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ABSTRACT

A polynomial $p(t_1,\ldots,t_n)$ in n real variables is a Selberg polynomial if (1) $p(t_1,\ldots,t_n)$ is homogeneous of (even) degree k, (2) $p(t_1,\ldots,t_n) = (t_1 t_2 \ldots t_n)^{\ell} \, p(t_1^{-1},\ldots,t_n^{-1})$ for some integer ℓ, (3) $p(t_1,\ldots,t_n)$ is symmetric in $t_1,\ldots,t_n$, and (4) $p(t_1,\ldots,t_n) = p(1-t_1,\ldots,1-t_n)$. The definition of these polynomials was motivated by a study of Selberg's (Norsk. Mat. Tidsskr. 26 (1944), 71-78) derivation of a multivariate beta-type integral formula. In this paper, we derive analogs of Selberg's and Mehta's integral formula for a large class of Selberg polynomials.

1. INTRODUCTION

In this paper, we evaluate multidimensional beta-type integrals, involving the "Selberg polynomials" introduced in [7].

For some necessary background recall that in [10], Selberg computed an important generalization of the beta integral. He proved the following result.

1.1. Theorem (Selberg [10]). Let x, y, and z be complex numbers with Re(x) > 0, Re(y) > 0 and Re(z) > max{-1/n, -Re(x)/(n-1), -Re(y)/(n-1)}. Further, let

$$\Delta(t_1,\ldots,t_n) = \prod_{i<j}^{n} (t_i-t_j)^2$$

be the discriminant polynomial in n real variables $t_1,\ldots,t_n$. Then,

$$\int_0^1\cdots\int_0^1 \Delta(t_1,\ldots,t_n)^z \prod_{i=1}^{n} t_i^{x-1}(1-t_i)^{y-1}\, dt_i$$

$$= \prod_{j=1}^{n} \frac{\Gamma(x+(j-1)z)\;\Gamma(y+(j-1)z)\;\Gamma(jz+1)}{\Gamma(x+y+(n+j-2)z)\;\Gamma(z+1)} . \qquad (1.1)$$

Recent work of Andrews [1], Askey [2], [3], Macdonald [6], Morris [9] and others have related (1.1) to topics as diverse as generalized hypergeometric series, orthogonal polynomials, the Dyson conjecture and the root systems of finite reflection groups. Following from (1.1) as a limiting case is the Mehta-Dyson integral [8; p. 42]: if Re(z) > -1/n, then

$$(2\pi)^{-n/2} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \exp\Big(-\tfrac{1}{2} \sum_{i=1}^{n} t_i^2\Big)\, \Delta(t_1,\ldots,t_n)^z\, dt_1\ldots dt_n$$

$$= \prod_{j=1}^{n} \frac{\Gamma(jz+1)}{\Gamma(z+1)} . \qquad (1.2)$$

In the course of proving (1.1), Selberg utilized the following four properties of the polynomial $p(\underset{\sim}{t}) = \Delta(\underset{\sim}{t})$ (we write $\underset{\sim}{t} = (t_1,\ldots,t_n)$).

(S1) $p(\underset{\sim}{t})$ is <u>homogeneous</u>:

$$p(s\underset{\sim}{t}) = s^k\, p(\underset{\sim}{t}) \;, \quad s \in \mathbb{R} \;,$$

for some nonnegative integer k;

(S2) $p(\underset{\sim}{t})$ is <u>translational</u>:

$$p(\underset{\sim}{t}) = p(\underset{\sim}{1}-\underset{\sim}{t})$$

where $\underset{\sim}{1} = (1,1,\ldots,1)$;

(S3) $p(\underset{\sim}{t})$ is ℓ-<u>reciprocal</u>: for some nonnegative integer ℓ,

$$p(\underset{\sim}{t}) = (t_1 t_2 \ldots t_n)^{\ell}\, p(\underset{\sim}{t}^{-1}) \;,$$

where $\underset{\sim}{t}^{-1} = (t_1^{-1},\ldots,t_n^{-1})$;

(S4) $p(\underset{\sim}{t})$ is <u>symmetric</u>: for any permutation σ of $\{1,2,\ldots,n\}$,

$$p(\underset{\sim}{t}) = p(\sigma\underset{\sim}{t})$$

where $\sigma\underset{\sim}{t} = (t_{\sigma(1)},\ldots,t_{\sigma(n)})$.

