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In a general univariate linear model, M-estimation of a subset of parameters is considered when the complementary subset is plausibly redundant. Both the preliminary test and shrinkage versions of the usual M-estimators are considered (along with the classical versions), and, in the light of their asymptotic distributional risks, the relative asymptotic risk-efficiency results are studied in detail. Though the shrinkage M-estimators may dominate their classical versions, they do not , in general, dominate the preliminary test versions.

1. Introduction. Consider the usual linear model :

$$X_n = (X_1,\ldots,X_n)' = A_n\beta + e_n \; ; \; e_n = (e_1,\ldots,e_n)' , \qquad (1.1)$$

where $\beta = (\beta_1,\ldots,\beta_p)'$ is a vector of unknown (regression) parameters, A_n is an $n \times p$ (design) matrix of known regression constants, $n > p \geq 1$, and the errors e_i are independent and identically distributed

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(i·i·d·) random variables (r·v.) with a distribution function (d·f·) F, defined on the real line R. Without any loss of generality, we may assume that $\underset{\sim}{A}_n$ is of rank p, and consider the following partitioning (where $p = p_1 + p_2$, $p_1 \geq 0$, $p_2 \geq 0$):

$$\underset{\sim}{\beta}' = (\underset{p_1 \times 1}{\underset{\sim}{\beta}_1'}, \underset{p_2 \times 1}{\underset{\sim}{\beta}_2'}) \quad \text{and} \quad \underset{\sim}{A}_n = (\underset{n \times p_1}{\underset{\sim}{A}_{n1}}, \underset{n \times p_2}{\underset{\sim}{A}_{n2}}), \tag{1.2}$$

so that (1.1) may also be expressed as $\underset{\sim}{X}_n = \underset{\sim}{A}_{n1}\underset{\sim}{\beta}_1 + \underset{\sim}{A}_{n2}\underset{\sim}{\beta}_2 + \underset{\sim}{e}_n$. We are primarily interested in the estimation of $\underset{\sim}{\beta}_1$ when it is plausible that $\underset{\sim}{\beta}_2$ is "close to" $\underset{\sim}{0}$. This situation may arise, for example, in a <u>multi-factor design</u>, where $\underset{\sim}{\beta}_1$ stands for the <u>main-effects</u> and $\underset{\sim}{\beta}_2$ for the <u>interactions</u>; it may be quite likely (though can not be taken for granted) that all the interactions are insignificant and one may then be mainly interested in the estimation of the main-effects. Other examples of this type abound in linear models. Also, instead of the null <u>pivot</u> for β, if we have any other specified $\overset{\circ}{\underset{\sim}{\beta}}_2$, then working with $\overset{\circ}{\underset{\sim}{X}}_n = \underset{\sim}{X}_n - \underset{\sim}{A}_{n2}\overset{\circ}{\underset{\sim}{\beta}}_2$, we may reduce the pivot to $\underset{\sim}{0}$.

Instead of the classical <u>least squares estimators</u> (LSE) (optimal for normal F) or the <u>maximum likelihood estimators</u> (MLE) (based on some assumed form of F), we shall be more interested in some general robust estimators, namely, the <u>M-estimators</u> (which contain both the LSE and MLE as special cases). For the global (unrestrained) model in (1.2), we denote an M-estimator of $\underset{\sim}{\beta}$ by $\tilde{\underset{\sim}{\beta}}_n = (\tilde{\underset{\sim}{\beta}}_{1n}', \tilde{\underset{\sim}{\beta}}_{2n}')'$ so that $\tilde{\underset{\sim}{\beta}}_{1n}$ is an <u>unrestrained M-estimator</u> (<u>UME</u>) of $\underset{\sim}{\beta}_1$. For various properties of $\tilde{\underset{\sim}{\beta}}_{1n}$, we may refer to Jurečková (1977), Yohai and Maronna (1979) and Singer and Sen (1985), among others. Secondly, for the restrained model: $\underset{\sim}{X}_n = \underset{\sim}{A}_{n1}\underset{\sim}{\beta}_1 + \underset{\sim}{e}_n$ (i.e., $\underset{\sim}{\beta}_2 = \underset{\sim}{0}$), let $\hat{\underset{\sim}{\beta}}_{1n}$

be the corresponding M-estimator of β_1; $\hat{\beta}_{1n}$ is termed a restrained M-estimator (RME) of β_1. This RME generally performs better than the UME when β_2 is 0 (or very close to 0). But, for β_2 away from the pivot 0, the RME may be considerably biased, inefficient, and, even, possibly, inconsistent, while the UME retains its performance characteristics steadily over the variation of β_2. For this reason, often, to incorporate the rather uncertain prior information on β_2 in the estimation of β_1, a suitable (M-) test statistic (for testing $H_o:\beta_2=0$) is taken into consideration. In a preliminary test M-estimation (PTME) formulation, the PTME $\hat{\beta}_{1n}^{PT}$ is chosen as the RME or UME, according as this preliminary test leads to the acceptance or rejection of H_o. The shrinkage M-estimator (SME), based on the usual James-Stein (1961) rule, incorporates the same test statistic in a more smoother manner. When β_2 is very "close to" 0, generally, both the PTME and SME perform better than the UME, but the RME may still be better than either of them. On the other hand, for β_2 away from 0, the RME may perform rather poorly, while both the PTME and SME are robust. This relative picture on the performance characteristics of all the four versions of M-estimators can best be studied in an asymptotic set up similar to that in Sen (1984) or Sen and Saleh (1985). Shrinkage M- estimation of the multivariate location has also been studied in the same vein by Saleh and Sen (1985). The object of the present study is to focus mainly on the linear models. In passing, we may remark that for the particular case of $p_1=0$, i.e., $p_2=p$, we have the classical shrinkage model, while for $p_1 \geq 1$, we have a partial shrinkage model, not treated in this generality in other places.

The proposed PTME and SME, along with the preliminary notions, are presented in Section 2. The notion of asymptotic distributional risk (ADR) is considered in Section 3, and, in this light, the ADR results for the various versions of the M-estimators are considered in the same section. The main results on the asymptotic risk-efficiency (ARE) of the different versions of M-estimators are presented in Section 4. The concluding section deals with some general discussions (including the asymptotic (distributional) minimax character of these estimators).

