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Autoregressive Conditionally Heteroscedastic Models

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Abstract

Two linear regression models, where the independent variables are (a) fixed and bounded, and (b) the lagged values of the dependent variable, with autoregressive conditionally heteroscedastic (ARCH) errors are considered. A series representation and some ergodic properties of the first order ARCH errors are derived. The strong consistency and the asymptotic normality of the maximum likelihood estimators are established. Asymptotic distributions of the least squares estimator and an estimated generalized least squares estimator are also derived.

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1. Introduction:

Consider a process $\{Y_t\}$ satisfying the equation

$$Y_t = \underset{\sim}{X}_t \underset{\sim}{\alpha} + e_t , \tag{1.1}$$

where $\underset{\sim}{X}_t$: 1 x p is a known row vector and $\underset{\sim}{\alpha}$: p x 1 is a vector of unknown parameters. Traditional econometric models assume that e_t is a sequence of uncorrelated $(0,\sigma^2)$ random variables and the conditional mean of Y_t given the past is $\underset{\sim}{X}_t \underset{\sim}{\alpha}$, where $\underset{\sim}{X}_t$ may contain lagged values of Y_t. Under a symmetric loss function, the best forecast of Y_t, based on the past information, is the conditional mean of Y_t given the past and is denoted by $E[Y_t | F_{t-1}]$, where F_{t-1} = information up to time (t-1). Therefore for the model (1.1), the one step forecast error is

$$e_t = Y_t - E[Y_t | F_{t-1}] .$$

The unconditional variance of the one period forecast is given by σ^2. Note that

$$E[e_t | F_{t-1}] = 0 \quad \text{a.s.}$$

and

$$E[e_t^2 | F_{t-1}] = V(Y_t | F_{t-1}) \quad \text{a.s.}$$

For the conventional econometric models, however, the conditional variance does not depend on F_{t-1}. For some processes one might expect better forecast intervals if additional information from the past were allowed to affect the forecast variance.

A model which allows the conditional variance to depend on the past realization is the bilinear model described by Granger and Anderson (1978). Jones (1965), Granger and Anderson (1978), and Priestly (1978) considered nonlinear time series models.

Engle (1982) proposed a class of models, called autoregressive conditionally heteroscedastic (ARCH) models, for which both the conditional mean and the conditional variance of a time series are functions of the past behavior of the time series. If the conditional density is normal, then a general expression for the ARCH model is

$$Y_t | F_{t-1} \sim N[g(F_{t-1}),\ h(F_{t-1})],$$

where g and h are measurable. A special case we consider assumes that the mean can be expressed as a linear combination of variables in the information set, while the variance is a qth order weighted average of the squares of past disturbances. More precisely,

$$Y_t | F_{t-1} \sim N(\underset{\sim}{X}_t \underset{\sim}{\alpha},\ h_t)$$

where

$$h_t = \beta_0 + \beta_1 e_{t-1}^2 + \ldots + \beta_q e_{t-q}^2$$

and

$$e_t = Y_t - \underset{\sim}{X}_t \underset{\sim}{\alpha} .$$

Assume that $\beta_1, \beta_2, \ldots, \beta_q$ are nonnegative and β_0 is positive. Although Y_t is conditionally normal, Engle (1982) established that the Y_t are not jointly normal and the marginal distribution of Y_t is not normal. In this manuscript we restrict to the case q = 1.

Engle (1982) derived the moments of the $\{e_t\}$ process for q = 1. In section 2 we derive a representation for the $\{e_t\}$ process and use it to derive the moments of the $\{e_t\}$ process. We also derive the ergodic properties of the $\{e_t\}$ process. Engle (1982) also considered the maximum likelihood estimation of the parameters. He established that the information matrix is block diagonal, indicating that the maximum likelihood estimators of $\underset{\sim}{\alpha}$ and $\underset{\sim}{\beta}$ are independent. He also indicated that the maximum likelihood estimators are asymptotically normal. In section 3 we formally derive the asymptotic distribution of the maximum likelihood estimators. We also derive the asymptotic properties of the least squares and estimated generalized least squares estimators of $\underset{\sim}{\alpha}$ and $\underset{\sim}{\beta}$.

2. Properties of the ARCH Models

The simplest ARCH model is the first order linear model given by

$$Y_t = \underset{\sim}{X}_t\underset{\sim}{\alpha} + e_t , \tag{2.1}$$

where

$$e_t | F_{t-1} \sim N(0,\ \beta_0 + \beta_1 e_{t-1}^2), \tag{2.2}$$

and

$$F_{t-1} = \sigma\text{-field generated by } Y_s \text{ and } e_s,\ s \le t-1,\ t = 1, 2, \ldots .$$

It is assumed that β_0 is strictly positive and β_1 is nonnegative. In the following theorem we obtain a representation for the $\{e_t^2\}$ sequence which is useful in deriving the properties of the model (2.1).

Theorem 2.1: Let $\{e_t\}$ be a sequence of random variables satisfying (2.2). Assume that $\beta_0 > 0$ and $0 \le \beta_1 < 1$. It is assumed that the process began indefinitely in the past with a finite initial variance. Then,

$$e_t^2 = \beta_0 \sum_{\ell=0}^{\infty} \beta_1^{\ell} \left(\prod_{i=0}^{\ell} Z_{t-i}^2 \right) \quad \text{a.s.} , \tag{2.3}$$

where $\{Z_t;\ t = 0, \pm 1, \pm 2, \ldots\}$ is a sequence of normal independent (0,1) variables.

Proof: Since e_t is conditionally normal, we get

$$e_t = Z_t(\beta_0 + \beta_1 e_{t-1}^2)^{\frac{1}{2}}$$

where Z_t is a standard normal random variable independent of e_{t-1}. Therefore,

$$\begin{aligned} e_t^2 &= Z_t^2(\beta_0 + \beta_1 e_{t-1}^2) \\ &= Z_t^2[\beta_0 + \beta_1 Z_{t-1}^2(\beta_0 + \beta_1 e_{t-2}^2)] \\ &= \beta_0 \sum_{\ell=0}^{j-1} \beta_1^{\ell} \left(\prod_{i=0}^{\ell} Z_{t-i}^2\right) + \beta_1^j \left(\prod_{i=0}^{j-1} Z_{t-i}^2\right) e_{t-j}^2 . \end{aligned}$$

Note that,

$$E\left[\sum_{\ell=0}^{\infty} \beta_1^{\ell} \left(\prod_{i=0}^{\ell} Z_{t-i}^2\right)\right] = (1 - \beta_1)^{-1} < \infty .$$

Therefore, by result (xi) of Chung (1974, p. 42),

$$\sum_{\ell=0}^{\infty} \beta_1^{\ell} \left(\prod_{i=0}^{\ell} Z_{t-i}^2\right) < \infty \quad \text{a.s. .}$$

Also, assuming that the process was initiated in the infinite past with a finite initial variance,

$$E\left[\sum_{j=0}^{\infty} \beta_1^j \left(\prod_{i=0}^{j-1} Z_{t-i}^2\right) e_{t-j}^2\right] < \infty .$$

Using the same argument,

$$\sum_{j=0}^{\infty} \beta_1^j \left(\prod_{i=0}^{j-1} Z_{t-i}^2\right) e_{t-j}^2 < \infty \quad \text{a.s. ,}$$

and

$$\beta_1^j \left(\prod_{i=0}^{j-1} Z_{t-i}^2\right) e_{t-j}^2 \longrightarrow 0 \quad \text{a.s., as } j \to \infty .$$

Therefore,

$$e_t^2 = \beta_0 \sum_{\ell=0}^{\infty} \beta_1^{\ell} \left(\prod_{i=0}^{\ell} Z_{t-i}^2\right) \quad \text{a.s.} \qquad \square$$

A similar expression for e_t^2 can be obtained in case $q > 1$. Note that e_t has a symmetric distribution and that the sequence $\{e_t\}$ is strictly stationary. The probability density function f of e_t can be obtained by solving

$$f(x) = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2(\beta_0+\beta_1 u^2)}} \{2\pi(\beta_0 + \beta_1 u^2)\}^{-\frac{1}{2}} f(u)du \ . \quad (2.4)$$

We have not been able to obtain a solution for (2.4). We use (2.3) to obtain the moments of the $\{e_t\}$ process. See also Engle (1982).

<u>Theorem 2.2</u>: Let $\{e_t\}$ be a sequence of random variables satisfying the conditions of Theorem 2.1. Then,

$$m_{2r} = E[e_t^{2r}] < \infty$$

if and only if

$$\theta_r = \beta_1^r \prod_{j=1}^{r} (2j-1) < 1 \ .$$

Also, if $\theta_r < 1$, then

$$m_{2r} = \theta_r(1-\theta_r)^{-1} \sum_{j=0}^{r-1} \binom{r}{j} \left(\frac{\beta_0}{\beta_1}\right)^{r-j} m_{2j} \ .$$

<u>Proof</u>: Note that,

$$e_t^{2r} = \beta_0^r \lim_{n\to\infty} [\sum_{j=0}^{n} \beta_1^j (\prod_{i=0}^{j} Z_{t-i}^2)]^r ,$$

and by Monotone convergence theorem (see Chung, 1974, p. 42),

$$E[e_t^{2r}] = \beta_0^r \lim_{n\to\infty} E[\sum_{j=0}^{n} \beta_1^j (\prod_{i=0}^{j} Z_{t-i}^2)]^r \ .$$

Using Minkouski's inequality,

$$\begin{aligned} E[e_t^{2r}] &\leq \beta_0^r \lim_{n\to\infty} \left[\sum_{j=0}^{n} \beta_1^j \{E \prod_{i=0}^{j} Z_{t-i}^{2r}\}^{\frac{1}{r}}\right]^r \\ &= \beta_0^r \lim_{n\to\infty} \left[\sum_{j=0}^{n} \beta_1^{-1} \theta_r^{(j+1)/r}\right]^r \\ &< \infty \quad \text{if } \theta_r < 1 \ . \end{aligned}$$

Also,

$$E[e_t^{2r}] \geq \beta_0^r \sum_{j=0}^{\infty} \beta_1^{jr} E(\prod_{i=0}^{j} Z_{t-i}^{2r})$$
$$= \beta_0^r \sum_{j=0}^{\infty} \beta_1^{-j} \theta_r^{(j+1)}$$
$$= \infty \text{ if } \theta_r \geq 1 .$$

Therefore, m_{2r} is finite iff $\theta_r < 1$. Now,

$$m_{2r} = E[e_t^{2r}]$$
$$= E[Z_t^{2r}(\beta_0 + \beta_1 e_{t-1}^2)^r]$$
$$= \sum_{j=0}^{r} \binom{r}{j} \beta_0^{r-j} \beta_1^j m_{2j} \theta_r \beta_1^{-r}$$
$$= \theta_r(1-\theta_r)^{-1} \sum_{j=0}^{r-1} \binom{r}{j} \frac{\beta_0}{\beta_1}^{r-j} m_{2j} . \qquad \square$$

Note that if $3\beta_1^2 < 1$, then

$$V(e_t^2) = 2\sigma^4(1-3\beta_1^2)^{-1}$$

and

$$\text{Cov}(e_t^2, e_{t-j}^2) = \beta_1^j V(e_t^2)$$

where

$$\sigma^2 = \beta_0(1-\beta_1)^{-1} .$$

Also, $\{e_t\}$ is a sequence uncorrelated $(0,\sigma^2)$ random variables.

We now give two lemmas that are useful in obtaining the ergodic properties of the $\{e_t\}$ process.