A polynomial $p(\underset{\sim}{t})$ which satisfies (S1)-(S4) is a <u>Selberg polynomial of type</u> (n,ℓ,k). Thus, $\Delta(\underset{\sim}{t})$ is a Selberg polynomial of type $(n,2(n-1),n(n-1))$.

The paper [7] contains a complete description of, and algorithms for constructing, the Selberg polynomials. It is easily shown, for example, that (S1)-(S3) entail (i) k is even, and (ii) $2k = n\ell$. A much deeper result is that the dimension of $V(n,\ell,k)$, the vector space of Selberg polynomials of type (n,ℓ,k), equals $\mathrm{par}(n,\ell,k) - \mathrm{par}(n,\ell,k-1)$; here, par(n,ℓ,k) equals the number of partitions [5] of k into at most n parts with no part exceeding ℓ. In particular, when n=2, $\Delta(\underset{\sim}{t})^{k/2}$ is (up to constant multiples) the unique Selberg polynomial of degree k.

In point of fact, [7] required as an alternative to (S2), the criterion

$$(S2)' \quad p(\underset{\sim}{t}) = p(\underset{\sim}{1}+\underset{\sim}{t}) .$$

We chose in [7] not to work with (S2) since it is, in conjunction with (S1), more restrictive than (S2)'.

In this paper, we show that Selberg's proof of Theorem 1.1 extends to an evaluation of the integral (1.1) with $\Delta(\underset{\sim}{t})$ replaced by a monomially bounded Selberg polynomial (Definition 2.3). Consequently, we obtain a plethora of Selberg-type integrals.

In Section 2, we abstract from [7] some preliminary material on the Selberg polynomials. Then, Section 3 contains the appropriate generalizations of (1.1) and (1.2).

Although we do not pursue any applications of our results, it is clear that we can relate them to the derivation of constant term identities as in [6] and [9].

2. SELBERG POLYNOMIALS

2.1. Lemma ([7; Theorem 2.3]). A polynomial $p(\underset{\sim}{t})$ is translational if and only if there exists a polynomial $q(t_1,\ldots,t_{n-1})$ such that $p(\underset{\sim}{t}) = q(t_1-t_2,\ldots,t_1-t_n)$.

Let us sketch the proof. First, we note that the translational property (S2) entails (indeed, is equivalent to) the criterion

$$p(s\underset{\sim}{1} - \underset{\sim}{t}) = p(\underset{\sim}{t}) \tag{2.1}$$

for all $s \in \mathbb{R}$. Indeed, for fixed $\underset{\sim}{t}$, (S2) shows that the polynomial $r(s) = p(s\underset{\sim}{1} - \underset{\sim}{t}) - p(\underset{\sim}{t})$ has a zero at every positive integer. Hence, $r(s)$ is identically zero. Substituting $s=t_1$ completes the proof in one direction. Since the converse is trivial, then the proof is complete.

Thus, every Selberg polynomial is (by symmetry) a polynomial in the differences t_i-t_j, $1 \le i < j \le n$. Using these observations, we are able to construct a large number of Selberg polynomials. The following result is a simple version of some general construction procedures developed in [7].

2.2. Theorem. Let $A = (a_{ij})$ be a symmetric $n \times n$ matrix with non-negative, even, integer entries. Assume that A has zero on the main diagonal and $\sum_{j=1}^{n} a_{ij} = \ell$ for all $i=1,\ldots,n$, where $n\ell \equiv 0 \pmod 4$. Then, the polynomial

$$p(\underset{\sim}{t}) = \prod_{i<j}^{n} (t_i-t_j)^{a_{ij}} + \text{symm} \tag{2.2}$$

is a nontrivial Selberg polynomial of type $(n,\ell, n\ell/2)$.

Here and throughout, the term "symm" in (2.2) denotes all terms required to ensure that the polynomial $p(\underset{\sim}{t})$ is symmetric. Then $p(\underset{\sim}{t})$ is clearly homogeneous (of degree $k=n\ell/2$), symmetric and translational. Further, $p(\underset{\sim}{t})$ is ℓ-reciprocal since each variable t_i appears exactly ℓ times in the product $\prod_{i<j}^{n} (t_i-t_j)^{a_{ij}}$.

In order to describe the class of Selberg polynomials to be treated in the next section, we recall some basic facts on symmetric functions [5].