2. The Proposed PTME and SME. First, we introduce the score function $\psi = \{\psi(x),\ x \in R\}$ needed for the definition of M-estimators. We assume that

$$(2.1) \qquad \psi(x) = \psi_1(x) + \psi_2(x), \quad x \in R,$$

where ψ_1 and ψ_2 are both nondecreasing and skew-symmetric (i.e., $\psi_j(-x) = -\psi_j(x)$, $\forall x$, $j=1,2$); ψ_1 is absolutely continuous on any bounded interval in R and ψ_2 is a step-function having only finitely many jumps. Also, we assume that there exists a positive and finite constant k, such that $\psi(x) = \psi(k)\operatorname{sign} x$, for $|x| \geq k$, and ψ is nonconstant on $[-k,k]$, so that

$$(2.2) \qquad 0 < \sigma_\psi^2 = \int_R \psi^2(x)\,dF(x) < \infty.$$

Let then $\underset{\sim}{A}_n' = (\underset{\sim}{a}_1', \ldots, \underset{\sim}{a}_n')$, and for every $n(\geq 1)$ and $\underset{\sim}{b} \in R^p$, define

$$(2.3) \qquad \underset{\sim}{M}_n(\underset{\sim}{b}) = (M_{n1}(\underset{\sim}{b}), \ldots, M_{np}(\underset{\sim}{b}))'$$
$$= \Sigma_{i=1}^n \underset{\sim}{a}_i' \,\psi\,(x_i - \underset{\sim}{b}'\underset{\sim}{a}_i'), \quad \underset{\sim}{b} \in R^p.$$

Also, we assume that the d.f. F (of the e_i) is symmetric about 0, so that

$$(2.4) \qquad \bar{\psi} = \int_R \psi(x)\,dF(x) = 0$$

Further, we let

$$(2.5) \qquad \underset{\sim}{C}_n = \underset{\sim}{A}_n'\underset{\sim}{A}_n = \begin{pmatrix} \underset{\sim}{A}_{n1}'\underset{\sim}{A}_{n1} & \underset{\sim}{A}_{n1}'\underset{\sim}{A}_{n2} \\ \underset{\sim}{A}_{n2}'\underset{\sim}{A}_{n1} & \underset{\sim}{A}_{n2}'\underset{\sim}{A}_{n2} \end{pmatrix} = \begin{pmatrix} \underset{\sim}{C}_{n11}, & \underset{\sim}{C}_{n12} \\ \underset{\sim}{C}_{n21}, & \underset{\sim}{C}_{n22} \end{pmatrix}$$

and assume that as n increases,

$$(2.6) \qquad n^{-1}\underset{\sim}{C}_n \longrightarrow \underset{\sim}{C} = \begin{pmatrix} \underset{\sim}{C}_{11} & \underset{\sim}{C}_{12} \\ \underset{\sim}{C}_{21} & \underset{\sim}{C}_{22} \end{pmatrix} \qquad \text{(positive definite)},$$

$$(2.7) \qquad n^{-1} \Sigma_{i=1}^{n} (\underset{\sim}{a}_i\underset{\sim}{a}_i')^2 = O(1)$$

Note that (2.6) and (2.7) ensure that

$$(2.8) \qquad \max_{1\le i\le n} \{ \underset{\sim}{a}_i\underset{\sim}{C}_n^{-1}\underset{\sim}{a}_i' \} = O(n^{-\frac{1}{2}}) = o(1), \text{ as } n \longrightarrow \infty.$$

Now, the UME $\tilde{\underset{\sim}{\beta}}_n = (\tilde{\underset{\sim}{\beta}}_{1n}', \tilde{\underset{\sim}{\beta}}_{2n}')'$ of $\underset{\sim}{\beta}$ is a solution to

$$(2.9) \qquad \underset{\sim}{M}_n(\underset{\sim}{b}) = \underset{\sim}{0}.$$

We also write $\underset{\sim}{M}_n(\underset{\sim}{b}) = (\underset{\sim}{M}_{n(1)}'(\underset{\sim}{b}_1,\underset{\sim}{b}_2), \underset{\sim}{M}_{n(2)}'(\underset{\sim}{b}_1,\underset{\sim}{b}_2))'$ where for the $\underset{\sim}{M}_n$ and $\underset{\sim}{b}$, we use the same partitioning as in (1.2). Then the RME $\hat{\underset{\sim}{\beta}}_{1n}$ of $\underset{\sim}{\beta}_1$ is a solution of

$$(2.10) \qquad \underset{\sim}{M}_{n(1)}(\underset{\sim}{b}_1,\underset{\sim}{0}) = \underset{\sim}{0}.$$

For the PTME and SME, we need to introduce a suitable (M-) test statistic for testing the null hypothesis $H_o:\underset{\sim}{\beta}_2 = \underset{\sim}{0}$. Towards this, we proceed as in Sen (1982) and Singer and Sen (1985), and let

$$(2.11) \qquad \hat{\underset{\sim}{M}}_{n(2)} = \underset{\sim}{M}_{n(2)}(\hat{\underset{\sim}{\beta}}_{1n},\underset{\sim}{0}),$$

where $\hat{\underset{\sim}{\beta}}_{1n}$, the RME of $\underset{\sim}{\beta}_1$, is defined by (2.10). Also, let

$$(2.12) \qquad S_n^2 = n^{-1} \Sigma_{i=1}^{n} \psi^2 (\underset{\sim}{X}_i - \underset{\sim}{a}_{i(1)}\hat{\underset{\sim}{\beta}}_{1n}); \quad \underset{\sim}{a}_i = (\underset{\sim}{a}_{i(1)},\underset{\sim}{a}_{i(2)}),$$

$$(2.13) \qquad \underset{\sim}{C}_{n22\cdot 1} = \underset{\sim}{C}_{n22} - \underset{\sim}{C}_{n21}\underset{\sim}{C}_{n11}^{-1}\underset{\sim}{C}_{n12}.$$

Then, an appropriate (aligned M-) test statistic is

$$(2.14) \qquad \mathcal{L}_n = S_n^{-2} \{\hat{\underset{\sim}{M}}_{n(2)}'\underset{\sim}{C}_{n22\ 1}\underset{\sim}{M}_{n(2)}\}.$$