Lemma 2.1: Let $\{U_n\}$ be a sequence of F_n measurable random variables and $F_n \subseteq F_{n+1}$. Suppose there exists an integrable random variable U and a constant c such that

$$P[|U_n| > x] \leq c\, P[|U| > x] .$$

Then

$$n^{-1} \sum_{k=1}^{n} [U_k - E(U_k|F_{k-1})] \xrightarrow{P} 0 .$$

If $E[|U|\log^{+}|U|] < \infty$ or if $\{U_k, k \geq 1\}$ and $\{E(U_k|F_{k-1}), k \geq 1\}$ are strictly stationary sequences then the covergence is almost sure.

Proof: See Hall and Heyde (1980, p. 36). □

Lemma 2.2: A covariance stationary process $\{X_t\}$ obeys the mean law of large numbers, i.e.,

$$\bar{X}_n = n^{-1} \sum_{t=1}^{n} X_t$$

converges in mean square to a square integrable random variable X. If the process is strictly stationary and integrable, then $\bar{X}_n$ converges almost surely to an integrable random variable.

Proof: See Révész (1968, p. 99). □

In the following theorem we obtain the ergodic properties of $\{e_t\}$.

Theorem 2.3: Consider $\{e_t\}$ satisfying the conditions of Theorem 2.1. If $\theta_r < 1$, then

$$n^{-1} \sum_{t=1}^{n} e_t^{2r} \rightarrow E[e_t^{2r}] \quad \text{a.s. ,}$$

$$n^{-1} \sum_{t=1}^{n} e_t^{2r-1} \rightarrow 0 \qquad \text{a.s. .}$$

If $3\beta_1^2 < 1$, then for a fixed $j \neq 0$,

$$n^{-1} \sum_{t=1}^{n} e_t e_{t-j} \rightarrow 0 \quad \text{a.s.}$$

and

$$n^{-1} \sum_{t=1}^{n} e_t^2 e_{t-j}^2 \rightarrow E(e_t^2 e_{t-j}^2) \quad \text{a.s. .}$$

Proof: From Lemma 2.1 and Lemma 2.2,

$$n^{-1} \sum_{t=1}^{n} e_t^2 \longrightarrow M_2 \quad \text{a.s.} \quad \text{(say)} ,$$

and

$$n^{-1} \sum_{t=1}^{n} [e_t^2 - E(e_t^2 | F_{t-1})] \longrightarrow 0 \quad \text{a.s.} ,$$

where

$$E[e_t^2 | F_{t-1}] = \beta_0 + \beta_1 e_{t-1}^2 \quad \text{a.s.} .$$

Therefore,

$$n^{-1} \sum_{t=1}^{n} E[e_t^2 | F_{t-1}] \longrightarrow M_2 \quad \text{a.s.} .$$

Also,

$$n^{-1} \sum_{t=1}^{n} E[e_t^2 | F_{t-1}] = \beta_0 + \beta_1 \, n^{-1} \sum_{t=1}^{n} e_{t-1}^2$$

$$\longrightarrow \beta_0 + \beta_1 M_2 \quad \text{a.s.},$$

and hence

$$M_2 = \beta_0 + \beta_1 M_2$$

or

$$M_2 = \beta_0 (1 - \beta_1)^{-1}$$

$$= E[e_t^2] .$$

Therefore

$$n^{-1} \sum_{t=1}^{n} e_t^2 \longrightarrow E(e_t^2) \quad \text{a.s.} .$$

For $r \geq 2$, we use induction procedure to prove the result. Assume that

$$n^{-1} \sum_{t=1}^{n} e_t^{2s} \longrightarrow E(e_t^{2s}) \quad \text{a.s.} ,$$

for $s = 1, 2, \ldots, r - 1$. We know from Lemma 1 and Lemma 2 that

$$n^{-1} \sum_{t=1}^{n} e_t^{2r} \longrightarrow M_{2r} \quad \text{a.s.} \quad \text{(say)},$$

and

$$n^{-1} \sum_{t=1}^{n} E[e_t^{2r} | F_{t-1}] \longrightarrow M_{2r} \quad \text{a.s.} .$$

Note that

$$E[e_t^{2r}|F_{t-1}] = \theta_r \sum_{j=0}^{r} \binom{r}{j} \left(\frac{\beta_0}{\beta_1}\right)^{r-j} e_{t-1}^{2j} \quad ,$$

Therefore,

$$M_{2r} = \theta_r \sum_{j=0}^{r-1} \binom{r}{j} \left(\frac{\beta_0}{\beta_1}\right)^{r-j} M_{2j} + \theta_r M_{2r} \quad \text{a.s.}$$

or

$$\begin{aligned} M_{2r} &= \theta_r (1 - \theta_r)^{-1} \sum_{j=0}^{r-1} \binom{r}{j} \left(\frac{\beta_0}{\beta_1}\right)^{r-j} M_{2j} \quad \text{a.s.} \\ &= E[e_t^{2r}] \ . \end{aligned}$$

Therefore,

$$n^{-1} \sum_{t=1}^{n} e_t^{2r} \longrightarrow E(e_t^{2r}) \quad \text{a.s.} \ .$$

Note that, if $\theta_r < 1$, then

$$E[e_t^{2r-1}|F_{t-1}] = 0 \quad \text{a.s.}$$

and hence

$$n^{-1} \sum_{t=1}^{n} e_t^{2r-1} \longrightarrow 0 \quad \text{a.s.} \quad \text{as } n \to \infty.$$

Similarly,

$$E[e_t e_{t-j}|F_{t-1}] = 0 \quad \text{a.s.}$$

and we get

$$n^{-1} \sum_{t=1}^{n} e_t \, e_{t-j} \longrightarrow 0 \quad \text{a.s.} \quad \text{as } n \to \infty.$$

Consider,

$$\begin{aligned} n^{-1} \sum_{t=1}^{n} E[e_t^2 \, e_{t-1}^2|F_{t-1}] &= n^{-1} \sum_{t=1}^{n} (\beta_0 + \beta_1 e_{t-1}^2) \, e_{t-1}^2 \\ &\longrightarrow \beta_0 \, E(e_{t-1}^2) + \beta_1 \, E(e_{t-1}^4), \quad \text{a.s.} \\ &= E(e_t^2 \, e_{t-1}^2). \end{aligned}$$

Therefore, by Lemma 1,

$$n^{-1} \sum_{t=1}^{n} e_t^2 \, e_{t-1}^2 \longrightarrow E(e_t^2 \, e_{t-1}^2) \quad \text{a.s.} \ .$$

Similarly,

$$n^{-1}\sum_{t=1}^{n} e_t^2\, e_{t-j}^2 \longrightarrow E(e_t^2\, e_{t-j}^2) \text{ a.s.}$$

as $n \to \infty$.

In the next section we consider estimation of the parameters $\underset{\sim}{\alpha}$ and $\underset{\sim}{\beta}$.

3. Estimation of ARCH Models

Given $Y_1, Y_2, \ldots, Y_n$ satisfying (2.1), we desire to estimate $\underset{\sim}{\alpha}$ and $\underset{\sim}{\beta} = (\beta_0, \beta_1)'$. Engle (1982) used the method of scoring to obtain the maximum likelihood estimates, but did not formally derive the asymptotic properties of the maximum likelihood estimators. Engle (1982) indicated that if the conditions of Crowder (1976) are satisfied then the maximum likelihood estimates are asymptotically normal. We derive the limiting distribution of the maximum likelihood estimators by verifying the conditions of Hall and Heyde (1980, p. 174). We also consider an estimated generalized least squares estimator and derive its limiting distribution.

We consider two particular choices for $\underset{\sim}{X}_t$

(a) $\underset{\sim}{X}_t$ is fixed and bounded;

and

(b) $\underset{\sim}{X}_t = (Y_{t-1}, Y_{t-1}, \ldots, Y_{t-p})$.

For case (b), we also assume that $\underset{\sim}{X}_1 = (Y_0, Y_{-1}, \ldots, Y_{-p+1})$ is given and we consider the likelihood given $\underset{\sim}{X}_1$. We will also indicate how the results may be extended for the case where $\underset{\sim}{X}_t$ consists of both fixed and lagged values and also for the case with $\underset{\sim}{X}_t$ that are not necessarily bounded.

3.1. Maximum Likelihood Estimation

Consider the log likelihood conditional on $\underset{\sim}{X}_1$,

$$L_n(\underset{\sim}{\gamma}) = -n^{-1} \sum_{t=1}^{n}\left[\ell n\ f_t(Y_t | F_{t-1})\right]$$

where $f_t(Y_t|F_{t-1})$ is the conditional density of Y_t given the past and $\gamma' = (\alpha', \beta')$. From (2.2),

$$f_t(Y_t|F_{t-1}) = (2\pi h_t)^{-\frac{1}{2}} \exp[-(Y_t - X_t\alpha)^2/2h_t] \tag{3.1}$$

where

$$h_t = \beta_0 + \beta_1(Y_{t-1} - X_{t-1}\alpha)^2 .$$

Therefore,

$$L_n(\gamma) = \text{constant} + (2n)^{-1} \sum_{t=1}^{n} \ln\left(h_t\right) + (2n)^{-1} \sum_{t=1}^{n} h_t^{-1}(Y_t - X_t\alpha)^2 \tag{3.2}$$

The maximum likelihood estimator $\bar{\gamma}_n$ of γ is the value of γ that minimizes $L_n(\gamma)$. Let $\gamma_0' = (\alpha_0', \beta')$ be the true value of γ. All probabilities and expectations are taken with respect to the true value γ_0. We assume that

$$0 < m_1 \leq \beta_0 \leq m_2 < \infty$$

and

$$0 \leq \beta_1 \leq 0.2 , \tag{3.3}$$

so that $E[e_t^{12}] < \infty$. (If X_t is fixed and bounded then we may assume that $0 \leq \beta_1 \leq 0.3$, so that $E[e_t^8] < \infty$.) We also assume that α_0 is in the interior of a compact set L. Let

$$\Gamma = L \times [m_1, m_2] \times [0, 0.2] .$$

Therefore γ_0 is assumed to be in the interior of Γ. All neighborhoods defined below will be taken to be contained in Γ. For $\delta > 0$, and $||\gamma - \gamma_0|| < \delta$, we obtain from the Taylor series expansion that

$$L_n(\gamma) = L_n(\gamma_0) + (\gamma - \gamma_0)' \left(\frac{\partial L_n}{\partial \gamma}\right)_{\gamma=\gamma_0} + \tfrac{1}{2}(\gamma - \gamma_0)' H_n(\gamma - \gamma_0)$$

$$+ \tfrac{1}{2}(\gamma - \gamma_0)' T_n(\gamma^*)(\gamma - \gamma_0) , \tag{3.4}$$

where

$$\underset{\sim}{H}_n = \left(\frac{\partial^2 L_n}{\partial \underset{\sim}{\gamma} \partial \underset{\sim}{\gamma}'}\right)_{\underset{\sim}{\gamma}=\underset{\sim}{\gamma}_0} ,$$

$$T_n(\underset{\sim}{\gamma}^*) = \left(\frac{\partial^2 L_n}{\partial \underset{\sim}{\gamma}\ \partial \underset{\sim}{\gamma}'}\right)_{\underset{\sim}{\gamma}=\underset{\sim}{\gamma}^*} - \underset{\sim}{H}_n ,$$

and $\underset{\sim}{\gamma}^*$ is a point between $\underset{\sim}{\gamma}$ and $\underset{\sim}{\gamma}_0$ (not necessarily the same at each occurrence). We include a result from Hall and Heyde (1980) that we use to obtain the asymptotic properties of $\bar{\underset{\sim}{\gamma}}_n$.