A <u>partition</u> λ of a nonnegative integer k is a set of nonnegative integers $\lambda_1,\ldots,\lambda_n$ such that $\lambda_1+\lambda_2+\ldots+\lambda_n=k$. Customarily, we write $\lambda=(\lambda_1,\ldots,\lambda_n)$ where $\lambda_1\geq\ldots\geq\lambda_n$ and refer to the λ_i as the <u>parts</u> of λ. The monomial symmetric function, in the variables $\underset{\sim}{t} = (t_1,\ldots,t_n)$, corresponding to λ is the polynomial

$$m_\lambda(\underset{\sim}{t}) = t_1^{\lambda_1} t_2^{\lambda_2} \ldots t_n^{\lambda_n} + \text{symm} .$$

Thus, $m_\lambda(\underset{\sim}{t})$ is symmetric and homogeneous of degree $k = \lambda_1+\ldots+\lambda_n$. It is well known that every symmetric, homogeneous polynomial can be expressed as a linear combination of monomial symmetric functions.

2.3. <u>Definition</u>. Let the symmetric polynomial $p(\underset{\sim}{t})$ be homogeneous of degree k, P be a set of partitions of k, and

$$p(\underset{\sim}{t}) = \sum_{\lambda\epsilon P} a_\lambda m_\lambda(\underset{\sim}{t}) \tag{2.3}$$

be the expansion of $p(\underset{\sim}{t})$ in monomial symmetric functions. (We assume that $a_\lambda\neq 0$ for all λ in P). Then $p(\underset{\sim}{t})$ is <u>monomially bounded</u> if there exist nonnegative integers $\alpha_1\geq\ldots\geq\alpha_n$, $\beta_1\geq\ldots\geq\beta_n$ such that

(i) $\alpha_i = \min_P \lambda_i$, $\beta_i = \max_P \lambda_i$, $i=1,\ldots,n$;

(ii) $\sum_{i=1}^{n} \beta_i - \sum_{i=1}^{n} \alpha_i = k$.

2.4. Examples. (i) Selberg [9] showed that $\Lambda(\underset{\sim}{t})$ is monomially bounded with $\alpha_i = n-i$, $\beta_i = 2n-i-1$; $i=1,\ldots,n$.

(ii) Let n be even and

$$p(\underset{\sim}{t}) = \prod_{i=1}^{n/2} (t_i - t_{n-i+1})^{\ell} + \text{symm} .$$

It may be shown that only partitions λ which satisfy $\lambda = \tilde{\lambda}$, $\tilde{\lambda} = (\ell-\lambda_n, \ell-\lambda_{n-1}, \ldots, \ell-\lambda_1)$, can appear in the decomposition (2.3); moreover, every such partition appears. Therefore, $p(\underset{\sim}{t})$ is monomially bounded with $\alpha_i = \beta_{\frac{1}{2}n+i} = \frac{1}{2}\ell$, $\beta_i = \ell$, and $\alpha_{\frac{1}{2}n+i} = 0$, $i=1,2,\ldots,\frac{1}{2}n$.

(iii) A Selberg polynomial (of type (4,6,12)) which is not monomially bounded is

$$p(\underset{\sim}{t}) = (t_1-t_2)^4 (t_1-t_3)^2 (t_2-t_4)^2 (t_3-t_4)^4 + \text{symm} .$$

However, $p(\underset{\sim}{t}) = p_1(\underset{\sim}{t}) - 6\,\Delta(\underset{\sim}{t})$, where $\Delta(\underset{\sim}{t})$ and

$$p_1(\underset{\sim}{t}) = (t_1-t_2)^6 (t_3-t_4)^6 + \text{symm}$$

are both monomially bounded Selberg polynomials. The computations underlying this and other examples were carried out using the symbolic manipulator REDUCE [4].

As the following result shows, we can readily construct monomially bounded Selberg polynomials of arbitrary type.

2.5. Proposition. (i) Let $p_1(\underset{\sim}{t})$ and $p_2(\underset{\sim}{t})$ be monomially bounded Selberg polynomials of types (n,ℓ_1,k_1) and (n,ℓ_2,k_2) respectively. Then the product $p_1(\underset{\sim}{t})p_2(\underset{\sim}{t})$ is a monomially bounded Selberg polynomial of type $(n,\ell_1+\ell_2,k_1+k_2)$.

(ii) Let $p_1(t_1,\dots,t_m)$ and $p_2(t_{m+1},\dots,t_n)$ be monomially bounded, homogeneous (of even degrees k_1 and k_2, respectively), translational, ℓ-reciprocal polynomials. Then the polynomial

$$p(\underset{\sim}{t}) = p_1(t_1,\dots,t_m)\, p_2(t_{m+1},\dots,t_n) + \text{symm}$$

is monomially bounded Selberg polynomial of type (n,ℓ,k_1+k_2).