Under H_o, $\mathcal{L}_n$ has asymptotically chi-square d·f· with p_2 degrees of freedom (DF). Thus, corresponding to a prescribed level of significance α: $0<\alpha<1$, the preliminary test for H_o may be based on the following:

$$(2.15) \qquad \mathcal{L}_n \begin{cases} \geq \chi^2_{p_2,\alpha}, \text{reject } H_o, \\ < \chi^2_{p_2,\alpha} \text{ accept } H_o, \end{cases}$$

where $\chi^2_{p_2,\alpha}$ is the upper $100\alpha\%$ point of the chi square d.f. with p_2DF. The PTME is then defined by

$$(2.16) \qquad \hat{\beta}^{PT}_{1n} = \tilde{\beta}_{1n} \, I(\mathcal{L}_n > \chi^2_{p_2,\alpha}) + \hat{\beta}_{1n} \, I(\mathcal{L}_n \leq \chi^2_{p_2,\alpha}),$$

where I(A) stands for the indicator function of the set A. Note that for defining the PTME, it suffices to assume that $p_2 \geq 1$.

For $p_2 \geq 3$ and C_{n12} non-null, we may consider the SMLE as follows. First, proceeding as in Singer and Sen (1985), we obtain that for large n,

$$(2.17) \qquad n^{\frac{1}{2}} (\tilde{\beta}_{1n} - \beta_1) \sim \mathcal{N}_{p_1}(0, \sigma^2_\psi \gamma^{-2} C^{-1}_{11\cdot 2}),$$

where $C_{11\cdot 2} = C_{11} - C_{12}C^{-1}_{22}C_{21}$ and

$$(2.18) \qquad \gamma = \int_R \psi(x)\{-f'(x)/f(x)\}dF(x),$$

and it assumed that the d.f. F has an absolutely continuous density function f with a finite Fisher information $I(f) = \int_R (f'(x)/f(x))^2 \cdot dF(x)$. Let then W be a given positive definite (p.d.) matrix (which we adopt in the definition of the risk, later on), and let

$$(2.19) \qquad d_n = ch_{p_1}(nWC^{-1}_{n11.2}) = \text{smallest characteristic root of } nWC^{-1}_{n11.2}\,.$$

Also, let $c : 0 < c < 2(p_2 - 2)$, $_2 \geq 3$, be a positive <u>shrinkage factor</u>.

Define then

$$(2.20) \qquad \hat{\beta}^S_{1n} = \hat{\beta}_{1n} + (I_{p_1} - cd_n W^{-1} C_{n11\cdot 2} n^{-1} \mathcal{L}_n^{-1})(\tilde{\beta}_{1n} - \hat{\beta}_{1n}).$$

Note that the Mahalanobis distance of $\tilde{\beta}_{1n}$ from β_1 is

$$(2.21) \qquad L(\tilde{\beta}_{1n}, \beta_1) = \{(\tilde{\beta}_{1n} - \beta_1)' C_{n11\cdot 2} (\tilde{\beta}_{1n} - \beta_1)\}\gamma^2/\sigma_\psi^2.$$

With this interpretation of the loss function, it may be quite natural to choose $\underset{\sim}{W} = n^{-1}\underset{\sim}{C}_{n11\cdot 2}$ $(\sim C_{11\cdot 2})$, in which case, by (2.19), $d_n = 1$, and hence, (2.20) reduces to

$$(2.22) \qquad \hat{\underset{\sim}{\beta}}{}^S_{1n} = \hat{\underset{\sim}{\beta}}_{1n} + (\underset{\sim}{I}_{p_1} - c\mathcal{L}_n^{-1})(\tilde{\underset{\sim}{\beta}}_{1n} - \hat{\underset{\sim}{\beta}}_{1n}).$$

In the sequel, we shall mainly use the SMLE in (2.22), though in the last section we shall comment on the general case in (2.20). Note that in the PTME, the indicator functions are 0-1 valued r.v., while in (2.20) or (2.22), we have a smoother version for the SME.

We may note that the test for H_o based on $\mathcal{L}_n$ is consistent for any (fixed) $\underset{\sim}{\beta}_2 \neq \underset{\sim}{0}$, so that both the PTME and SME would be asymptotically equivalent to the UME $\tilde{\underset{\sim}{\beta}}_{1n}$. Hence, to avoid this asymptotic degeneracy, we consider the case when $\underset{\sim}{\beta}_2$ is "close to" $\underset{\sim}{0}$ and where the different versions of the M-estimators have non-equivalent performance characteristics.

3. ADR of PTME and SME. In the classical normal theory model, with a loss function defined as in (2.21), the risk is computed as the expected loss. In our case, to retain the simplicity of the assumed regularity conditions, we shall compute the risk by reference to the asymptotic distribution, and term the same as the <u>asymptotic distributional risk</u> (ADR). Under additional regularity conditions, ensuring the existence of the negative moments of $\mathcal{L}_n$, the <u>asymptotic risk</u> may also be computed, and these two would generally yield comparable results. As such, we shall mainly confine ourselves to the study of the ADR properties of all the versions of M-estimators, and comment on their asymptotic dominance in the light of the ADR too.

To avoid the limiting degeneracy, we consider a shrinking neighborhood of the pivot ($\underset{\sim}{0}$) while studying these ADR results. Specifically, we consider the sequence $\{K_n\}$ of alternatives, where

(3.1) $\quad K_n : \beta_2 = \beta_{2(n)} = n^{-\frac{1}{2}}\xi, \; \xi = (\xi_{p_1+1}, \ldots, \xi_p)' \varepsilon R^{p_2},$

so that the null hypothesis H_o reduces to $H_o: \xi = 0$.

For a suitable estimator β^*_{1n} of β_1, we denote by

(3.2) $\quad G^*(x) = \lim_{n\to\infty} P\{n^{\frac{1}{2}}(\tilde{\beta}^*_{1n} - \beta_1) \le x \mid K_n\}, \; x \varepsilon R^{p_1},$

where we assume that G^* is non-degenerate. Then, with a loss function $n(\beta^*_{1n} - \beta_1)' W (\beta^*_{1n} - \beta_1)$, for a suitable W, the ADR of β^*_{1n} is given by

(3.3) $\quad R(\beta_1^*; W) = \mathrm{Tr}\{W \int \cdots \int_{R^p} xx' dG^*(x)\}$

$\quad\quad = \mathrm{Tr}\{WV^*\}$, say,

where V^* is the dispersion matrix for the asymptotic distribution G^*.