Theorem 3.1: Suppose that

$$\lim_{n\to\infty} \sup_{\delta \downarrow 0} \ \delta^{-1}|T_n(\underset{\sim}{\gamma}^*)|_{ij} < \infty \text{ a.s. }, \quad \begin{array}{l} 1 \le i \le p+2 \\ 1 \le j \le p+2 , \end{array} \tag{3.5}$$

$$\lim_{n\to\infty} \underset{\sim}{H}_n = \underset{\sim}{H} \quad \text{a.s. }, \tag{3.6}$$

and

$$\lim_{n\to\infty} \left(\frac{\partial L_n}{\partial \underset{\sim}{\gamma}}\right)_{\underset{\sim}{\gamma}=\underset{\sim}{\gamma}_0} = \underset{\sim}{0} \quad \text{a.s.} \tag{3.7}$$

where $\underset{\sim}{H}$ is a positive definite matrix of constants. Then, there exists a sequence of estimators $\{\bar{\underset{\sim}{\gamma}}_n\}$ such that $\bar{\underset{\sim}{\gamma}}_n$ converges to $\underset{\sim}{\gamma}_0$ almost surely, and for $\varepsilon > 0$ there is an event E with $P(E) > 1 - \varepsilon$ and n_0 such that on E, for $n > n_0$, $\bar{\underset{\sim}{\gamma}}_n$ satisfies

$$\left(\frac{\partial L_n}{\partial \underset{\sim}{\gamma}}\right)_{\underset{\sim}{\gamma}=\bar{\underset{\sim}{\gamma}}_n} = \underset{\sim}{0} ,$$

and $L_n(\underset{\sim}{\gamma})$ attains a relative minimum at $\bar{\underset{\sim}{\gamma}}_n$. If, in addition,

$$n^{-\frac{1}{2}} \left(\frac{\partial L_n}{\partial \underset{\sim}{\gamma}}\right)_{\underset{\sim}{\gamma}=\underset{\sim}{\gamma}_0} \xrightarrow{L} N(\underset{\sim}{0}, \underset{\sim}{W}) \tag{3.8}$$

where $\underset{\sim}{W}$ is a positive definite matrix, then

$$n^{\frac{1}{2}}(\tilde{\gamma}_n - \gamma_0) \xrightarrow{L} N(0, H^{-1} W H^{-1}) .$$

Proof: See Hall and Heyde (1980, p. 174). □

We now compute the partial derivatives of $L_n(\gamma)$. Note that,

$$\frac{\partial L_n}{\partial \alpha} = \frac{1}{n} \sum_{t=1}^{n} \frac{(Y_t - X_t\alpha)}{h_t} X_t' + \beta_1 \frac{1}{n} \sum_{t=1}^{n} \left[\frac{(Y_t - X_t\alpha)^2 - h_t}{h_t^2}\right] (Y_{t-1} - X_{t-1}\alpha) X_{t-1}' ,$$

$$\frac{\partial L_n}{\partial \beta} = \frac{-1}{2n} \sum_{t=1}^{n} \left[\frac{(Y_t - X_t\alpha)^2 - h_t}{h_t^2}\right] \begin{bmatrix} 1 \\ (Y_{t-1} - X_{t-1}\alpha)^2 \end{bmatrix} ,$$

$$\frac{\partial^2 L_n}{\partial \alpha \partial \alpha'} = \frac{1}{n} \sum_{t=1}^{n} \frac{X_t' X_t}{h_t} - \beta_1 \frac{1}{n} \sum_{t=1}^{n} \left[\frac{(Y_t - X_t\alpha)^2 - h_t}{h_t^2}\right] X_{t-1}' X_{t-1}$$

$$+ 2\beta_1^2 \frac{1}{n} \sum_{t=1}^{n} \left[\frac{2(Y_t - X_t\alpha)^2 - h_t}{h_t^3}\right] (Y_{t-1} - X_{t-1}\alpha)^2 X_{t-1}' X_{t-1}$$

$$- 2\beta_1 \frac{1}{n} \sum_{t=1}^{n} \left[\frac{(Y_t - X_t\alpha)}{h_t^2}\right] (Y_{t-1} - X_{t-1}\alpha) [X_t' X_{t-1} + X_{t-1}' X_t] ,$$

$$\frac{\partial^2 L_n}{\partial \beta \partial \beta'} = \frac{1}{2n} \sum_{t=1}^{n} \left[\frac{2(Y_t - X_t\alpha)^2 - h_t}{h_t^3}\right] \begin{bmatrix} 1 & (Y_{t-1} - X_{t-1}\alpha)^2 \\ (Y_{t-1} - X_{t-1}\alpha)^2 & (Y_{t-1} - X_{t-1}\alpha)^4 \end{bmatrix} ,$$

and

$$\frac{\partial^2 L_n}{\partial \beta \partial \alpha'} = \frac{1}{n} \sum_{t=1}^{n} \left[\frac{(Y_t - X_t\alpha)^2 - h_t}{h_t^2}\right] \begin{bmatrix} 0 \\ 1 \end{bmatrix} (Y_{t-1} - X_{t-1}\alpha) X_{t-1}$$

$$- \frac{1}{n} \sum_{t=1}^{n} \left[\frac{2(Y_t - X_t\alpha)^2 - h_t}{h_t^2}\right] \begin{bmatrix} (Y_{t-1} - X_{t-1}\alpha) \\ (Y_{t-1} - X_{t-1}\alpha)^3 \end{bmatrix} X_{t-1}$$

$$+ \frac{1}{n} \sum_{t=1}^{n} \left[\frac{(Y_t - X_t\alpha)}{h_t^2}\right] \begin{bmatrix} 1 \\ (Y_{t-1} - X_{t-1}\alpha)^2 \end{bmatrix} X_{t-1} .$$

Recall that

$$e_t = Z_t v_t^{1/2}$$

where

$$v_t = \theta_0 + \theta_1 e_{t-1}^2 ,$$

$\underset{\sim}{\theta}' = (\theta_0, \theta_1)$ is the true value of $\underset{\sim}{\beta}' = (\beta_0, \beta_1)$ and Z_t is a sequence of independent $N(0,1)$ variables. Therefore,

$$\left(\frac{\partial L_n}{\partial \underset{\sim}{\alpha}}\right)_{\underset{\sim}{\gamma} = \underset{\sim}{\gamma}_0} = -\frac{1}{n} \sum_{t=1}^{n} v_t^{-1/2} Z_t \underset{\sim}{X}_t' + \frac{1}{n} \theta_1 \sum_{t=1}^{n} \frac{(Z_t^2 - 1)}{v_t} e_{t-1} \underset{\sim}{X}_{t-1}' ,$$

and

$$\left(\frac{\partial L_n}{\partial \underset{\sim}{\beta}}\right)_{\underset{\sim}{\gamma} = \underset{\sim}{\gamma}_0} = -\frac{1}{2n} \sum_{t=1}^{n} \frac{(Z_t^2 - 1)}{v_t} \begin{bmatrix} 1 \\ e_{t-1}^2 \end{bmatrix} .$$

In the following theorem we verify the conditions (3.5) - (3.8) for the case (a) where we assume that $\underset{\sim}{X}_t$ is fixed and bounded.

<u>Theorem 3.2</u>: Assume that $\underset{\sim}{X}_t$ is fixed and bounded. Also, assume that

$$n^{-1} \sum_{t=1}^{n} \underset{\sim}{X}_t' \underset{\sim}{X}_t \longrightarrow \underset{\sim}{A} , \quad \text{as } n \rightarrow \infty ,$$

where $\underset{\sim}{A}$ is a positive definite matrix. Then, the conditions (3.5) - (3.8) are satisfied with

$$\underset{\sim}{H} = \underset{\sim}{W} = \begin{bmatrix} \underset{\sim}{H}_{11} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{H}_{22} \end{bmatrix} . ,$$

where

$$\underset{\sim}{H}_{11} = \underset{\sim}{A}(c_1 + 2\theta_1^2 c_3) ,$$

$$\underset{\sim}{H}_{22} = \tfrac{1}{2} \begin{bmatrix} c_2 & c_3 \\ c_3 & c_4 \end{bmatrix} ,$$

$$c_1 = E[v_2^{-1}] ,$$

$$c_2 = E[v_2^{-2}] ,$$

$$c_3 = E[e_1^2 v_2^{-2}] = \theta_1^{-1}(c_1 - \theta_0 c_2) ,$$

and

$$c_4 = E[e_1^4 v_2^{-2}] = \theta_1^{-2}(1-2\theta_0 c_1 + \theta_0^2 c_2) .$$

<u>Proof</u>: For a fixed i, let

$$U_n = v_n^{-\frac{1}{2}} Z_n M^{-1} X_{n,i} ,$$

where $M = \sup_{i,n} \{X_{n,i}\}$.

Then,

$$E[U_n | F_{n-1}] = 0 \quad \text{a.s.}$$

and $E(U_n^2)$ is finite. Therefore, by Lemma 2.1,

$$n^{-1} \sum_{t=1}^{n} U_t \longrightarrow 0 \quad \text{a.s.} , \quad \text{as } n \to \infty .$$

Similarly,

$$\lim_{n\to\infty} n^{-1} \sum_{t=1}^{n} v_t^{-2}(e_t^2 - v_t)\ e_{t-1}^k\ X_{t-1,i}^{\ell}\ X_{t-1,j}^{s} = 0 , \quad \text{a.s.}$$

for k = 0, 1, 2, 3, 4 and ℓ,s = 0,1 .

Also,

$$\lim_{n\to\infty} n^{-1} \sum_{t=1}^{n} v_t^{-2} e_t\ X_{t-1,i} = 0 \quad \text{a.s.} ,$$

$$\lim_{n\to\infty} n^{-1} \sum_{t=1}^{n} v_t^{-2} e_t e_{t-1}^2 X_{t-1,i} = 0 \quad \text{a.s.} ,$$

$$\lim_{n\to\infty} n^{-1} \sum_{t=1}^{n} v_t^{-2}(e_t^2 - v_t) e_{t-1}^2 = 0 \quad \text{a.s.},$$

and

$$\lim_{n\to\infty} n^{-1} \sum_{t=1}^{n} v_t^{-3}(e_t^2 - v_t) e_{t-1}^4 = 0 \quad \text{a.s.} .$$

Note also that,

$$E[v_t^{-1} e_{t-1} | F_{t-2}] = v_{t-1}^{\frac{1}{2}} E[(\theta_0 + \theta_1 v_{t-1} Z_{t-1}^2)^{-1} Z_{t-1} | F_{t-2}]$$

$$= 0 \quad \text{a.s.} ,$$

and

$$E[v_t^{-1} e_{t-1}^3 | F_{t-2}] = 0 \quad \text{a.s.} .$$

Therefore,

$$\lim_{n\to\infty} n^{-1} \sum_{t=1}^{n} v_t^{-1} e_{t-1} \underset{\sim}{X}_{t-1} = \underset{\sim}{0} \quad \text{a.s.} ,$$

and

$$\lim_{n\to\infty} n^{-1} \sum_{t=1}^{n} v_t^{-1} e_{t-1}^3 \underset{\sim}{X}_{t-1} = \underset{\sim}{0} \quad \text{a.s} .$$

Now consider,

$$\text{Cov}(v_t^{-1}, v_{t-j}^{-1}) = E[v_t^{-1} v_{t-j}^{-1}] - \{E[v_t^{-1}]\}^2$$

$$\leq E[v_t^{-1}] \, E[\{\theta_0 + \theta_1 \theta_0 \sum_{\ell=0}^{j-1} \theta_1^{\ell} \prod_{i=0}^{\ell} z_{t-1-i}^2\}^{-1} - v_t^{-1}]$$

$$\leq \theta_0^{-3} \theta_1^{j+1} E(e_{t-1-j}^2) \quad .$$

Therefore,

$$n^{-1} \sum_{t=1}^{n} v_t^{-1} \xrightarrow{P} E[v_t^{-1}] \qquad \text{as } n \to \infty \quad ,$$

and since v_t^{-1} is bounded the convergence is almost sure.

