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POSITIVE DEFINITE SYMMETRIC FUNCTIONS ON FINITE - DIMENSIONAL SPACES, II

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ABSTRACT

Integral representations for the density functions of absolutely continuous α-symmetric random vectors are derived, and general methods for constructing new α-symmetric distributions are presented. An explicit formula for determining the spectral measure of a symmetric stable random vector from its characteristic function, is obtained.

1. INTRODUCTION

In this paper, we discuss a circle of ideas related to positive definite (or characteristic) functions defined on a finite-dimensional space and satisfying certain symmetry relations. In part, our results are a continuation of the work of Cambanis, Keener and Simons [2] and Richards [9]. In addition, we present results for some problems on the multivariate symmetric stable laws and their associated spectral measures.

An n-dimensional real random vector X is said to have an α-symmetric distribution, $\alpha > 0$, if its characteristic function is of the form $\phi(|\xi_1|^\alpha + \ldots + |\xi_n|^\alpha)$ for some function ϕ. The class $\Phi_n(\alpha)$ of admissible functions ϕ was characterized by Schoenberg [12] in the case $\alpha = 2$, and by Cambanis et al [2] when $\alpha = 1$.

Using the Radon transform, Richards [9] obtained integral representations for the density functions of certain absolutely continuous α-symmetric vectors. The Radon transform also led to the appearance of a new class of special functions generalizing the classical spherical Bessel functions. One of our main results shows that the integral representation referred to above is valid for arbitrary absolutely continuous α-symmetric distributions. The approach used here is more direct, and requires a generalization of the standard polar coordinates decomposition of the Lebesegue measure on $\mathbf{R}^n$.

Surprisingly, no α-symmetric distributions are known for the case when $\alpha > 2$ and $n > 2$. Herz [6] shows, for example, that the function $\exp(-(|\xi_1|^\alpha + |\xi_2|^\alpha)^{1/\alpha})$ is positive definite on $\mathbb{R}^2$; however this results does not extend to three or more dimensions. We shall collect some known examples and present general methods for constructing new examples.

The second aspect of our work deals with the symmetric α-stable (SαS) distributions on $\mathbb{R}^n$. Recall that if a random vector X(in $\mathbb{R}^n$) is SαS, $0 < \alpha < 2$, then (by Lévy's theorem) the characteristic function of X is of the form $\exp(-F(x))$, where

$$F(x) = \int_{S^{n-1}} |\langle x,y\rangle|^\alpha \, d\mu(y), \quad x \in \mathbb{R}^n.$$

Here, $\langle\cdot,\cdot\rangle$ is the usual inner product on $\mathbb{R}^n$; S^{n-1} is the unit sphere; and the spectral measure μ is a finite, symmetric Borel measure on S^{n-1}. The general theory of stable laws ensures that there is a one-one correspondence between F and μ, but does not provide explicit methods for the determination of μ from F. We prove that if $F(x)$ is sufficiently smooth, then μ is absolutely continuous with respect to the surface measure on S^{n-1}. Further we express the density of μ in terms of the spherical harmonic expansion of $F(x)$, and obtain explicit results for the α-sub-Gaussian distributions.

2. α-SYMMETRIC DISTRIBUTIONS

This section extends and completes certain results of [9], and we shall use the notation established there. Thus if $\xi = (\xi_1,\ldots,\xi_n) \in \mathbb{R}^n$, then $\|\xi\|_\alpha = (|\xi_1|^\alpha + \ldots + |\xi_n|^\alpha)^{1/\alpha}$ for $\alpha > 0$, and $S_\alpha^{n-1} = \{\xi \in R^n : \|\xi\|_\alpha = 1\}$. Define the differential form

$$\omega(\xi) = \sum_{j=1}^{n} (-1)^{j-1} \xi_j \, d\xi_1 \ldots d\xi_{j-1} d\xi_{j+1} \ldots d\xi_n, \tag{2.1}$$

then

$$J_{n,\alpha}(x) = \int_{S_\alpha^{n-1}} e^{i\langle x,\xi\rangle} \omega(\xi) \tag{2.2}$$

is the generalized Bessel function introduced in [9]. When restricted to the unit sphere S_2^{n-1}, $\omega(\xi)$ is simply the Lebesgue measure; then as in [12], $J_{n,2}(x)$ may be expressed in terms of the classical spherical Bessel functions.

Under more restrictive assumptions, the following result was established in [9].