Now, by virtue of (2.17) and (3.2)-(3.3), for the UME, we have

(3.4) $\quad R(\tilde{\beta}_1; W) = (\sigma_\psi^2 \gamma^{-2}) \; \mathrm{Tr}(WC^{-1}_{11\cdot 2}).$

For the RME, we may use the linearity results of Jurečková (1977) along with those of Jurečková and Sen (1984) and Singer and Sen (1985), and claim that under $\{K_n\}$,

(3.5) $\quad n^{\frac{1}{2}}(\hat{\beta}_{1n} - \beta_1) \sim \mathcal{N}_{p_1}(C^{-1}_{11} C_{12}\xi, \sigma^2_\psi \gamma^{-2} C^{-1}_{11}),$

so that the ADR of the RME is equal to

(3.6) $\quad R(\hat{\beta}_1; W) = (\sigma_\psi^2 \gamma^{-2}) \; \mathrm{Tr}(WC^{-1}_{11}) + \xi' M \xi \;;$

(3.7) $\quad M = C_{21} C^{-1}_{11} W C^{-1}_{11} C_{12}.$

We may further note that by virtue of the same linearity results on aligned M-statistics, under $\{K_n\}$,

(3.8) $\quad \hat{\beta}_{1n} = \tilde{\beta}_{1n} + C_{11} C_{12} \tilde{\beta}_{2n} + o_p(n^{-\frac{1}{2}}),$

(3.9) $\quad \mathcal{L}_n = (\tilde{\beta}'_{2n} C_{n22\cdot 1} \tilde{\beta}_{2n}) \gamma^2 \sigma_\psi^{-2} + o_p(1),$

so that the PTME and SME may both be expressed in terms of the UME

$\underset{\sim}{\beta}_n$. Recall that under $\{K_n\}$,

(3.10) $$n^{\frac{1}{2}}(\tilde{\underset{\sim}{\beta}}{}'_{1n} - \underset{\sim}{\beta}'_1, \tilde{\underset{\sim}{\beta}}{}'_{2n} - n^{-\frac{1}{2}}\underset{\sim}{\xi}')' \sim \mathcal{N}_p(\underset{\sim}{0}, \sigma_\psi^2 \gamma^{-2} \underset{\sim}{C}^{-1}).$$

Thus, by virtue of (2.16), (3.8), (3.9) and (3.10), we obtain by some standard steps that for the PTME

(3.11) $$R(\hat{\underset{\sim}{\beta}}{}^{PT}; W) = \sigma_\psi^2\gamma^{-2}\{\mathrm{Tr}(\underset{\sim}{W}\underset{\sim}{C}_{11\cdot 2}^{-1})[1-H_{p_2+2}(\chi^2_{p_2,\alpha};\Delta] +$$

$$\mathrm{Tr}(WC_{11}^{-1})\, H_{p_2+2}(\chi^2_{p_2,\alpha};\Delta)\} +$$

$$(\underset{\sim}{\xi}'\underset{\sim}{M}\underset{\sim}{\xi})\{2H_{p_2+2}(\chi^2_{p_2,\alpha};\Delta) - H_{p_2+4}(\chi^2_{p_2,\alpha};\Delta)\},$$

where $H_q(x;\delta)$ stands for the noncentral chi square d.f. with q DF and noncentrality parameter δ, and

(3.12) $$\Delta = (\underset{\sim}{\xi}'\underset{\sim}{C}_{22\cdot 1}\underset{\sim}{\xi})\,\gamma^2/\sigma_\psi^2.$$

Now, by virtue of (2.22), (3.8) and (3.9), we obtain that under $\{K_n\}$,

(3.13) $$n^{\frac{1}{2}}(\hat{\underset{\sim}{\beta}}_{1n} - \underset{\sim}{\beta}_1) \xrightarrow{\mathcal{D}} \underset{\sim}{D}_1\underset{\sim}{U} + \frac{c\,\sigma_\psi^2\; \underset{\sim}{C}_{11}^{-1}\underset{\sim}{C}_{12}\,(\underset{\sim}{D}_2\underset{\sim}{U} + \underset{\sim}{\xi})}{\gamma^2\,\{(\underset{\sim}{D}_2\underset{\sim}{U} + \underset{\sim}{\xi})^1\,\underset{\sim}{C}_{22\cdot 1}\,(\underset{\sim}{D}_2\underset{\sim}{U} + \underset{\sim}{\xi})\}},$$

where

(3.14) $$\underset{\sim}{D} = \begin{pmatrix}\underset{\sim}{D}_1\\ \underset{\sim}{D}_2\end{pmatrix} = \underset{\sim}{C}^{-1}, \quad \underset{\sim}{U} \sim \mathcal{N}_p(\underset{\sim}{0}, \sigma^2\gamma^{-2}\underset{\sim}{C}).$$

Therefore, we have

(3.15) $$R(\hat{\underset{\sim}{\beta}}{}_1^S; \underset{\sim}{W}) = \sigma_\psi^2\gamma^{-2}\mathrm{Tr}(WD_1CD'_1) +$$

$$2\sigma_\psi^2\gamma^{-2}cE\left\{\frac{(\underset{\sim}{D}_2\underset{\sim}{U} + \underset{\sim}{\xi})'\underset{\sim}{C}_{21}\underset{\sim}{C}_{11}\underset{\sim}{W}\underset{\sim}{D}_1\underset{\sim}{U}}{(\underset{\sim}{D}_2\underset{\sim}{U} + \underset{\sim}{\xi})'\underset{\sim}{C}_{22\cdot 1}(\underset{\sim}{D}_2\underset{\sim}{U} + \underset{\sim}{\xi})}\right\} +$$

$$c^2\sigma_\psi^4\gamma^{-4}E\left\{\frac{(\underset{\sim}{D}_2\underset{\sim}{U} + \underset{\sim}{\xi})'\underset{\sim}{C}_{21}\underset{\sim}{C}^{-1}\underset{\sim}{W}\underset{\sim}{C}_{11}^{-1}\underset{\sim}{C}_{12}(\underset{\sim}{D}_2\underset{\sim}{U} + \underset{\sim}{\xi})}{(\underset{\sim}{D}_2\underset{\sim}{U} + \underset{\sim}{\xi})'\underset{\sim}{C}_{22\cdot 1}(\underset{\sim}{D}_2\underset{\sim}{U} + \underset{\sim}{\xi})}\right\}.$$