Similarly,

$$\lim_{n\to\infty} n^{-1} \sum_{t=1}^{n} v_t^{-1} \underset{\sim}{X}_t' \underset{\sim}{X}_t = \underset{\sim}{A} c_1 \quad \text{a.s.} ,$$

and

$$\lim_{n\to\infty} n^{-1} \sum_{t=1}^{n} v_t^{-2} = E[v_t^{-2}] \quad \text{a.s.} \quad .$$

Therefore,

$$\underset{n\to\infty}{\ell im} \left(\frac{\partial L_n}{\partial \gamma}\right)_{\gamma=\gamma_0} = 0 \quad \text{a.s.} ,$$

and

$$\underset{n\to\infty}{\ell im} \left(\frac{\partial^2 L_n}{\partial \gamma \partial \gamma'}\right)_{\gamma=\gamma_0} = H \quad \text{a.s.} .$$

Since

$$(c_1 + 2\theta_1^2 c_3) > 0$$

and

$$(c_4 c_2 - c_3^2) = \frac{1}{\theta_1^2} V(v_t^{-1}) > 0 ,$$

we observe that H is positive definite.

Now we establish (3.5). Note that $|T_n(\gamma)|_{ij}$ is a linear combination of terms of the form

$$f_t(\gamma; a, b, k) = h_t^{-k} (Y_t - X_t\alpha)^a (Y_{t-1} - X_{t-1}\alpha)^b$$

where a = 0, 1, 2; b = 0, 1, 2, 3, 4; and k = 1, 2, 3.

Consider for $||\gamma - \gamma_0|| < \delta$,

$$|f_t(\gamma; a, b, k) - f_t(\gamma_0; a, b, k)| \leq \delta \sum_{i=1}^{p+2} \left| \frac{\partial f_t}{\partial \gamma_i} \right|_{\gamma^*}$$

where $\gamma_i^* = \lambda_i \gamma_i + (1-\lambda_i)\gamma_{i,0}$ and $0 \leq \lambda_i \leq 1$.

Note that,

$$|\gamma_i^* - \gamma_{i,0}| \leq \lambda_i |\gamma_i - \gamma_{i,0}| < \delta .$$

We will show that

$$\left| \frac{\partial f_t}{\partial \gamma_i} \right|_{\gamma^*} < g_t(a,b,k) ,$$

for all γ^* , $||\gamma^* - \gamma_0|| < \delta$,

and $n^{-1} \sum_{t=1}^{n} g_t(a,b,k)$ converges almost surely to a finite limit.

For example, consider

$$f_t(\gamma;\ 2,4,3) = h_t^{-3}(Y_t - \underset{\sim}{X}_t\underset{\sim}{\alpha})^2(Y_{t-1} - \underset{\sim}{X}_{t-1}\underset{\sim}{\alpha})^4 \quad .$$

Note that,

$$\frac{\partial f_t}{\partial \beta_1} = -\ 3h_t^{-4}(Y_t - \underset{\sim}{X}_t\underset{\sim}{\alpha})^2(Y_{t-1} - \underset{\sim}{X}_{t-1}\underset{\sim}{\alpha})^6 \ .$$

Now,

$$Y_t - \underset{\sim}{X}_t\underset{\sim}{\alpha} = e_t - a_t$$

where

$$a_t = \underset{\sim}{X}_t(\underset{\sim}{\alpha} - \underset{\sim}{\alpha}_0) \ .$$

Then,

$$\left|\frac{\partial f_t}{\partial \beta_1}\right|_{\underset{\sim}{\gamma}} \le 3m_1^{-4}(e_t - a_t)^2(e_{t-1} - a_{t-1})^6$$

$$\le 2^8 \cdot 3m_1^{-4}[e_t^2e_{t-1}^6 + M^6p^3\delta^6e_t^2 + M^2p\ e_{t-1}^6\delta^2 + M^8p^4\delta^8] \quad .$$

Note that, since we assumed that $E[e_t^8]$ is finite, $n^{-1}\sum_{t=1}^{n} e_t^2e_{t-1}^6$, $n^{-1}\sum_{t=1}^{n} e_t^2$ and $n^{-1}\sum_{t=1}^{n} e_{t-1}^6$ converge almost surely to finite constants. Similarly, other terms of $|T_n(\underset{\sim}{\gamma})|_{ij}$ can be bounded to establish (3.5).

Now, using Scott's martingale central limit theorem (see Scott (1973)) we will show that $n^{\frac{1}{2}}\left(\frac{\partial L_n}{\partial \underset{\sim}{\gamma}}\right)_{\underset{\sim}{\gamma}=\underset{\sim}{\gamma}_0}$ is asymptotically normal.

Define,

$$S_n = \underset{\sim}{\eta}'\left(\frac{\partial L_n}{\partial \underset{\sim}{\gamma}}\right)_{\underset{\sim}{\gamma}=\underset{\sim}{\gamma}_0}$$

where $\underset{\sim}{\eta}' = (\underset{\sim}{\eta}_0', \eta_1, \eta_2)$ is an arbitrary column vector such that $\underset{\sim}{\eta}'\underset{\sim}{\eta} \neq 0$. Note that,

$$S_n = \theta_1 \sum_{t=1}^{n} v_t^{-1}(Z_t^2-1)e_{t-1}\underset{\sim}{\eta}_0'\underset{\sim}{X}_{t-1}' - \sum_{t=1}^{n} v_t^{-\frac{1}{2}}\, Z_t\underset{\sim}{\eta}_0'\underset{\sim}{X}_t'$$

$$-\ \tfrac{1}{2}\sum_{t=1}^{n} v_t^{-1}(Z_t^2 - 1)(\eta_1 + \eta_2e_{t-1}^2) \ .$$

It is clear that $\{S_n, F_n\}$ is a martingale. Let

$$V_n^2 = E[S_n^2 | F_{n-1}]$$

and

$$s_n^2 = E[S_n^2] .$$

Then,

$$V_n^2 = 2\,\theta_1^2 \sum_{t=1}^{n} v_t^{-2} e_{t-1}^2\, \underset{\sim}{\eta}_0' \underset{\sim}{X}_{t-1}' \underset{\sim}{X}_{t-1}\, \underset{\sim}{\eta}_0 + \sum_{t=1}^{n} v_t^{-1}\, \underset{\sim}{\eta}_0' \underset{\sim}{X}_t' \underset{\sim}{X}_t \underset{\sim}{\eta}_0$$

$$+ \tfrac{1}{2} \sum_{t=1}^{n} v_t^{-2} (\eta_1 + \eta_2 e_{t-1}^2)^2 ,$$

and

$$s_n^2 = 2\theta_1^2\, c_3 \sum_{t=1}^{n} \underset{\sim}{\eta}_0' \underset{\sim}{X}_{t-1}' \underset{\sim}{X}_{t-1}\, \underset{\sim}{\eta}_0 + c_1\, \underset{\sim}{\eta}_0' \sum_{t=1}^{n} \underset{\sim}{X}_t' \underset{\sim}{X}_t\, \underset{\sim}{\eta}_0$$

$$+ n(\eta_1\ \eta_2)\, \underset{\sim}{H}_2 (\eta_1\ \eta_2)' .$$

Note that,

$$\frac{V_n^2}{s_n^2} = \frac{V_n^2/n}{s_n^2/n} \xrightarrow{P} 1 \quad \text{as } n \to \infty$$

and

$$\lim_{n\to\infty} n^{-1} s_n^2 = \underset{\sim}{\eta}' \underset{\sim}{H}\, \underset{\sim}{\eta} .$$

Since we assumed that $E[e_t^6] < \infty$, it follows that

$$E|(Z_{t-1}^2)^3\, e_{t-1}^3\, v_t^{-3}| < \infty ,$$

$$E|v_t^{-3/2}\, Z_t^3| < \infty ,$$

$$E|v_t^{-3} (Z_{t-1}^2)^3 (\eta_1 + \eta_2 e_{t-1}^2)^3| < \infty$$

and the Lindeberg condition is satisfied. Therefore,

$$s_n^{-1} S_n \xrightarrow{L} N(0,1)$$

and

$$n^{\frac{1}{2}}(\bar{\underset{\sim}{Y}}_n - \underset{\sim}{Y}_0) \xrightarrow{L} N(\underset{\sim}{0}, \underset{\sim}{H}^{-1}) \qquad \square$$

Now we consider the case $\underset{\sim}{X}_t = (Y_{t-1}, Y_{t-2}, \ldots, Y_{t-p})$. We assume that the roots of the characteristic equation

$$m^p - \alpha_1 m^{p-1} - \ldots - \alpha_p = 0 \tag{3.9}$$

lie inside the unit circle. Then, Y_t can be written as an infinite moving average as,

$$Y_t = \sum_{j=0}^{\infty} w_j e_{t-j} \tag{3.10}$$

where $\{w_j\}$ satisfy

$$\begin{aligned} w_j &= \alpha_1 w_{j-1} + \ldots + \alpha_p w_{j-p} \quad , \; j > 0 \\ &= 1 \quad , \; j = 0 \\ &= 0 \quad , \; j < 0 \;. \end{aligned}$$

In the following theorem we obtain the asymptotic properties of the maximum likelihood estimator.

Theorem 3.3: Assume that $\underset{\sim}{X}_t = (Y_{t-1}, Y_{t-2}, \ldots, Y_{t-p})$ and that the roots of the equation (3.9) lie inside the unit circle. Then the conditions (3.5) - (3.8) are satisfied with

$$\underset{\sim}{H} = \underset{\sim}{W} = \begin{pmatrix} \underset{\sim}{H}_{11}^* & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{H}_{22} \end{pmatrix}$$

where

$$\underset{\sim}{H}_{11}^* = E[v_t^{-1} \underset{\sim}{X}_t' \underset{\sim}{X}_t] + 2\theta_1 E[v_t^{-1} \underset{\sim}{X}_{t-1}' \underset{\sim}{X}_{t-1}] - 2\theta_1\theta_0 E[v_t^{-2} \underset{\sim}{X}_{t-1}' \underset{\sim}{X}_{t-1}]$$

and $\underset{\sim}{H}_{22}$ is as defined in Theorem 3.2.

Proof: For a fixed i, let

$$U_n = v_n^{-1} e_n Y_{n-i} \;.$$

Then,

$$E[U_n | F_{n-1}] = 0 \quad \text{a.s.} ,$$

and $E[U_n^2]$ is finite. Therefore,

$$n^{-1} \sum_{t=1}^{n} U_t \longrightarrow 0 \quad \text{a.s.} \quad .$$

Also,

$$n^{-1} \sum_{t=1}^{n} v_t^{-2}(e_t^2 - v_t)\, e_{t-1} X_{t-1,i} = n^{-1} \sum_{t=1}^{n} v_t^{-1}(Z_t^2 - 1) e_{t-1} Y_{t-1-i}$$

$$\longrightarrow 0 \quad \text{a.s.} ,$$

and

$$n^{-1} \sum_{t=1}^{n} v_t^{-3}(e_t^2 - v_t)\, Y_{t-1-i}\, e_{t-1} \longrightarrow 0 \quad \text{a.s.}$$

as $n \to \infty$.