Proof. (i) It is evident, from (S1)-(S4), that $p_1(\underset{\sim}{t})p_2(\underset{\sim}{t})$ is a Selberg polynomial of type (n,k_1+k_2,ℓ). Therefore we need only check the monomial boundedness. Since $p_1(\underset{\sim}{t})$ (resp. $p_2(\underset{\sim}{t})$) is monomially bounded, then every monomial $t_1^{\lambda_1}\dots t_n^{\lambda_n}$ (resp. $t_1^{\mu_1}\dots t_n^{\mu_n}$) appearing in the expansion of $p_1(\underset{\sim}{t})$ (resp. $p_2(\underset{\sim}{t})$) satisfies $\alpha_i \le \lambda_i \le \beta_i$ (resp. $\alpha_i' \le \mu_i \le \beta_i'$), $i=1,\dots,n$, for some fixed set of integers α_i, β_i (resp. α_i', β_i'), where $\sum_1^n(\beta_i-\alpha_i) = k_1$ (resp. $\sum_1^n(\beta_i'-\alpha_i') = k_2$). In the expansion of $p_1(\underset{\sim}{t})p_2(\underset{\sim}{t})$, every monomial will then be of the form

$$t_1^{\lambda_{\rho(1)}+\mu_{\sigma(1)}} \dots t_n^{\lambda_{\rho(n)}+\mu_{\sigma(n)}}$$

where ρ and σ are permutations on $\{1,2,\dots,n\}$. Since

$$\alpha_{\rho(i)} + \alpha'_{\sigma(i)} \le \lambda_{\rho(i)} + \mu_{\sigma(i)} \le \beta_{\rho(i)} + \beta'_{\sigma(i)} ,$$

$i=1,\dots,n$, and obviously,

$$\sum_{i=1}^{n} [(\beta_{\rho(i)} + \beta'_{\sigma(i)}) - (\alpha_{\rho(i)} + \alpha'_{\sigma(i)})] = k_1 + k_2 ,$$

then $p_1(\underset{\sim}{t})p_2(\underset{\sim}{t})$ is monomially bounded.

The proof of (ii) follows from similar arguments.

3. INTEGRALS OF SELBERG POLYNOMIALS

Throughout, we set

$$c(p) = \int_0^1 \cdots \int_0^1 p(\underset{\sim}{t})\, dt_1 \ldots dt_n, \tag{3.1}$$

where $p(\underset{\sim}{t})$ is a Selberg polynomial. When evaluating $c(p)$, we shall use an expansion of $p(\underset{\sim}{t})$ which follows directly from Lemma 2.1.:

$$p(\underset{\sim}{t}) = \sum_{|\underset{\sim}{j}|=k} c_{\underset{\sim}{j}} (t_1-t_2)^{j_2} (t_1-t_3)^{j_3} \ldots (t_1-t_n)^{j_n}, \tag{3.2}$$

where $\underset{\sim}{j} = (j_2,\ldots,j_n)$ is a multi-index, $|\underset{\sim}{j}| = j_2+\ldots+j_n$, and $c_{\underset{\sim}{j}}$ are constants.

3.1. Theorem (Selberg's Integral). Let $p(\underset{\sim}{t})$ be a monomially bounded Selberg polynomial, and the integers α_i, β_i $(i=1,\ldots,n)$ be specified by Definition 2.3. If $\mathrm{Re}(x) > -\alpha_n$, $\mathrm{Re}(y) > -\alpha_n$, then

$$\int_0^1 \cdots \int_0^1 p(\underset{\sim}{t}) \prod_{i=1}^{n} t_i^{x-1}(1-t_i)^{y-1} dt_i$$

$$= c(p) \prod_{j=1}^{n} \frac{\Gamma(x+\alpha_j)\, \Gamma(y+\alpha_j)\, \Gamma(\beta_j+2)}{\Gamma(x+y+\beta_j)\, \Gamma^2(\alpha_j+1)}. \tag{3.3}$$

Further,.

$$c(p) = \frac{n}{n+k} \sum_{|\underset{\sim}{j}|=k} c_{\underset{\sim}{j}} \prod_{i=2}^{n} (j_i+1)^{-1}. \tag{3.4}$$