Then, we note that

$$\text{(3.16)} \qquad \mathrm{Tr}(\underset{\sim}{W}\underset{\sim}{D}_1\underset{\sim}{C}\underset{\sim}{D}_1') = \mathrm{Tr}(\underset{\sim}{W}\underset{\sim}{C}_{11\cdot 2}^{-1}).$$

Also, by (3.7), (3.14) and the Stein identity [viz., Appendix B of Judge and Bock (1978)], the last term on the right hand side of (3.15) is equal to

$$\text{(3.17)} \qquad \sigma_\psi^2\, \gamma^{-2} c^2 \mathrm{Tr}(\underset{\sim}{M}\underset{\sim}{C}_{22\cdot 1}^{-1})\, E(\chi_{p_2+2}^{-4}(\Delta)) + c^2(\underset{\sim}{\xi}'\underset{\sim}{M}\underset{\sim}{\xi})E(\chi_{p_2+4}^{-4}(\Delta)),$$

while the second term reduces to

$$\text{(3.18)} \qquad -2\gamma^{-2}\sigma_\psi^2 c\mathrm{Tr}(\underset{\sim}{M}\underset{\sim}{C}_{22\cdot 1}^{-1})E(\chi_{p_2+2}^{-2}(\Delta) + 4c(\underset{\sim}{\xi}'\underset{\sim}{M}\underset{\sim}{\xi})E(\chi_{p_2+4}^{-4}(\Delta)).$$

Thus, we have

$$\text{(3.19)} \qquad R(\hat{\underset{\sim}{\beta}}{}_1^S;\underset{\sim}{W}) = \sigma_\psi^2\gamma^{-2}\{\mathrm{Tr}(\underset{\sim}{W}\underset{\sim}{C}_{11\cdot 2}^{-1}) - c\mathrm{Tr}(\underset{\sim}{M}C_{22\cdot 1}^{-1})[2E(\chi_{p_2+2}^{-2}(\Delta) - c\,E(\chi_{p_2+2}^{-4}(\Delta))]\} + c(c+4)\ (\underset{\sim}{\xi}'\underset{\sim}{M}\underset{\sim}{\xi})E(\chi_{p_2+4}^{-4}(\Delta)).$$

In this context, it may be recalled that

$$\text{(3.20)} \qquad (\xi'M\xi)/\Delta = \sigma_\psi^2\gamma^{-2}\ \{(\underset{\sim}{\xi}'\underset{\sim}{M}\underset{\sim}{\xi})/(\underset{\sim}{\xi}'\underset{\sim}{C}_{22\cdot 1}\underset{\sim}{\xi})\}$$
$$\leq \sigma_\psi^2\gamma^{-2} ch_1(\underset{\sim}{M}\underset{\sim}{C}_{22\cdot 1}^{-1})\ (\leq \sigma_\psi^2\gamma^{-2}\mathrm{Tr}(\underset{\sim}{M}\underset{\sim}{C}_{22\cdot 1}^{-1})).$$

Also, note that for the general SME in (2.20), if we let $d = ch_p(\underset{\sim}{W}\underset{\sim}{C}_{11\cdot 2}^{-1})$, then the second term on the right hand side of (3.13) will be

$$\text{(3.21)} \qquad (\gamma^{-2}cd\,\sigma_\psi^2)\{\underset{\sim}{W}^{-1}\underset{\sim}{C}_{11\cdot 2}\underset{\sim}{C}_{11}\underset{\sim}{C}_{12}(\underset{\sim}{D}_2\underset{\sim}{U}+\underset{\sim}{\xi})/(\underset{\sim}{D}_2\underset{\sim}{U}+\underset{\sim}{\xi})'\underset{\sim}{C}_{22\cdot 1}(\underset{\sim}{D}_2\underset{\sim}{U}+\underset{\sim}{\xi})\},$$

so that for the ADR, we have an expression parallel to (3.19) where $\mathrm{Tr}(\underset{\sim}{M}\underset{\sim}{C}_{22\cdot 1}^{-1})$ and $(\underset{\sim}{\xi}'\underset{\sim}{M}\underset{\sim}{\xi})$ are to be replaced by $\mathrm{Tr}(\underset{\sim}{M}^o\underset{\sim}{C}_{22\cdot 1}^{-1})$ and $(\underset{\sim}{\xi}'\underset{\sim}{M}^o\underset{\sim}{\xi})$, respectively, and where

$$\text{(3.22)} \qquad \underset{\sim}{M}^o = \underset{\sim}{C}_{21}\underset{\sim}{C}_{11}^{-1}\underset{\sim}{C}_{11}\ {}_2\underset{\sim}{W}^{-1}\underset{\sim}{C}_{11\cdot 2}\underset{\sim}{C}_{11}^{-1}\underset{\sim}{C}_{12}.$$

In the light of the ADR, the asymptotic distributional risk-efficiency (ADRE) results are considered in the next section.

4. ADRE results. Note that for $\underset{\sim}{C}_{12} = \underset{\sim}{O}$, (3.4), (3.6), (3.11) and (3.19) all have the common value $\sigma_\psi^2\gamma^{-2}\mathrm{Tr}(\underset{\sim}{W}\underset{\sim}{C}_{11}^{-1})$ $(\forall\underset{\sim}{\xi})$, and hence, these are all ADR equivalent. Hence, in the sequel, it will be assumed that $\underset{\sim}{C}_{12} \neq \underset{\sim}{O}$. Note that $\underset{\sim}{C}_{11\cdot 2}^{-1} - \underset{\sim}{C}_{11}^{-1}$ is then p.s.d. and hence, by (3.4) and (3.6),

$$(4.1) \qquad R(\tilde{\underset{\sim}{\beta}}_1;\underset{\sim}{W}) - R(\hat{\underset{\sim}{\beta}}_1;\underset{\sim}{W}) = (\gamma^{-2}\sigma_\psi^2)\mathrm{Tr}(\underset{\sim}{W}(\underset{\sim}{C}_{11\cdot 2}^{-1} - \underset{\sim}{C}_{11}^{-1})) - \underset{\sim}{\xi}'\underset{\sim}{M}\underset{\sim}{\xi},$$

and hence,

$$(4.2) \qquad R(\tilde{\underset{\sim}{\beta}}_1;\underset{\sim}{W}) \gtreqless R(\tilde{\underset{\sim}{\beta}}_1;\underset{\sim}{W}) \text{ according as}$$

$$(\underset{\sim}{\xi}'\underset{\sim}{M}\underset{\sim}{\xi}) \lesseqgtr (\gamma^{-2}\sigma_\psi^2)\mathrm{Tr}(\underset{\sim}{W}(\underset{\sim}{C}_{11\cdot 2}^{-1} - \underset{\sim}{C}_{11}^{-1})).$$