Using the arguments similar to those in Theorem 3.2, we get

$$\lim_{n\to\infty} \left(\frac{\partial^2 L_n}{\partial \beta \partial \underset{\sim}{\alpha}'} \right)_{\underset{\sim}{\gamma} = \underset{\sim}{\gamma}_0} = \underset{\sim}{0} \quad \text{a.s.} ,$$

$$\lim_{n\to\infty} \left(\frac{\partial^2 L_n}{\partial \beta \partial \underset{\sim}{\beta}'} \right)_{\underset{\sim}{\gamma} = \underset{\sim}{\gamma}_0} = \underset{\sim}{H}_{22} \quad \text{a.s.} \quad .$$

Also,

$$\lim_{n\to\infty} \left(\frac{\partial^2 L_n}{\partial \underset{\sim}{\alpha} \partial \underset{\sim}{\alpha}'} \right)_{\underset{\sim}{\gamma} = \underset{\sim}{\gamma}_0} = \lim_{n\to\infty} n^{-1} \sum_{t=1}^{n} v_t^{-1} \underset{\sim}{X}_t' \underset{\sim}{X}_t$$

$$+ \lim_{n\to\infty} 2\theta_1^2 \, n^{-1} \sum_{t=1}^{n} v_t^{-2} e_{t-1}^2 \underset{\sim}{X}_{t-1}' \underset{\sim}{X}_{t-1} \quad .$$

Consider, for fixed i and j

$$n^{-1} \sum_{t=1}^{n} v_t^{-1} Y_{t-i} Y_{t-j} = \sum_{k=0}^{\infty} \sum_{s=0}^{\infty} w_k w_s n^{-1} \sum_{t=1}^{n} v_t^{-1} e_{t-i-k}\, e_{t-j-s} \quad .$$

Now for fixed q and r, consider

$$n^{-1} \sum_{t=1}^{n} v_t^{-1} e_{t-q}\, e_{t-r} \quad .$$

If q=r=1, then

$$n^{-1} \sum_{t=1}^{n} v_t^{-1} e_{t-1}^2 = (n\theta_1)^{-1} \sum_{t=1}^{n} v_t^{-1}(v_t - \theta_0)$$

$$\longrightarrow E[v_t^{-1} e_{t-1}^2] \quad \text{a.s.} \quad .$$

If q=r>1, then it can be shown that

$$|\mathrm{cov}(v_t^{-1} e_{t-q}^2,\ v_t^{-1} e_{t-q-j}^2)| \le \text{constant} \cdot \theta_1^{j-q} \quad ,$$

for j > q. Therefore,

$$\lim_{n\to\infty} n^{-1} \sum_{t=1}^{n} v_t^{-1} e_{t-q}^2 = E[v_t^{-1} e_{t-q}^2] \quad \text{a.s.} \quad .$$

Now if $q \neq r$, then $v_t^{-1} e_{t-q} e_{t-r}$ are uncorrelated and hence

$$\lim_{n\to\infty} n^{-1} \sum_{t=1}^{n} v_t^{-1} e_{t-q} e_{t-r} = E[v_t^{-1} e_{t-q} e_{t-r}]$$

$$= 0 \quad \text{a.s.} \quad .$$

Using Lemma 6.3.1 of Fuller (1976), we get

$$\lim_{n\to\infty} n^{-1} \sum_{t=1}^{n} v_t^{-1} Y_{t-i} Y_{t-j} = E[v_t^{-1} Y_{t-i} Y_{t-j}] \quad \text{a.s.} \ .$$

Similarly,

$$\lim_{n\to\infty} n^{-1} \sum_{t=1}^{n} v_t^{-2} Y_{t-i} Y_{t-j} = E[v_t^{-2} Y_{t-i} Y_{t-j}] \quad \text{a.s.} \ ,$$

and hence

$$\lim_{n\to\infty} \left(\frac{\partial^2 L_n}{\partial \underset{\sim}{\alpha} \partial \underset{\sim}{\alpha}} \right)_{\underset{\sim}{\gamma} = \underset{\sim}{\gamma}_0} = E[v_t^{-1} \underset{\sim}{X}_t' \underset{\sim}{X}_t] + 2\theta_1 \, E[v_t^{-1} \underset{\sim}{X}_{t-1}' \underset{\sim}{X}_{t-1}]$$

$$- 2\theta_0 \theta_1 \, E[v_t^{-2} \underset{\sim}{X}_{t-1}' \underset{\sim}{X}_{t-1}] \quad .$$

Note that

$$\underset{\sim}{\eta}_0' \, \underset{\sim}{H}_{11}^* \underset{\sim}{\eta}_0 \geq \underset{\sim}{\eta}_0' \, E[v_t^{-1} \underset{\sim}{X}_t' \underset{\sim}{X}_t] \, \underset{\sim}{\eta}_0$$

for any arbitrary $\underset{\sim}{\eta}_0$. Now, $E[v_t^{-1} \underset{\sim}{X}_t' \underset{\sim}{X}_t]$ is the variance covariance matrix of $v_{p+1}^{-1}(Y_p, Y_{p-1}, \ldots, Y_1)$. If

$$\underset{\sim}{\eta}_0' \, E[v_t^{-1} \underset{\sim}{X}_t' \underset{\sim}{X}_t] \, \underset{\sim}{\eta}_0 = 0$$

then

$$\underset{\sim}{\eta}_0' \, \underset{\sim}{X}_{p+1}' v_{p+1}^{-\frac{1}{2}} = 0 \quad \text{a.s.}$$

or

$$\underset{\sim}{\eta}_0' \, \underset{\sim}{X}_{p+1}' = 0 \quad \text{a.s. .}$$

However, we know that the variance covariance matrix of $\underset{\sim}{X}_{p+1} = (Y_p, Y_{p-1}, \ldots, Y_1)$ is positive definite. Therefore, $\underset{\sim}{H}_{11}^*$ is positive definite.

Now to establish (3.5), note that $|\underset{\sim}{T}_n(\underset{\sim}{\gamma})|_{ij}$ is a linear combination of terms of the form

$$f_t(\underset{\sim}{\gamma}; a,b,k,\underset{\sim}{q}) = h_t^{-k}(Y_t - \underset{\sim}{X}_t \underset{\sim}{\alpha})^a (Y_{t-1} - \underset{\sim}{X}_{t-1} \underset{\sim}{\alpha})^b \, X_{t-1,i}^{q_1} X_{t-1,j}^{q_2} X_{t,r}^{q_3} X_{t,\ell}^{q_4}$$

where $q_i = 0$ or 1 and a, b and k range from 0 to 4.

Again, for example, consider

$$f_t(\underset{\sim}{\gamma}; 2,4,3,\underset{\sim}{0}) = h_t^{-3}(Y_t - \underset{\sim}{X}_t \underset{\sim}{\alpha})^2 (Y_{t-1} - \underset{\sim}{X}_{t-1} \underset{\sim}{\alpha})^4 \quad .$$

Then

$$\left|\frac{\partial f_t}{\partial \beta_1}\right|_{\underset{\sim}{\gamma}} \leq \text{constant } [e_t^2 \, e_{t-1}^6 + \delta^6 p^2 \sum_{i=1}^{p} Y_{t-1-i}^6 e_t^2 + \delta^2 \sum_{i=1}^{p} Y_{t-i}^2 \, e_{t-1}^6$$
$$+ \delta^8 \, p^2 \sum_{i=1}^{p} \sum_{j=1}^{p} Y_{t-i}^2 \, Y_{t-j-1}^2] \quad ,$$

and

$$\left|\frac{\partial f_t}{\partial \alpha_i}\right|_{\underset{\sim}{\gamma}} \le \text{constant } [|(e_t - a_t)(e_{t-1}-a_{t-1})^4 Y_{t-i}|$$

$$+ |(e_t - a_t)^2(e_{t-1} - a_{t-1})^3 Y_{t-i-1}|$$

$$+ |(e_t - a_t)^2(e_{t-1} - a_{t-1})^5 Y_{t-i-1}|] .$$

Since $E[e_t^{12}]$ is assumed to be finite, we get

$$\lim_{n\to\infty} \sup_{\delta \downarrow 0} \delta^{-1} |\underset{\sim}{T}_n(\underset{\sim}{\gamma}^*)|_{ij} < \infty \quad \text{a.s.} \quad .$$

Now we verify (3.8). Let

$$S_n = \underset{\sim}{\eta}' \left(\frac{\partial L_n}{\partial \underset{\sim}{\gamma}}\right)_{\underset{\sim}{\gamma}=\underset{\sim}{\gamma}_0} .$$

where $\underset{\sim}{\eta}' = (\underset{\sim}{\eta}_0', \eta_1, \eta_2)$ is an arbitrary vector of constants such that $\underset{\sim}{\eta}'\underset{\sim}{\eta} \neq 0$.

Then,

$$S_n = \theta_1 \sum_{t=1}^{n} v_t^{-1}(Z_t^2-1)e_{t-1} \underset{\sim}{\eta}_0'\underset{\sim}{X}_{t-1}' - \sum_{t=1}^{n} v_t^{-\frac{1}{2}} Z_t \underset{\sim}{\eta}_0'\underset{\sim}{X}_t'$$

$$- \tfrac{1}{2} \sum_{t=1}^{n} v_t^{-1}(Z_t^2 - 1)(\eta_1 + \eta_2 e_{t-1}^2) ,$$

$$V_n^2 = E[S_n^2|F_{n-1}]$$

$$= 2 \theta_1^2 \sum_{t=1}^{n} v_t^{-2} e_{t-1}^2 \underset{\sim}{\eta}_0'\underset{\sim}{X}_{t-1}'\underset{\sim}{X}_{t-1}\underset{\sim}{\eta}_0 + \sum_{t=1}^{n} v_t^{-1} \underset{\sim}{\eta}_0'\underset{\sim}{X}_t'\underset{\sim}{X}_t\underset{\sim}{\eta}_0$$

$$+ \tfrac{1}{2} \sum_{t=1}^{n} v_t^{-2}(\eta_1 + \eta_2 e_{t-1}^2)^2 ,$$

and

$$s_n^2 = n\underset{\sim}{\eta}_0' \, [2\theta_1^2 \, E(v_t^{-2}e_{t-1}^2\underset{\sim}{X}_{t-1}'\underset{\sim}{X}_{t-1}) + E[(v_t^{-1}\underset{\sim}{X}_t'\underset{\sim}{X}_t)]\underset{\sim}{\eta}_0$$

$$+ n(\eta_1\eta_2) \, \underset{\sim}{H}_{22}(\eta_1\eta_2)' \quad .$$

Note that

$$\frac{V_n^2}{s_n^2} = \frac{V_n^2/n}{s_n^2/n} \xrightarrow{P} 1 \ , \quad \text{as } n \to \infty \ .$$

Since we assumed $E[e_t^{12}] < \infty$,

$$E|(Z_t^2-1)^3 \ e_{t-1}^3 Y_{t-1-i}^3 v_t^{-3}| < \infty \ ,$$

$$E|Z_t^3 v_t^{-3/2} Y_{t-i}^3| < \infty \ ,$$

and

$$E|v_t^{-3}(Z_t^2-1)^3(\eta_1 + \eta_2 e_{t-1}^2)^3| < \infty \ .$$

Therefore, by Scott's martingale central limit theorem,

$$s_n^{-1} S_n \xrightarrow{L} N(0,1) \ ,$$

and hence

$$n^{\frac{1}{2}}(\tilde{\gamma}_n - \gamma_0) \xrightarrow{L} N(0, H^{-1}) \ . \quad \square$$

It is easy to see that if $X_t = (1, Y_{t-1}, Y_{t-2}, \ldots, Y_{t-p+1})$ the maximum likelihood estimator is still consistent and asymptotically normal. If X_t is fixed but not necessarily bounded then $\bar{\alpha}$ may converge to α_0 at a rate faster than $n^{-\frac{1}{2}}$. For example, if $X_t = t$, then $(\bar{\alpha} - \alpha_0)$ is $O_p(n^{-1})$.