2.1. <u>Theorem</u>. Let X be an n-dimensional α-symmetric random vector with characteristic function $\phi(\|\xi\|_\alpha^\alpha)$. If X is absolutely continuous with density function f(x) then

$$f(x) = (2\pi)^{-n} \int_0^\infty t^{n-1} \phi(t^\alpha) J_{n,\alpha}(tx)dt, \quad x \in \mathbb{R}^n. \tag{2.3}$$

<u>Proof</u>. Since the hypersurface S_α^{n-1} intersects every ray from the origin exactly once, then every non-zero vector y in $\mathbb{R}^n$ may

uniquely represented by generalized polar coordinates: $y=t\xi$, where $t>0$ and $\xi \in S_\alpha^{n-1}$. Under this decomposition, the Lebesgue measure dy decomposes as

$$dy = \omega(\xi)\, t^{n-1}dt. \tag{2.4}$$

Therefore by Fourier inversion,

$$f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i<x,y>} \phi(||y||_\alpha^\alpha)dy$$

$$= (2\pi)^{-n} \int_0^\infty \int_{S_\alpha^{n-1}} e^{it<x,\xi>} \phi(t^\alpha||\xi||_\alpha^\alpha) \quad \omega(\xi)t^{n-1}dt$$

$$= (2\pi)^{-n} \int_0^\infty t^{n-1} J_{n,\alpha}(tx)\ \phi(t^\alpha)dt,$$

and this completes the proof.

The decomposition (2.4) is established by Gel'fand and Shilov [5; pp. 394-395]. Further, their results show that if in Theorem 2.1 $\phi(||\xi||_\alpha^\alpha)$ is replaced by $\phi(c(\xi))$ where $c(\xi)$ is continuous, scalar-valued and homogeneous, then (2.3) remains valid with $J_{n,\alpha}(x)$ replaced by

$$J_{n,c}(x) = \int_{c(\xi)=1} e^{i<x,\xi>}\,\omega(\xi).$$

Another consequence of (2.4) is that the differential form $\omega(\xi)$, regarded as a measure on S_α^{n-1}, is a positive measure. This result closes a problem raised in [9] and which we state as

2.2. Corollary. The measure $\omega(\xi)$ is positive on S_{α}^{n-1}. Equivalently, the function $J_{n,\alpha}(x)/J_{n,\alpha}(0)$ is positive definite on $\mathbb{R}^n$ for all $\alpha > 0$.

3. EXAMPLES OF α-SYMMETRIC DISTRIBUTIONS

Definition. A function $\psi: \mathbb{R}^n \to \mathbb{C}$ is negative definite if $\exp(-t\psi(x))$ is positive definite for all $t > 0$.

A thorough treatment of negative definite functions and their generalizations has been provided by Berg and Forst [1]. The relation between negative definite functions and α-symmetric distributions is given in the following result which is essentially the combination of results due to Herz [6] and others.

3.1 Proposition On $\mathbb{R}^2$, $||x||_\alpha^\beta$ is negative definite if (i) $0 < \beta \leq 1$, $1 \leq \alpha \leq \infty$, or (ii) $0 < \beta \leq \alpha \leq 2$.

On $\mathbb{R}^n$, $n \geq 3$, $||x||_\alpha^\beta$ is negative definite for $0 < \beta \leq \alpha \leq 2$ and if $\alpha > 2$ it is not negative definite for $\beta > 1$.

Proof. That $||x||_\alpha$ is negative definite on $\mathbb{R}^2$ for $\alpha \geq 1$ is due to a theorem of Herz [6]: on a two-dimensional space, all norms are negative definite. The proof of (i) is completed using a well-known result of Bochner: if ψ is negative definite then ψ^β is negative definite for all $0 < \beta \leq 1$. In the case of (ii), $||x||_\alpha^\alpha = |x_1|^\alpha + |x_2|^\alpha$ is well known to be negative definite for all $0 < \alpha \leq 2$ (Lévy's theorem). Again using the result of Bochner stated above we see that $||x||_\alpha^\beta$ is negative definite if in addition $0 < \beta/\alpha \leq 1$. This argument also works if $n \geq 3$.

When $n \geq 3$, Dor [3] showed that $\|x\|_\alpha^\beta$ is not negative definite if $\alpha > 2$. Therefore if $\|x\|_\alpha^\beta$ is negative definite with $\beta > 1$ then we apply Bochner's theorem to obtain a contradiction.

Using the techniques of Berg and Forst [1], Chapter 2, we can now produce a large number of other α-symmetric characteristic functions. Recall that if F is an abstract set of functions and the composition $g \circ f$ is defined for all f in F, then the function g is said to operate on F if $g \circ f$ belongs to F for all f in F. Then the following result is based on the well-known descriptions of the functions which operate on the sets of negative or positive definite functions on $\mathbb{R}^n$ ([1; pp. 14 and 70]).