In particular, under H_o, $\underset{\sim}{\xi}=\underset{\sim}{O}$, so that $R(\tilde{\underset{\sim}{\beta}}_1;\underset{\sim}{W}) > R(\hat{\underset{\sim}{\beta}}_1;\underset{\sim}{W})$, while as $\underset{\sim}{\xi}$ moves away from $\underset{\sim}{O}$ i.e., $\underset{\sim}{\xi}'\underset{\sim}{M}\underset{\sim}{\xi} \to +\infty$, $R(\hat{\underset{\sim}{\beta}}_1;\underset{\sim}{W}) \to +\infty$, but $R(\tilde{\underset{\sim}{\beta}}_1;\underset{\sim}{W})$ remains the same. Thus, excepting in a neighborhood of $\underset{\sim}{\xi}= \underset{\sim}{O}$, the RME has generally higher ADR than the UME.

Since the ADR of the UME is fixed, we may choose $\underset{\sim}{W} = \gamma^2\sigma_\psi^{-2}C_{11\cdot 2}$, which will lead to $R(\tilde{\underset{\sim}{\beta}}_1;W) = p_1$. We study the ADR properties of the PTME and SME under this setup. The general case of $\underset{\sim}{W}$ will be treated briefly in the next section. By (3.11) and the above choice of $\underset{\sim}{W}$, we have

$$(4.3) \qquad R(\hat{\underset{\sim}{\beta}}_1^{PT};\underset{\sim}{W}) = p_1 - \mathrm{Tr}(\underset{\sim}{C}_{12}\underset{\sim}{C}_{22}^{-1}\underset{\sim}{C}_{21}\underset{\sim}{C}_{11}^{-1})\, H_{p_2+2}(\chi^2_{p_2,\alpha};\Delta) +$$

$$(\underset{\sim}{\xi}'\underset{\sim}{C}_{21}\underset{\sim}{C}_{11}^{-1}\underset{\sim}{C}_{11\cdot 2}\underset{\sim}{C}_{11}^{-1}\underset{\sim}{C}_{12}\underset{\sim}{\xi})(\gamma^2/\sigma_\psi^2)\{2H_{p_2+2}(\chi^2_{p_2,\alpha};\Delta) - H_{p_2+4}(\chi^2_{p_2,\alpha};\Delta)\}.$$

Therefore, we have

$$(4.4) \qquad R(\tilde{\underset{\sim}{\beta}}_1;\underset{\sim}{W}) = R(\hat{\underset{\sim}{\beta}}_1^{PT};\underset{\sim}{W}) \text{ according as}$$

$$(\underset{\sim}{\xi}'\underset{\sim}{C}_{21}\underset{\sim}{C}_{11}^{-1}\underset{\sim}{C}_{11\cdot 2}\underset{\sim}{C}_{11}^{-1}\underset{\sim}{C}_{12}\underset{\sim}{\xi}) \lesseqgtr \frac{\sigma_\psi^2}{\gamma^2} \cdot \frac{\mathrm{Tr}(C_{12}C_{22}^{-1}C_{21}C_{11}^{-1})H_{p_2+2}(\chi^2_{p_2,\alpha};\Delta)}{2H_{p_2+2}(\chi^2_{p_2,\alpha};\Delta) - H_{p_2+4}(\chi^2_{p_2,\alpha};\Delta)}.$$

Thus, in a neighborhood of $\mathbf{\xi}=\mathbf{0}$, the PTME has a smaller (larger) ADR than the UME (RME), and this neighborhood is contained in the neighborhood in which the RME has a smaller ADR than the UME. However, both the second and third terms on the right hand side of (4.3) are bounded functions, each converging to 0 as $\mathbf{\xi}$ moves away from $\mathbf{0}$ (i.e., $\Delta\to\infty$). Thus, unlike the case of the RME, the ADR of the PTME does not blow up as $\Delta\to\infty$); rather, this ADR has the asymptote p_1, although the maximum ADR of the PTME is generally (slightly) larger than p_1. This later feature deprives the PTME from having the asymptotic minimax character (in the light of the ADR).

For the SME, we consider the case of $\mathbf{W} = \gamma^2 \sigma_\psi^{-2} \mathbf{C}_{11\cdot 2}$, so that by (3.19), $R(\hat{\mathbf{\beta}}_1^S;\mathbf{W}) = p_1 - c\mathrm{Tr}(\mathbf{C}_{21}\mathbf{C}_{11}^{-1}\mathbf{C}_{11\cdot 2}\mathbf{C}_{11}^{-1}\mathbf{C}_{12}\mathbf{C}_{22\cdot 1}^{-1})[2E(\chi^{-2}_{p_2+2}(\Delta)) -$

(4.5) $$cE(\chi^{-4}_{p_2+2}(\Delta))] + c(c+4)(\mathbf{\xi}'\mathbf{C}_{21}\mathbf{C}_{11}^{-1}\mathbf{C}_{11\cdot 2}\mathbf{C}_{11}^{-1}\mathbf{C}_{12}\mathbf{\xi})\cdot$$
$$\gamma^2 \sigma_\psi^{-2} E(\chi^{-4}_{p_2+4}(\Delta)).$$

In this content, letting $\mathbf{M}^* = \mathbf{C}_{21}\mathbf{C}_{11}^{-1}\mathbf{C}_{11\cdot 2}\mathbf{C}_{11}^{-1}\mathbf{C}_{12}\mathbf{C}_{22\cdot 1}^{-1}$, we have