Now we consider the least squares estimation of γ.

3.2. <u>Least Squares Estimation</u>:

The maximum likelihood estimates considered in 3.1 do not have explicit expressions and are estimated using iterative procedures. We now consider the ordinary and estimated generalized least squares estimates of γ. The least squares estimates are obtained as follows:

<u>Step 1</u>: Regress Y_t on X_t to obtain the ordinary least squares estimator $\tilde{\alpha}$ of α. Let $e_t = Y_t - X_t\tilde{\alpha}$.

Step 2: Regress $\tilde{e}_t^2$ on a column of ones and $\tilde{e}_{t-1}^2$ to get $\tilde{\theta}_0$ and $\tilde{\theta}_1$. Let $\tilde{v}_t = \tilde{\theta}_0 + \tilde{\theta}_1 \tilde{e}_{t-1}^2$.

Step 3: Regress $\tilde{v}_t^{-1} \tilde{e}_t^2$ on $\tilde{v}_t^{-1}$ and $\tilde{v}_t^{-1} \tilde{e}_{t-1}^2$ to get an estimated generalized least squares estimates $\hat{\theta}_0$ and $\hat{\theta}_1$. Let $\hat{v}_t = \hat{\theta}_0 + \hat{\theta}_1 \tilde{e}_{t-1}^2$.

Step 4: Regress $\hat{v}_t^{-\frac{1}{2}} Y_t$ on $\hat{v}_t^{-\frac{1}{2}} \underset{\sim}{X}_t$ to get an estimated generalized least squares estimate $\hat{\underset{\sim}{\alpha}}$ of $\underset{\sim}{\alpha}$.

We now study the properties of $\tilde{\underset{\sim}{\alpha}}$, $\tilde{\underset{\sim}{\theta}}$, $\hat{\underset{\sim}{\alpha}}$ and $\hat{\underset{\sim}{\theta}}$. We first consider the case where $\underset{\sim}{X}_t$ is fixed and bounded.

Theorem 3.4: Let $\{\underset{\sim}{X}_t\}$ and $\{Y_t\}$ satisfy the conditions of Theorem 3.2. Let $\underset{\sim}{\gamma}_0$ be in the interior of Γ. Then

$$n^{\frac{1}{2}}(\tilde{\underset{\sim}{\gamma}} - \underset{\sim}{\gamma}_0) \xrightarrow{L} N(\underset{\sim}{0}, \underset{\sim}{B}_0) ,$$

and

$$n^{\frac{1}{2}}(\hat{\underset{\sim}{\gamma}} - \underset{\sim}{\gamma}_0) \xrightarrow{L} N(\underset{\sim}{0}, \underset{\sim}{B}_2) ,$$

where

$$\tilde{\underset{\sim}{\gamma}}' = (\tilde{\underset{\sim}{\alpha}}', \tilde{\underset{\sim}{\theta}}') ,$$

$$\hat{\underset{\sim}{\gamma}}' = (\hat{\underset{\sim}{\alpha}}', \hat{\underset{\sim}{\theta}}') ,$$

$$B_0 = \begin{bmatrix} \sigma^2 \underset{\sim}{A}^{-1} & \underset{\sim}{0} \\ \underset{\sim}{0} & 2\, \underset{\sim}{A}_1^{-1} \underset{\sim}{B}_1 \underset{\sim}{A}_1^{-1} \end{bmatrix} ,$$

$$\underset{\sim}{B}_1 = E \begin{bmatrix} v_2^2 & e_1^2 v_2^2 \\ e_1^2 v_2^2 & e_1^4 v_2^2 \end{bmatrix} ,$$

$$v_2 = \theta_0 + \theta_1 e_1^2 ,$$
$$\sigma^2 = \theta_0 (1 - \theta_1)^{-1} ,$$

$$\underset{\sim}{B}_2 = \begin{bmatrix} (c_1\underset{\sim}{A})^{-1} & \underset{\sim}{0} \\ \underset{\sim}{0} & 2\,\underset{\sim}{H}_{22}^{-1}\,\underset{\sim}{A}_1\,\underset{\sim}{H}_{22}^{-1} \end{bmatrix} ,$$

$$\underset{\sim}{A}_1 = E \begin{bmatrix} 1 & e_1^2 \\ e_1^2 & e_1^4 \end{bmatrix} ,$$

and c_1 and $\underset{\sim}{H}_{22}$ as defined in Theorem 3.2.

Proof: Note that

$$n^{\frac{1}{2}}(\tilde{\underset{\sim}{\alpha}} - \underset{\sim}{\alpha}_0) = (n^{-1}\underset{\sim}{X}'\underset{\sim}{X})^{-1}\, n^{-\frac{1}{2}}\underset{\sim}{X}'e$$

where $\underset{\sim}{e} = (e_1, e_2, \ldots, e_n)'$ and $\underset{\sim}{X}' = (\underset{\sim}{X}_1', \underset{\sim}{X}_2', \ldots, \underset{\sim}{X}_n')$.

We know that

$$\lim_{n\to\infty} (n^{-1}\underset{\sim}{X}'\underset{\sim}{X})^{-1} = \underset{\sim}{A}^{-1} .$$

Consider,

$$S_n = \underset{\sim}{\eta}_0'\, \underset{\sim}{X}'\underset{\sim}{e}$$

$$= \sum_{t=1}^{n} b_t e_t$$

where

$$b_t = \sum_{i=1}^{p} \eta_{i,0}\, X_{t,i}$$

and $\underset{\sim}{\eta}_0$ is an arbitrary vector of constants with $\eta_0'\,\eta_0 \neq 0$. Note that $\{S_n, F_n\}$ is a martingale with,

$$V_n^2 = E[S_n^2 | F_{n-1}]$$

$$= \sum_{t=1}^{n} b_t^2\, v_t \quad \text{a.s.} ,$$

and

$$s_n^2 = \sigma^2 \sum_{t=1}^{n} b_t^2 .$$

Therefore,

$$s_n^{-2} V_n^2 \xrightarrow{P} 1$$

and

$$n^{-1} s_n^2 \longrightarrow \underset{\sim}{\eta}_0' \sigma^2 \underset{\sim}{A} \underset{\sim}{\eta}_0 .$$

Since $E[e_t^4]$ is finite, the Lindeberg condition is satisfied and

$$s_n^{-1} S_n \xrightarrow{L} N(0,1) .$$

Therefore,

$$n^{\frac{1}{2}}(\tilde{\underset{\sim}{\alpha}} - \underset{\sim}{\alpha}_0) \xrightarrow{L} N(\underset{\sim}{0}, \sigma^2 \underset{\sim}{A}^{-1}).$$

Now consider,

$$\tilde{\underset{\sim}{\theta}} = \begin{bmatrix} (n-1) & \sum_{t=2}^{n} \tilde{e}_{t-1}^2 \\ \sum_{t=2}^{n} \tilde{e}_{t-1}^2 & \sum_{t=2}^{n} \tilde{e}_{t-1}^4 \end{bmatrix}^{-1} \begin{bmatrix} \sum_{t=2}^{n} \tilde{e}_t^2 \\ \sum_{t=2}^{n} \tilde{e}_{t-1}^2 \tilde{e}_t^2 \end{bmatrix}$$

where

$$\tilde{e}_t = Y_t - \underset{\sim}{X}_t \tilde{\underset{\sim}{\alpha}}$$

$$= e_t - \tilde{a}_t$$

and

$$\tilde{a}_t = \underset{\sim}{X}_t (\tilde{\underset{\sim}{\alpha}} - \underset{\sim}{\alpha}_0) .$$

Note that,

$$n^{-1}\sum_{t=2}^{n}\tilde{e}_{t-1}^{2} = n^{-1}\sum_{t=2}^{n} e_{t-1}^{2} - 2n^{-1}\sum_{t=2}^{n} e_{t-1}\tilde{a}_{t-1} + n^{-1}\sum_{t=2}^{n}\tilde{a}_{t-1}^{2}$$

$$= n^{-1}\sum_{t=2}^{n} e_{t-1}^{2} + 0_p(n^{-\frac{1}{2}}) \ .$$

Similarly,

$$n^{-1}\sum_{t=2}^{n}\tilde{e}_{t-1}^{4} = n^{-1}\sum_{t=2}^{n} e_{t-1}^{4} + 0_p(n^{-\frac{1}{2}})$$

and

$$n^{-1}\sum_{t=2}^{n}\tilde{e}_{t-1}^{2}\,\tilde{e}_{t}^{2} = n^{-1}\sum_{t=2}^{n} e_{t-1}^{2}e_{t}^{2} + 0_p(n^{-\frac{1}{2}}) \ .$$

Therefore,

$$n^{\frac{1}{2}}(\tilde{\underset{\sim}{\theta}}-\underset{\sim}{\theta}) = \begin{bmatrix} 1 & n^{-1}\sum_{t=2}^{n} e_t^2 \\ n^{-1}\sum_{t=2}^{n} e_t^2 & n^{-1}\sum_{t=2}^{n} e_t^4 \end{bmatrix}^{-1} \begin{bmatrix} n^{-\frac{1}{2}}\sum_{t=2}^{n} d_t \\ n^{-\frac{1}{2}}\sum_{t=2}^{n} d_t e_{t-1}^2 \end{bmatrix} + 0_p(n^{-\frac{1}{2}}) \ ,$$

where

$$d_t = (Z_t^2 - 1)v_t \ .$$

Note that,

$$\begin{bmatrix} 1 & n^{-1}\sum_{t=2}^{n} e_t^2 \\ n^{-1}\sum_{t=2}^{n} e_t^2 & n^{-1}\sum_{t=2}^{n} e_t^4 \end{bmatrix}^{-1} \longrightarrow \underset{\sim}{A}_1 \quad \text{a.s.} \ ,$$

as $n \to \infty$.

Consider,

$$S_n = \eta_1 \sum_{t=2}^{n} d_t + \eta_2 \sum_{t=2}^{n} d_t e_{t-1}^2$$

$$= \sum_{t=2}^{n} (\eta_1 + \eta_2 e_{t-1}^2)(\theta_0 + \theta_1 e_{t-1}^2)(Z_t^2 - 1)$$

where η_1 and η_2 are two arbitrary constants. Then S_n is a martingale with

$$V_n^2 = 2 \sum_{t=2}^{n} (\eta_1 + \eta_2 e_{t-1}^2)^2 (\theta_0 + \theta_1 e_{t-1}^2)^2$$

and

$$s_n^2 = 2(n-1) E[(\eta_1 + \eta_2 e_{t-1}^2)^2 (\theta_0 + \theta_1 e_{t-1}^2)^2] .$$

Using the usual arguments we get

$$s_n^{-1} S_n \xrightarrow{L} N(0,1) ,$$

and hence

$$n^{\frac{1}{2}}(\tilde{\underset{\sim}{\theta}} - \underset{\sim}{\theta}) \xrightarrow{L} N(\underset{\sim}{0}, 2\underset{\sim}{A}_1^{-1} \underset{\sim}{B}_1 \underset{\sim}{A}_1^{-1}) .$$

Note that (Z_t^2-1) and e_t are uncorrelated and hence $\tilde{\underset{\sim}{\alpha}}$ and $\tilde{\underset{\sim}{\theta}}$ are asymptotically independent. Now we consider $\hat{\underset{\sim}{\theta}}$ obtained by step 3.