3.2. Proposition. Let α, β be such that $\|x\|_\alpha^\beta$ is negative definite. (i) If $g: [0,\infty) \to \mathbb{C}$ is the canonical extension of a Bernstein function, then, $g(\|x\|_\alpha^\beta)$ is negative definite. (ii) If $g: \mathbb{C} \to \mathbb{C}$ such that $g(z) = \sum_{j=0}^{\infty} a_j z^j$ is holomorphic in the open disc $|z| < \rho$, $\rho > 0$, and $a_j \geq 0$ for all j, then $g(\exp(-\|x\|_\alpha^\beta))$ is positive definite.

3.3 Remark. There are some striking geometrical reasons underlining the failure of Herz's theorem on $\mathbb{R}^3$. A review of the topic and its relationship with the geometry of convex sets is given in [11].

Finally, we remark that the existence of α-symmetric characteristic functions remains an open question for the case $n > 2$ and $\alpha > 2$.

4. DETERMINATION OF SPECTRAL MEASURES

This section deals with the following question: Given a symmetric α-stable ($S\alpha S$) law on $\mathbb{R}^n$ with characteristic function $\exp(-F_0(x))$, where

$$F_0(x) = \int_{S^{n-1}} |\langle x,y\rangle|^\alpha \, d\mu(y), \quad x \in \mathbb{R}^n,$$

how is the spectral measure μ determined from F_0? By homogeneity it suffices to work instead with F, the restriction of F_0 to the unit sphere. Throughout, we denote by $C^r(S^{n-1})$ the space of functions $f: S^{n-1} \to \mathbb{C}$ which are r-times continuously differentiable. Then, we have the following result.

4.1. <u>Theorem</u>. Suppose that $F \in C^{N+3}(S^{n-1})$. Then, there exists a unique, even, continuous function g such that

$$F(x) = \int_{S^{n-1}} |\langle x,y\rangle|^\alpha \, g(y) \, dy, \quad x \in S^{n-1}, \tag{4.1}$$

where dy is the surface area element on S^{n-1}.

After proving this result, it was discovered from [11] that Schneider [10] had established the case $\alpha=1$ using almost identical arguments. In the interests of brevity, we shall simply outline the main features of the argument.

For $j=0,1,2,\ldots$, let $\{Y_{jk}\}$, $k=1,2,\ldots,d(n,j)$, be an orthonormal system of surface spherical harmonics on S^{n-1}, with $\deg(Y_{jk})=j$. Here, $d(n,j)$ is the dimension of the vector space of polynomials harmonic and homogeneous of degree j (cf. Erdélyi et al. [4]). Define

$$a_{jk} = \int_{S^{n-1}} F(x)\, Y_{2j,k}(x)\, dx,$$

and

$$Y_{2j}(x) = \sum_{k=1}^{d(n,2j)} a_{jk}\, Y_{2j,k}(x), \quad x \in S^{n-1}.$$

Since $F(x)$ is even, we obtain the expansion

$$F(x) = \sum_{j=0}^{\infty} Y_{2j}(x),$$

with equality holding in the L^2-norm. Let $\nu = \frac{1}{2}(n-2)$ and $C_j^{\nu}(\cdot)$ be the usual Gegenbauer polynomial. Then by the Funk-Hecke theorem,

$$\int_{S^{n-1}} |\langle x,y\rangle|^{\alpha}\, Y_{2j}(y)\, dy = \lambda_j\, Y_{2j}(x)$$

where

$$\lambda_j = c(n,j) \int_{-1}^{1} |t|^{\alpha}\, (1-t^2)^{\nu-1/2}\, C_{2j}^{\nu}(t)\, dt, \tag{4.2}$$

and $c(n,j)$ is a constant depending only on n,j. The integral in (4.2) has been evaluated by Richards [9], Proposition A.1. Then, estimates for λ_j and $Y_{2j}(x)$ prove that the series

$$g(x) = \sum_{j=0}^{\infty} \lambda_j^{-1} Y_{2j}(x)$$

is uniformly convergent on S^{n-1} and represents the unique, continuous, even solution to (4.1).

It is interesting to note, as Schneider does, that explicit closed form solutions to (4.1) are available when $\alpha=1$ and $n=2,3$. Also, the proof of Theorem 4.1 shows that starting with the function $g(x)$, expansions can be obtained for $F(x)$. This generalizes results obtained by Richards [9] for the zonally symmetric stable laws.

Finally, we compute the Fourier coefficients a_{jk} for the α-sub-Gaussian stable laws. If Σ is a $n \times n$ covariance matrix, then $F(x)=\exp\{-\langle x,\Sigma x\rangle^{\alpha/2}\}$ is known to be $S\alpha S$ (cf. Paulauskas [7]), and the distributions having these characteristic functions are known as <u>α-sub-Gaussian</u>. In computing the coefficients a_{jk}, it is enough to assume that Σ is of full rank; otherwise, the problem can be reduced to one of lower dimension. We shall only provide explicit details for the case $n=2$; if $n \geq 3$, our approach also works and the results can be obtained using explicit formulas for the higher dimensional spherical harmonics appearing in [4]. As a further reduction of the problem, we note that it even suffices to assume that Σ is diagonal with eigenvalues σ_1 and σ_2; this is equivalent to choosing a suitable basis for the space of spherical harmonics.