(4.6) $$(\mathbf{\xi}'\mathbf{C}_{21}\mathbf{C}_{11}\mathbf{C}_{11\cdot 2}\mathbf{C}_{11}\mathbf{C}_{12}\mathbf{\xi})\gamma^2 \sigma_\psi^{-2}/\Delta$$
$$= (\mathbf{\xi}'\mathbf{M}^*\mathbf{C}_{11\cdot 2}\mathbf{\xi})/(\mathbf{\xi}'\mathbf{C}_{11\cdot 2}\mathbf{\xi})$$
$$\leq ch_1(\mathbf{M}^*) = h\mathrm{Tr}(\mathbf{M}^*), \text{ say,}$$

where

(4.7) $$0 \leq h = ch_1(\mathbf{M}^*)/\mathrm{Tr}(\mathbf{M}^*) \quad (\leq 1)$$

Thus, in order that $R(\hat{\mathbf{\beta}}^S{}_1;\mathbf{W}) \leq R(\tilde{\mathbf{\beta}}_1;\mathbf{W}), \forall \mathbf{\xi}$, a sufficient condition is that

(4.8) $$2E(\chi^{-2}_{p_2+2}(\Delta)) - cE(\chi^{-4}_{p_2+2}(\Delta) - (c+4)h\Delta E(\chi^{-4}_{p_2+4}(\Delta)) \geq 0, \ \forall \Delta \geq 0.$$

Since, we have the identity that

(4.9) $$E(\chi^{-2}_{p_2+2}(\Delta) - (p_2-2)E(\chi^{-4}_{p_2+2}(\Delta)) = \Delta E(\chi^{-4}_{p_2+4}(\Delta)),$$

it suffices to show that

(4.10) $$(2-h(c+4))E[\chi^{-2}_{p_2+2}(\Delta)) - (p_2-2\chi^{-4}_{p_2+2}(\Delta)] + (2(p_2-2)-c)E(\chi^{-4}_{p_2+2}(\Delta) \geq 0, \ \forall\Delta \geq 0.$$

In particular, if we choose (in(2.22) $c=p_2-2$, it follows that under H_0 ($\Delta=0$), (4.8) is equal to $1/p_2$, and (4.10) is positive for all $\Delta\geq 0$ whenever $h\leq 2/(p_2+2)$. On the other hand, if $h > 2/(p_2+2)$, (4.10) can still be made positive by choosing c smaller than p_2-2. For the classical multinormal mean problem, $h=1/p$ (So that $h\leq 2/(2+p_2)$), and hence, (4.10) is positive for every $0<c<2(p-2)$. However, in the model under consideration, the choice of c in (2.22) is dependent on h in (4.7), and smaller is the value of h, the larger is the range of c for which the SME dominates the UME (in the light of their ADR). In the same vein, we now compare the PTME and SME. First consider the case where $\underset{\sim}{\xi}=\underset{\sim}{0}$. Then, by (4.3) and (4.5), we have under $H_0:\underset{\sim}{\xi}=\underset{\sim}{0}$ and $\underset{\sim}{W}=\gamma^2\sigma_\psi^{-2}\underset{\sim}{C}_{11\cdot 2}$,

(4.11) $$R(\hat{\underset{\sim}{\beta}}{}^S_1;\underset{\sim}{W}) - R(\hat{\underset{\sim}{\beta}}{}^{PT}_1;\underset{\sim}{W})=\mathrm{Tr}(\underset{\sim}{C}_{12}\underset{\sim}{C}^{-1}_{22}\underset{\sim}{C}_{21}\underset{\sim}{C}^{-1}_{11})H_{p_2+2}(\chi^2_{p_2,\alpha};0) - c\mathrm{Tr}(\underset{\sim}{M}^*)[2p_2^{-1} - c/(p_2(p_2-2))],$$

where $\mathrm{Tr}(\underset{\sim}{C}_{12}\underset{\sim}{C}^{-1}_{22}\underset{\sim}{C}_{21}\underset{\sim}{C}^{-1}_{11}) = \mathrm{Tr}((\underset{\sim}{C}_{11}-\underset{\sim}{C}_{11\cdot 2})\underset{\sim}{C}^{-1}_{11}) = \mathrm{Tr}\ (\underset{\sim}{I}-\underset{\sim}{C}_{11\cdot 2}\underset{\sim}{C}^{-1}_{11})$ and

$\mathrm{Tr}(\underset{\sim}{M}^*) = \mathrm{Tr}(\underset{\sim}{C}_{21}\underset{\sim}{C}^{-1}_{11}(\underset{\sim}{C}_{11}-\underset{\sim}{C}_{12}\underset{\sim}{C}^{-1}_{22}\underset{\sim}{C}_{21})\ \underset{\sim}{C}^{-1}_{11}\underset{\sim}{C}_{12}\underset{\sim}{C}^{-1}_{22\cdot 1}) = \mathrm{Tr}(\underset{\sim}{C}_{21}\underset{\sim}{C}^{-1}_{11}\underset{\sim}{C}_{12}\underset{\sim}{C}^{-1}_{22\cdot 1}) - \mathrm{Tr}(\underset{\sim}{C}_{21}\underset{\sim}{C}^{-1}_{11}\underset{\sim}{C}_{12}\underset{\sim}{C}^{-1}_{22}\underset{\sim}{C}_{21}\underset{\sim}{C}^{-1}_{11}\underset{\sim}{C}_{12}\underset{\sim}{C}^{-1}_{22\cdot 1}) = \mathrm{Tr}(\underset{\sim}{C}_{21}\underset{\sim}{C}^{-1}_{11}\underset{\sim}{C}_{12}\underset{\sim}{C}^{-1}_{22\cdot 1}) - \mathrm{Tr}(\underset{\sim}{C}_{21}\underset{\sim}{C}^{-1}_{11}\underset{\sim}{C}_{12}\underset{\sim}{C}^{-1}_{22\cdot 1}) + \mathrm{Tr}(\underset{\sim}{C}_{21}\underset{\sim}{C}^{-1}_{11}\underset{\sim}{C}_{12}\underset{\sim}{C}^{-1}_{22}) = \mathrm{Tr}(\underset{\sim}{C}_{21}\underset{\sim}{C}^{-1}_{11}\underset{\sim}{C}_{12}\underset{\sim}{C}^{-1}_{22}) = \mathrm{Tr}(\underset{\sim}{C}_{12}\underset{\sim}{C}^{-1}_{22}\underset{\sim}{C}_{21}\underset{\sim}{C}^{-1}_{11}) = \mathrm{Tr}\ (\underset{\sim}{I}-\underset{\sim}{C}_{11\cdot 2}\underset{\sim}{C}^{-1}_{11})$. Therefore, (4.11) reduces to