Using arguments as above, we get

$$\hat{\underset{\sim}{\theta}} = \begin{bmatrix} \sum_{t=2}^{n} \tilde{\tilde{v}}_t^{-2} & \sum_{t=2}^{n} \tilde{\tilde{v}}_t^{-2} \tilde{e}_{t-1}^2 \\ \sum_{t=2}^{n} \tilde{\tilde{v}}_t^{-2} \tilde{e}_{t-1}^2 & \sum_{t=2}^{n} \tilde{\tilde{v}}_t^{-2} \tilde{e}_{t-1}^4 \end{bmatrix}^{-1} \begin{bmatrix} \sum_{t=2}^{n} \tilde{\tilde{v}}_t^{-1} \tilde{e}_t^2 \\ \sum_{t=2}^{n} \tilde{\tilde{v}}_t^{-1} \tilde{e}_{t-1}^2 \tilde{e}_t^2 \end{bmatrix}$$

and

$$n^{\frac{1}{2}}(\hat{\underset{\sim}{\theta}} - \underset{\sim}{\theta}) = \underset{\sim}{H}_{22}^{-1} \begin{bmatrix} n^{-\frac{1}{2}} \sum_{t=2}^{n} v_t^{-1} d_t \\ n^{-\frac{1}{2}} \sum_{t=2}^{n} v_t^{-1} e_{t-1}^2 d_t \end{bmatrix} + o_p(1) .$$

Consider,

$$S_n = \eta_1 \sum_{t=2}^{n} v_t^{-1} d_t + \eta_2 \sum_{t=2}^{n} v_t^{-1} e_{t-1}^2 d_t$$

$$= \sum_{t=2}^{n} (\eta_1 + \eta_2 e_{t-1}^2)(Z_t^2 - 1) .$$

Then, S_n is a martingale with

$$V_n^2 = 2 \sum_{t=2}^{n} (\eta_1 + \eta_2 e_{t-1}^2)^2 \quad \text{a.s.,}$$

and

$$s_n^2 = 2(n-1) E(\eta_1 + \eta_2 e_{t-1}^2)^2 .$$

Since $E[e_t^6]$ is finite, S_n satisfies the Lindeberg condition and

$$s_n^{-1} S_n \xrightarrow{L} N(0,1) .$$

Therefore,

$$n^{\frac{1}{2}}(\hat{\theta}-\theta_0) \xrightarrow{L} N(0, 2H_{22}^{-1} A_1 H_{22}^{-1}) .$$

Now we consider the regression of $\hat{v}_t^{-1} Y_t$ on $\hat{v}_t^{-1} X_t$ where $\hat{v}_t = \hat{\theta}_0 + \hat{\theta}_1 \tilde{e}_{t-1}^2$. Note that,

$$(\hat{\alpha}-\alpha_0) = (X' \hat{G}_n^{-1} X)^{-1} X' \hat{G}_n^{-1} e ,$$

where

$$\hat{G}_n = \text{diag}\{\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_n\} .$$

Let

$$G_n = \text{diag}\{v_1, v_2, \ldots, v_n\} .$$

Then, as $n \to \infty$,

$$n^{-1}(X' G_n^{-1} X) \xrightarrow{P} E[v_1^{-1}] A ,$$

$$n^{-1}[X'(G_n^{-1} - \hat{G}_n^{-1}) X] \xrightarrow{P} 0$$

and

$$n^{-\frac{1}{2}}[X'(G_n^{-1} - \hat{G}_n^{-1}) e] \xrightarrow{P} 0 .$$

Consider,

$$S_n = \eta_0' X' G_n^{-1} e$$

$$= \sum_{t=1}^{n} \left(\sum_{i=1}^{p} \eta_{i,0} X_{t,i} \right) v_t^{-\frac{1}{2}} Z_t .$$

Note again that S_n is a martingale with

$$V_n^2 = \eta_0' X' G_n^{-1} X \eta_0$$

and

$$s_n^2 = E[v_1^{-1}] \eta_0' X' X \eta_0 .$$

The Lindeberg condition is clearly satisfied and hence

$$s_n^{-1} S_n \xrightarrow{L} N(0,1)$$

and

$$n^{1/2}(\hat{\underset{\sim}{\alpha}} - \underset{\sim}{\alpha}_0) \xrightarrow{L} N(\underset{\sim}{0},\ (c \cdot \underset{\sim}{A})^{-1}) \ .$$

Since (Z_t^2-1) and Z_t are uncorrelated $\hat{\underset{\sim}{\alpha}}$ and $\hat{\underset{\sim}{\theta}}$ are asymptotically independent. □

Note that

$$(c_1 + 2\theta_1^2 c_3)^{-1} \le c_1^{-1} \le \sigma^2$$

and the equality hold only if $\theta_1 = 0$. Therefore, the maximum likelihood estimator is asymptotically the best among the three estimators considered and the estimated generalized least squares estimator is asymptotically better than the ordinary least squares estimator.

Here we have assumed that $\underset{\sim}{X}_t$ is fixed and bounded. Suppose $\underset{\sim}{X}_t$ is fixed and satisfies

$$\lim_{n\to\infty} \underset{\sim}{D}_n^{-1}(\underset{\sim}{X}'\underset{\sim}{X})\underset{\sim}{D}_n^{-1} = \underset{\sim}{A}_0 \ ,$$

$$\lim_{n\to\infty} \left\{ \sum_{t=1}^{n-h} X_{t,i}^2 \sum_{t=1}^{n-h} X_{t+h,j}^2 \right\}^{-1/2} \sum_{t=1}^{n} X_{t,i} X_{t+h,j} = a_{h,i,j} = a_{-h,i,j} \ ,$$

$$\lim_{n\to\infty} \sum_{t=1}^{n} X_{t,i}^2 = \infty$$

and

$$\lim_{n\to\infty} \left(\sum_{t=1}^{n} X_{t,i}^2 \right)^{-1} X_{n,i}^2 = 0 \ ,$$

for $i = 1, 2, \ldots, p$; $j = 1, 2, \ldots, p$,

where

$$\underset{\sim}{D}_n = \text{diag}\left\{ \left(\sum_{t=1}^{n} X_{ti}^2 \right)^{1/2}, \ldots, \left(\sum_{t=1}^{n} X_{t,p}^2 \right)^{1/2} \right\} \ .$$

Then, it can be shown that

$$\underset{\sim}{D}_n(\tilde{\underset{\sim}{\alpha}}-\underset{\sim}{\alpha}_0) \xrightarrow{L} N(\underset{\sim}{0},\sigma^2\underset{\sim}{A}_0^{-1})$$

and

$$\underset{\sim}{D}_n(\hat{\underset{\sim}{\alpha}}-\underset{\sim}{\alpha}_0) \xrightarrow{L} N(\underset{\sim}{0},(c_1\underset{\sim}{A}_0)^{-1}) .$$

Now we obtain the asymptotic properties of the least squares estimators for the case when $\underset{\sim}{X}_t = (Y_{t-1}, Y_{t-2}, \ldots, Y_{t-p})$.

<u>Theorem 3.5</u>: Assume that $\underset{\sim}{X}_t$ saisfies the conditions in Theorem 3.3. Then,

$$n^{\frac{1}{2}}(\tilde{\underset{\sim}{\gamma}}-\underset{\sim}{\gamma}_0) \xrightarrow{L} N(\underset{\sim}{0},\underset{\sim}{B}_3)$$

and

$$n^{\frac{1}{2}}(\hat{\underset{\sim}{\gamma}}-\underset{\sim}{\gamma}_0) \xrightarrow{L} N(\underset{\sim}{0},\underset{\sim}{B}_4) ,$$

where

$$\underset{\sim}{B}_3 = \begin{bmatrix} \underset{\sim}{Q} & \underset{\sim}{0} \\ \underset{\sim}{0} & 2\underset{\sim}{A}_1^{-1}\underset{\sim}{B}_1\underset{\sim}{A}_1^{-1} \end{bmatrix} ,$$

$$\underset{\sim}{B}_4 = \begin{bmatrix} \underset{\sim}{Q}_3^{-1} & \underset{\sim}{0} \\ \underset{\sim}{0} & 2\underset{\sim}{H}_{22}^{-1}\ \underset{\sim}{A}_1\underset{\sim}{H}_{22}^{-1} \end{bmatrix},$$

$$\underset{\sim}{Q} = \sigma^2\ \underset{\sim}{Q}_1^{-1} + \theta_1\underset{\sim}{Q}_1^{-1}\ \underset{\sim}{Q}_2\underset{\sim}{Q}_1^{-1} ,$$

$$\gamma_{e^2}(0) = V(e_t^2) = 2\sigma^4(1 - 3\theta_1^2)^{-1} ,$$

for $j \geq i$,

$$(\underset{\sim}{Q}_1)_{ij} = E[Y_{t-i}Y_{t-j}] = \sum_{k=0}^{\infty} w_k w_{k+j-i}\sigma^2$$
$$= (\underset{\sim}{Q}_1)_{ji} ,$$

$$(\underset{\sim}{Q}_2)_{ij} = E[Y_{t-i}Y_{t-j}e_{t-1}^2]$$
$$= \gamma_{e^2}(0)\sum_{k=0}^{\infty} w_k w_{k+j-i}\theta_1^{k+i-1}$$
$$= (\underset{\sim}{Q}_2)_{ji} ,$$

and

$$(Q_3)_{ij} = E[v_t^{-1} Y_{t-i} Y_{t-j}]$$

$$= \sum_{k=0}^{\infty} w_k \, w_{k+i-1} \, E[v_t^{-1} e_{t-i-k}^2] \; .$$

Proof: Note that,

$$n^{-1} \sum_{t=1}^{n} Y_{t-i} Y_{t-j} \rightarrow \sum_{k=0}^{\infty} w_k w_{k+j-i} \, \sigma^2 \quad \text{a.s.}$$

and hence

$$n^{-1} \sum_{t=1}^{n} \underset{\sim}{X}_t' \underset{\sim}{X}_t \rightarrow \underset{\sim}{Q}_1 = E[\underset{\sim}{X}_t' \underset{\sim}{X}_t] \qquad \text{a.s.} \quad .$$

Consider,

$$S_n = \underset{\sim}{\eta}_0' \underset{\sim}{X}' \underset{\sim}{e}$$

$$= \sum_{t=1}^{n} b_t e_t$$

where

$$b_t = \sum_{i=1}^{p} \eta_{i,0} Y_{t-i}$$

and $\underset{\sim}{\eta}_0$ is an arbitrary vector with $\underset{\sim}{\eta}_0' \underset{\sim}{\eta} \neq 0$. Then S_n is a martingale with,

$$V_n^2 = \theta_0 \, \underset{\sim}{\eta}_0' \, \underset{\sim}{X}' \underset{\sim}{X} \, \underset{\sim}{\eta}_0 + \theta_1 \sum_{t=1}^{n} \underset{\sim}{\eta}_0' \, \underset{\sim}{X}_t' \, \underset{\sim}{X}_t \, e_{t-1}^2 \, \underset{\sim}{\eta}_0 \qquad \text{a.s.} \; ,$$

and

$$s_n^2 = n \, \theta_0 \, \underset{\sim}{\eta}_0' \, \underset{\sim}{Q}_1 \underset{\sim}{\eta}_0 + n \, \theta_1 \, \underset{\sim}{\eta}_0' \, E[\underset{\sim}{X}_t' \underset{\sim}{X}_t e_{t-1}^2] \, \underset{\sim}{\eta}_0 \quad .$$