For n=2, transform to polar coordinates, $x=(\cos\theta, \sin\theta)$, $0<\theta\leq 2\pi$. The independent spherical harmonics are $\{e^{ij\theta}\}$ where $j=0,\pm1,\pm2,\ldots$. Writing $F(\theta)$ for $F(x)$, we obtain the Fourier series

$$F(\theta) = \frac{a_0}{2\pi} + \sum_{j=1}^{\infty} \frac{a_j \cos j\theta}{\pi},$$

holding since $F(x)$ is even in x. Further, the series converges uniformly and absolutely since $F(x)$ is infinitely differentiable. Using standard methods, we obtain

$$a_j = \int_0^{2\pi} (\sigma_1 \cos^2\theta + \sigma_2 \sin^2\theta)^{\alpha/2} \cos 2j\theta \, d\theta, \quad j=0,1,2,\ldots$$

and this shows that we may assume $\sigma_1=1\geq\sigma_2=\sigma$, without losing generality.

4.2. Proposition. For $0<\sigma\leq 1$,

$$a_j = \frac{2(-1)^j (1-\sigma)^j (-\frac{1}{2}\alpha)_j \Gamma(\frac{1}{2}) \Gamma(j+\frac{1}{2})}{\Gamma(j+1)} \, {}_2F_1(j-\tfrac{1}{2}\alpha, j+\tfrac{1}{2}, j+1; -(1-\sigma)),$$

where ${}_2F_1(a,b;c;t)$ is Gauss' hypergeometric series and $(t)_j = \Gamma(t+j)/\Gamma(t)$.

Proof. Clearly,

$$\tfrac{1}{2} a_j = \int_0^{\pi} (\cos^2\theta + \sigma \sin^2\theta)^{\alpha/2} \cos 2j\theta \, d\theta$$

$$= \int_0^\pi [1-(1-\sigma)\ (1-\cos^2\theta)]^{\alpha/2} \cos 2j\theta \, d\theta$$

$$= \int_{-1}^{1} [1-(1-\sigma)\ (1-t^2)]^{\alpha/2}\ T_{2j}(t)\ (1-t^2)^{-\frac{1}{2}}\ dt,$$

where $T_{2j}(\cdot)$ is the Tchebicheff polynomial of the first kind. Expanding the term $[1-(1-\sigma)\ (1-t^2)]^{\alpha/2}$ through the binomial theorem and integrating termwise, we have

$$\tfrac{1}{2}\, a_j = 2 \sum_{k=0}^{\infty} \frac{(-\frac{1}{2}\alpha)_k\ (1-\sigma)^k}{k!} \int_0^1 (1-t^2)^{k-\frac{1}{2}}\ T_{2j}(t)\ dt. \qquad (4.3)$$

To evaluate the k-th integral I_k in (4.3) we proceed as in [9]; substitute

$$T_{2j}(t) = {}_2F_1(-j,j;\ \tfrac{1}{2};\ 1-t^2),$$

expand the terminating hypergeometric series and integrate termwise. The value of I_k is seen to be

$$\tfrac{1}{2} \sum_{i=1}^{j} \frac{(-j)_i\ (j)_i\ \Gamma(k+i+\frac{1}{2})\ \Gamma(\frac{1}{2})}{i!\,(\frac{1}{2})_i\ \Gamma(k+i+1)} = \frac{\Gamma(\frac{1}{2})\ \Gamma(k+\frac{1}{2})}{2\Gamma(k+1)}\ {}_3F_2 \left[\begin{matrix} -j,j,k+\frac{1}{2} \\ \frac{1}{2},\ k+1 \end{matrix} \middle|\ 1 \right].$$

This ${}_3F_2$ is balanced (i.e. Saalschutzian) so by Saalschutz's theorem (Rainville [8], p. 87), $I_k=0$ for $k \le j-1$ and for $k \ge j$,

$$I_k = \frac{\Gamma(\frac{1}{2})\ \Gamma(k+\frac{1}{2})\ (-k)_j}{2\Gamma(k+1)\ (k+1)_j} = \frac{(-1)^j\ \Gamma(\frac{1}{2})\ \Gamma(k+\frac{1}{2})}{2\Gamma(k+1-j)}.$$

Using this in (4.3) and simplifying, we obtain the desired result.

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