$$(4.12)\qquad \{\mathrm{Tr}(\underset{\sim}{I}-\underset{\sim}{C}_{11\cdot 2}\underset{\sim}{C}_{11}^{-1})\}[H_{p_2+2}(\chi^2_{p_2,\alpha};0) - cp_2^{-1}(2-c/p_2-2)].$$

$$= \{\mathrm{Tr}(\underset{\sim}{I}-\underset{\sim}{C}_{11\cdot 2}\underset{\sim}{C}_{11}^{-1})\}\{H_{p_2+2}(\chi^2_{p_2,\alpha};\Delta) - \frac{p_2-2}{p_2}\frac{c}{p_2-2}(2-\frac{c}{p_2-2})\}.$$

Now, $c(p_2-2)^{-1}(2-c(p_2-2)^{-1}$, for $0<c<2(p_2-2)$ attains the maximum value 1 at $c = p_2-2$. So that

$$(4.13)\qquad \inf_{c\,\varepsilon\,(0,2(p_2-2))}\{(4.12)\} = [\mathrm{Tr}(\underset{\sim}{I}-\underset{\sim}{C}_{11\cdot 2}\underset{\sim}{C}_{11}^{-1})][H_{p_2+2}(\chi^2_{p_2,\alpha};0) -(1-2/p_2)].$$

Consequently, when α, the level of significance, for the preliminary test is not so large, in the sense that $H_{p_2+2}(\chi^2_{p_2,\alpha};0) > 1-2/p_2$ (note that $H_{p_2}(\chi^2_{p_2,\alpha};0) = 1-\alpha > H_{p_2+2}(\chi^2_{p_2,\alpha};0)$), then (4.13) is positive, and hence, the SME does not dominate the PTME, under H_0, unless α is large, This also shows that for any fixed α $(0<\alpha<1)$, as p_2 increases, the opposite inequality holds for $H_{p_2+2}(\chi^2_{p_2,\alpha};0)$, and hence, for large p_2, the SME may dominate the PTME, even under H_0. To complete the picture, we consider the general case where $\underset{\sim}{\xi}$ need not be equal to $\underset{\sim}{0}$. There, we have by (4.3) and (4.5),

$$(4.14)\qquad R(\hat{\underset{\sim}{\beta}}_1^{S};\underset{\sim}{W}) - R(\hat{\underset{\sim}{\beta}}_1^{PT};\underset{\sim}{W}) = \{\mathrm{Tr}(\underset{\sim}{I}-\underset{\sim}{C}_{11\cdot 2}\underset{\sim}{C}_{11}^{-1})\}\{H_{p_2+2}(\chi^2_{p_2,\alpha};\Delta) -c[2E(\chi^{-2}_{p_2+2}(\Delta)) -cE(\chi^{-4}_{p_2+2}(\Delta))]\}$$

$$-(\underset{\sim}{\xi}'\underset{\sim}{C}_{21}\underset{\sim}{C}_{11}^{-1}\underset{\sim}{C}_{11\cdot 2}\underset{\sim}{C}_{11}^{-1}\underset{\sim}{C}_{12}\underset{\sim}{\xi})(\gamma^2/\sigma^2_{\psi})\{2H_{p_2+2}(\chi^2_{p_2,\alpha};\Delta) -H_{p_2+4}(\chi_{p_2,\alpha};\Delta) - c(c+4)E(\chi^{-4}_{p_2+4}(\Delta))\}.$$

By using (4.6) and (4.9), it can be shown that as $\Delta\to +\infty$, (4.14) converges to 0. However, for intermediate values of Δ, the second term in (4.14) dominates the picture, and hence, $R(\hat{\underset{\sim}{\beta}}_1^{S};\underset{\sim}{W}) < R(\hat{\underset{\sim}{\beta}}_1^{PT};\underset{\sim}{W})$. Thus, the PTME generally fails to dominate the SME (for all $\Delta \geq 0$). This shows that none of the PTME and SME dominates the other unless either α is large or p_2 is large.

5. Some general remarks. It follows from the results of Section 4 that both the PTME and SME are robust from the risk-efficiency point of view. Of the two, the SME may have generally the asymptotic minimax character (See (4.10), while the PTME generally has the maximum ADR greater than that of the UME, and hence, is not minimax. The relative-risk of the PTME and SME is relatively close to 1 in the tail where $\Delta \to +\infty$, although, SME may have smaller ADR than the PTME in the tail. In a neighborhood of $\underset{\sim}{\xi}=\underset{\sim}{0}$, generally, the PTME dominates the SME, although just outside this domain, the PTME may have an ADR larger than both the UME and SME. This suggests that when we have apriori reasons to suspect that Δ is close to 0, the PTME may be preferred to the SME; on the contrary, for larger Δ, SME is preferred. However, for the PTME, we do not need that $p_2 \geq 3$, while for the SME to have good risk-efficiency property, we need that $p_2 \geq 3$ (actually, more than that: $h > 2/(p_2+2)$ for $c=(p_2-2)$). Finally, we may remark that in Section 4, we have mainly considered the SME in (2.22) and taken $\underset{\sim}{W}=\gamma^2 \sigma_\psi^{-2} \underset{\sim}{C}_{11\cdot 2}$. For the general SME in (2.20) with an arbitrary $\underset{\sim}{W}$, we have the ADR of the SME given by (3.19) with $\underset{\sim}{M}$ replaced by $\underset{\sim}{M}^o$ in (3.22). With this modification, we can proceed as in Section 4 and draw conclusions quite similar to those in the same section. If, however, we use the SME in (3.22) but use an arbitrary W, then the asymptotic minimax character (in the light of the ADR) of (2.22) may not hold. Also, in that case, the PTME may have a better performance characteristic than the SME in (2.22). But, the use of $\underset{\sim}{W} = \gamma^2 \sigma_\psi^{-2} \underset{\sim}{C}_{11\cdot 2}$ has already been justified earlier by the use of the Mahalanobis distance, and hence, our general conclusions of Section 4 stand well.

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