Note that, for $j \geq i$,

$$n^{-1} \sum_{t=1}^{n} Y_{t-i} Y_{t-j} e_{t-1}^2 = \sum_{k=0}^{\infty} \sum_{s=0}^{\infty} w_k w_s \, n^{-1} \sum_{t=1}^{n} e_{t-i-k} e_{t-s-j} e_{t-1}^2$$

$$\xrightarrow{P} \sum_{k=0}^{\infty} w_k w_{k+j-i} \, E(e_{t-1}^2 e_{t-i-k}^2)$$

$$= E[Y_{t-i} Y_{t-j} e_{t-1}^2]$$

$$= \sum_{k=0}^{\infty} w_k w_{k+j-i} \, [\gamma_{e^2}(0) \theta_1^{k+i-1} + \sigma^4] \; .$$

Therefore,

$$s_n^{-2}V_n^2 \xrightarrow{P} 1$$

and

$$n^{-1}s_n^2 \longrightarrow \sigma^2 \underset{\sim}{\eta}_0' \underset{\sim}{Q}_1 \underset{\sim}{\eta}_0 + \theta_1 \underset{\sim}{\eta}_0' \underset{\sim}{Q}_2 \underset{\sim}{\eta}_0 \quad .$$

Since $E[e_t^6]$ is finite, the Lindeberg condition is satisfied and

$$s_n^{-1}S_n \xrightarrow{L} N(0,1) \ .$$

Therefore,

$$n^{\frac{1}{2}}(\tilde{\underset{\sim}{\alpha}}-\underset{\sim}{\alpha}_0) \xrightarrow{L} N(\underset{\sim}{0}, \sigma^2 \underset{\sim}{Q}_1^{-1} + \gamma_{e^2}(0)\ \underset{\sim}{Q}_1^{-1}\underset{\sim}{Q}_2\underset{\sim}{Q}_1^{-1}) \ .$$

Using the arguments similar to those of Theorem 3.4, it follows that

$$n^{\frac{1}{2}}(\tilde{\underset{\sim}{\theta}}-\underset{\sim}{\theta}) \xrightarrow{L} N(\underset{\sim}{0},\ 2\ \underset{\sim}{A}_1^{-1}\underset{\sim}{B}_1\underset{\sim}{A}_1^{-1})$$

$$n^{\frac{1}{2}}(\hat{\underset{\sim}{\theta}}-\underset{\sim}{\theta}) \xrightarrow{L} N(\underset{\sim}{0},\ 2\underset{\sim}{H}_{22}^{-1}\ \underset{\sim}{A}_1\underset{\sim}{H}_{22}^{-1}) \ ,$$

and $\tilde{\underset{\sim}{\alpha}}$ and $\tilde{\underset{\sim}{\theta}}$ are asymptotically independent.

Now, to obtain the limiting distribution of $\hat{\underset{\sim}{\alpha}}$, consider

$$S_n = \underset{\sim}{\eta}_0'\underset{\sim}{X}'\underset{\sim}{G}_n^{-1}\underset{\sim}{e}$$

$$= \sum_{t=1}^{n} \left(\sum_{i=1}^{p} \eta_{i,0}Y_{t-i}\right)v_t^{-\frac{1}{2}}\ Z_t$$

where $\underset{\sim}{G}_n$ and $\underset{\sim}{\eta}_0$ are as defined in Theorem 3.4. Then, S_n is a martingale with

$$V_n^2 = \eta_0'\ \underset{\sim}{X}'\underset{\sim}{G}_n^{-1}\underset{\sim}{X}\underset{\sim}{\eta}_0 \quad \text{a.s.},$$

and

$$s_n^2 = \underset{\sim}{\eta}_0'\ E[\underset{\sim}{X}'\underset{\sim}{G}_n^{-1}\ \underset{\sim}{X}]\ \underset{\sim}{\eta}_0 \ .$$

From Theorem 3.3, we know that

$$n^{-1}\sum_{t=1}^{n} v_t^{-1}Y_{t-i}Y_{t-j} \longrightarrow E[v_t^{-1}Y_{t-i}Y_{t-j}] \quad \text{a.s.} \quad .$$

Therefore,

$$s_n^{-2}V_n^2 \xrightarrow{P} 1$$

and

$$n^{-1}s_n^2 \longrightarrow \underset{\sim}{\eta}_0' \, E[v_t^{-1}\underset{\sim}{X}_t'\underset{\sim}{X}_t] \, \underset{\sim}{\eta}_0 \ .$$

Using Scott's martingale central limit theorem, we get

$$s_n^{-1}S_n \xrightarrow{L} N(0,1)$$

and

$$n^{\frac{1}{2}}(\hat{\underset{\sim}{\alpha}}-\underset{\sim}{\alpha}_0) \xrightarrow{L} N(\underset{\sim}{0}, \underset{\sim}{Q}_3^{-1}) \ .$$

Note also that $\hat{\underset{\sim}{\alpha}}$ and $\hat{\underset{\sim}{\theta}}$ are asymptotically independent. □

If $\underset{\sim}{X}_t$ involves both fixed and lagged variables then one can obtain results similar to those of Fuller, Hasza and Goebel(1981). Also, if Y_t process has a unit root, we can obtain the asymptotic distribution of the least squares estimator. Consider, for example,

$$Y_t = \alpha_1 Y_{t-1} + e_t \ ,$$

$$Y_0 = 0 \ ,$$

$$\alpha_1 = 1 \ ,$$

and $\{e_t\}$ satisfies the conditions of Theorem 3.4. The least squares estimator of α_1 is given by,

$$\tilde{\alpha}_1 = \left[\sum_{t=2}^{n} Y_{t-1}^2\right]^{-1} \sum_{t=2}^{n} Y_t Y_{t-1} \ .$$

Then,

$$n(\tilde{\alpha}_1 - 1) = [n^{-2}\sum_{t=2}^{n} Y_{t-1}^2]^{-1}[n^{-1}\sum_{t=2}^{n} Y_{t-1}e_t]$$

$$= [2n^{-2}\sum_{t=2}^{n} Y_{t-1}^2]^{-1} \; [(n^{-\frac{1}{2}} Y_{n-1})^2 - n^{-1}\sum_{t=2}^{n} e_t^2] \; .$$

If $\theta_1 = 0$, Dickey and Fuller (1979) obtained the asymptotic distribution of $n(\tilde{\alpha}_1 - 1)$. We now show that even if $\theta_1 \neq 0$, $n(\tilde{\alpha}_1 - 1)$ has the same limiting distribution.

We know that

$$\ell im_{n\to\infty} n^{-1}\sum_{t=2}^{n} e_t^2 = \sigma^2 \quad \text{a.s.}$$

Now consider,

$$T_n = n^{-\frac{1}{2}} Y_{n-1}$$

$$= \sum_{i=1}^{n-1} a_{i,n} Z_{i,n}^*$$

and

$$\Gamma_n = n^{-2}\sum_{t=2}^{n} Y_{t-1}^2$$

$$= n^{-2}\sum_{i=1}^{n-1} \lambda_{i,n} Z_{i,n}^{*2}$$

where

$$\underset{\sim}{Z}^* = (Z_{1,n}^*, \ldots, Z_{n-1,n}^*)'$$

$$= \underset{\sim}{M}_n \, \underset{\sim}{e}_n \quad ,$$

$$\underset{\sim}{e}_n = (e_1, e_2, \ldots, e_{n-1})' \; ,$$

$$\lambda_{i,n} = \tfrac{1}{4} \sec^2[(n-i)\pi/(2n-1)]$$

$$m_{it}(n) = (i,t) - \text{th element of } \underset{\sim}{M}_n$$

$$= 2(2n-1)^{-\frac{1}{2}} \operatorname{Cos}[4n-2)^{-1}(2t-1)(2i-1)\pi]$$

and

$$a_{i,n} = i^{th} \text{ element of } n^{-\frac{1}{2}}(1,\ldots,1)\underset{\sim}{M}_n^{-1} \quad .$$

Using Scott's martingale central theorem, it follows that, for any fixed k,

$$(Z^*_{1,n},\ldots,Z^*_{k,n}) \xrightarrow{L} N(\underset{\sim}{0},\sigma^2 \underset{\sim}{I}_k)$$

where $\underset{\sim}{I}_k$ is an k x k identity matrix. Now using the arguments similar to Hasza (1977) and Pantula (1982) it follows that $n(\tilde{\alpha}_1 - 1)$ has the same limiting distribution as that obtained by Dickey and Fuller (1979). Similarly, the results for p^{th} order ARCH models may be obtained.

4. <u>Summary</u>:

We have considered linear regression models with autoregressive conditionally heteroscedastic errors, introduced by Engel (1982). We have obtained a series representation for the first order ARCH errors. We have used the representation to derive the ergodic properties of the errors. Similar representation can be obtained for the q^{th} $(q > 1)$ order ARCH errors but are not presented here. A special case where the conditional error variance is of the form $\beta_0 + \beta_1 \sum_{j=1}^{q} a_j e^2_{t-j}$, where $a_j = q^{-1}$ or $a_j = 2[q(q+1)]^{-1} \cdot [q+1-j]$ will be considered elsewhere.

We have considered the maximum likelihood estimation of ARCH regression models. The maximum likelihood estimators do not have explicit algebraic form and are computed using iterative methods. We have shown that the maximum likelihood estimators are strongly consistent and asymptotically normal. We have also shown that the least squares estimator and an estimated generalized least squares estimator are asymptotically normal. For a random walk model $(Y_t = \alpha_1 Y_{t-1} + e_t, \alpha_1 = 1)$ with ARCH errors, we have shown that the asymptotic distribution of the least squares estimator of α_1 is the distribution obtained by Dickey and Fuller (1979) for the homoscedastic case.

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BIBLIOGRAPHY

Chung, K. L. (1974). *A Course in Probability Theory*. Academic Press, NY.

Crowder, M. J. (1976). Maximum Likelihood Estimation for Dependent Observations, *Journal of the Royal Statistical Society, Series B*, 45-53.

Dickey, D. A. and W. A. Fuller (1979). Distribution of the Estimators for Autoregressive Time Series with a Unit Root, *Journal of American Statistical Association*, **74**, 427-531.

Engle, R. F. (1982). Autoregressive Conditional Heteroscedasticity with Estimates of United Kingdom Inflation, *Econometrica*, **50**, 987-1007.

Fuller, W. A. (1976). *Introduction to Statistical Time Series*. Wiley, NY.

Fuller, W. A., D. P. Hasza, and J. J. Goebel (1981). Estimation of the Parameters of Stochastic Difference Equations, *The Annals of Statistics*, **9**, 531-543.

Granger, C. W. J. and A. Anderson (1978). *An Introduction to Bilinear Time-Series Models*. Vandenhoeck and Ruprecht, Göttingen.

Hall, P. and C. C. Heyde (1980). *Martingale Limit Theory and Its Application*. Academic Press, NY.

Hasza, D. P. (1977). Estimation in Nonstationary Time Series. Unpublished Ph.D. Thesis, Iowa State University, Ames, Iowa.

Jones, R. H. (1965). An Experiment in Nonlinear Prediction, *Journal of Applied Meterology*, **4**, 701-705.

Pantula, S. G. (1982). Properties of Estimator of the Parameters of Autoregressive Time Series. Unpublished Ph.D. Thesis, Iowa State University, Ames, Iowa.

Priestly, M. B. (1978). Nonlinear Models in Time Series Analysis, *The Statistician*, **27**, 159-176.

Révész, P. (1968). The Laws of Large Numbers. Academic Press, NY.

Scott, D. J. (1973). Central Limit Theorems for Martingales and Processes with Stationary Increments Using a Skorokhod Representation Approach, Advances in Applied Probability, **5**, 119-137.