorname{Sin}^m\theta \operatorname{Cos}^n\theta \, d\theta .$$

and

$$E\left(|U_{n+2,\,n+2}|^m\right) = \left[\prod_{i=1}^{n} a_i^{-1} \int_{-\pi/2}^{\pi/2} \operatorname{Cos}^{m+i}\theta \, d\theta\right]$$

$$\times \left[(2a_0)^{-1} \int_{-\pi}^{\pi} |\operatorname{Cos}\theta|^m \, d\theta\right]$$

$$= \prod_{i=0}^{n} a_i^{-1} \int_{-\pi/2}^{\pi/2} \operatorname{Cos}^{m+i}\theta \, d\theta$$

$$= \frac{\prod_{i=m}^{n+m} a_i}{\prod_{i=0}^{n} a_i} = \frac{\prod_{i=0}^{n+m} a_i}{\prod_{i=0}^{n} a_i \prod_{i=0}^{m-1} a_i} .$$

By lemma 9 we have

$$E\left(|U_{n+2,\ 1}|^m\right) = E\left(|U_{n+2,\ m+2}|^m\right) ,$$

thus

$$\int_0^{\pi/2} \mathrm{Sin}^m\theta\ \mathrm{Cos}^n\theta\ d\theta = \frac{\prod_{i=0}^{n+m} a_i}{2\prod_{i=0}^{m-1} a_i \prod_{i=0}^{n-1} a_i} .$$

noting that $\prod_{i=0}^{k} a_i = \frac{1}{2} A(S^{k+1}) = \frac{\pi^{\frac{k+2}{2}}}{\Gamma\left(\frac{k+2}{2}\right)}$, the results follows.

Note: Other integral formulas and relations may be reached using similar arguments. Corollary 16 is known but the proof is new.

We conclude with the following remarks.

1. The problem of finding the distribution of product of random variables may be solved using the Mellin transforms. A complete coverage of the subject may be found in Springer (1979). In fact our first attempt to answer some of the questions raised above was tried using the Mellin transforms. The computational involvement led us to the geometric approach used in this paper.
2. Laha (1959, theorem 2.1) showed: If X and Y are (i) independent, (ii) identically distributed with common distribution function F, and (iii) W = X/Y is Cauchy, symmetric about 0, then F has the following four properties.

(1) Symmetric about 0 .

(2) F is absolutely continuous and has continous p.d.f. $f(x) > 0$.

(3) X has unbounded range .

(4) f satisfies

$$\int_0^\infty f(x)\ f(ux)x\,dx = C_0/(1+u^2)$$

for all u, where C_0 is a constant.

Consider the following problem: "Let X and Y be two random variables with joint distribution function $F_{X,Y}$ and marginals F_X and F_Y, such that X/Y is standard Cauchy. What kind of general conditions must hold on $F_{X,Y}$, F_X, and F_Y".

Some partial answers:

Theorem 17:

Let X and Y be identically distributed random variables with common distribution function F such that X/Y is standard Cauchy. Then X and Y are independent iff F has the four properties mentioned above.

Proof:

The if part follows from Laha's theorem. For only if: Let $X = U_{21}$ and $Y = U_{22}$. That shows if X and Y are not independent, then F does not satisfy these properties.

On the other hand, if we assume that X and Y are independent (but not identically distributed) such that X/Y is standard Cauchy, then F_X and F_Y do not have to satisfy these four properties simultaneously. This may be seen by considering U_{31}/U_{33}.

5. MOTIVATION: ESTIMATION IN ONE LESS THAN FULL RANK LINEAR MODEL

The problem posed in section 1 was motivated by the following. Consider the linear model $E(Y) = X\beta$, where Y is an $m \times 1$ random vector, X is $m \times n$ design matrix, and β $n \times 1$ parameter vector. The least square estimate $\hat{\beta}$ of β satisfies $(X^tX)\hat{\beta} = X^tY$. If $\text{Rank}(X^tX) = n - 1$, then in order to obtain an "estimate" for β we must add one more constraint. If there is no theory or previous knowledge to suggest the form of this new additional constraint, it is chosen more or less at random. Let $C^t\beta = 0$ be the additional random constraint such that $\|C\| = 1$. Let $\hat{\beta}_C$ be the solution satisfying $(X^tX)\hat{\beta}_C = X^tY$ and $C^t\beta_C = 0$. Let V be the eigenvector corresponding to the zero eigenvalue of X^tX. Then $\hat{\beta}_C$ may be written as $\hat{\beta}_C = \beta - (C^t\beta/C^tV)V$. Thus, the stochastic properties of $\hat{\beta}_C$ are essentially those of the quantity $C^t\beta/C^tV$, $C \in S^{n-1}$.

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BIBLIOGRAPHY

Laha, R.G. (1959). "On the laws of Cauchy and Gauss". Ann. of Math. Stat., 30, 1165-1174.

Springer, M.D. (1979). The Algebra of Random Variables. John Wiley, New York.

Springer, M.D. and Thompson, W.E. (1966). "The distribution of product of independent random variables." J. SIAM Appl. Math., Vol. 14, No. 3, 511-526.