Proof. Denote the integral in (3.3) by $I(p)$. By expanding $p(\underset{\sim}{t})$, we see that $I(p)$ is a linear combination of integrals of the form

$$\int_0^1 \cdots \int_0^1 \prod_{i=1}^{n} t_i^{x+\lambda_i-1} (1-t_i)^{y-1} dt_i = \prod_{j=1}^{n} \frac{\Gamma(x+\lambda_i)\, \Gamma(y)}{\Gamma(x+y+\lambda_i)},$$

where, without loss of generality, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Since $p(\underset{\sim}{t})$ is monomially bounded, then $\alpha_i \leq \lambda_i \leq \beta_i$ $(i=1,\ldots,n)$. Hence

$$\frac{\Gamma(x+\lambda_i)}{\Gamma(x+y+\lambda_i)} = \frac{\Gamma(x+\alpha_i)}{\Gamma(x+y+\beta_i)} \prod_{j=0}^{\lambda_i-\alpha_i-1} (x+\lambda_i+j) \prod_{j=0}^{\beta_i-\alpha_i-1} (x+y+\lambda_i+j)$$

$$= \frac{\Gamma(x+\alpha_i)}{\Gamma(x+y+\beta_i)} q_{\lambda_i}(x,y) ,$$

where $q_{\lambda_i}(x,y)$ is a polynomial of degree at most $\beta_i-\lambda_i$ in y and at most $\beta_i-\lambda_i$ in x. Therefore,

$$\int_0^1 \cdots \int_0^1 \prod_{i=1}^{n} t_i^{x+\lambda_i-1}(1-t_i)^{y-1}dt_i = Q_{\lambda_1,\ldots,\lambda_n}(x,y) \prod_{j=1}^{n} \frac{\Gamma(x+\alpha_i)\Gamma(y)}{\Gamma(x+y+\beta_i)}$$

where $Q_{\lambda_1,\ldots,\lambda_n}(x,y)$ is a polynomial of degree at most $\sum_1^n (\beta_i-\lambda_i) = (\sum_1^n \beta_i) - k$ in y. Therefore,

$$I(p) = Q(x,y) \prod_{j=1}^{n} \frac{\Gamma(x+\alpha_i)\ \Gamma(y)}{\Gamma(x+y+\beta_i)}$$

where $Q(x,y)$ is a polynomial of degree at most $(\sum_1^n \beta_i) - k$ in y. Let $R(y) = \prod_{i=1}^{n} y(y+1)\ldots(y+\alpha_i-1)$; then

$$I(p) = \frac{Q(x,y)}{R(y)} \prod_{j=1}^{n} \frac{\Gamma(x+\alpha_i)\ \Gamma(y+\alpha_j)}{\Gamma(x+y+\beta_j)} . \qquad (3.5)$$

Replacing $\underset{\sim}{t}$ by $1-\underset{\sim}{t}$ in (3.3), we see that

$$\frac{Q(x,y)}{R(y)} = \frac{Q(y,x)}{R(x)} .$$

Since $Q(y,x)/R(x)$ is a polynomial in y, then so is $Q(x,y)/R(y)$; hence, $R(y)$ divides $Q(x,y)$. Further,

$$\sum_{i=1}^{n} \alpha_i = \deg(R(y)) \leq \deg(Q(x,y)) \leq (\sum_1^n \beta_i) - k = \sum_{i=1}^{n} \alpha_i ,$$

the last equality holding since $p(\underset{\sim}{t})$ is monomially bounded. Therefore, $\deg(R(y)) = \deg(Q(x,y))$ and hence $Q(x,y)/R(y)$ is independent of y. By symmetry $Q(y,x)/R(x)$ is independent of x, so that

$$\frac{Q(x,1)}{R(1)} = \frac{Q(x,y)}{R(y)} = \frac{Q(y,x)}{R(x)} = \frac{Q(y,1)}{R(1)}$$

is independent of both x and y; say, $Q(x,y)/R(y) = c_1(p)$, a constant. Substituting $x=y=1$ in (3.5) we obtain

$$\int_0^1 \cdots \int_0^1 p(\underset{\sim}{t})dt_1 \ldots dt_n = c_1(p) \prod_{i=1}^{n} \frac{\Gamma^2(\alpha_i+1)}{\Gamma(\beta_i+2)} ,$$

which leads immediately to (3.3).

Since $p(\underset{\sim}{t})$ is symmetric, then

$$c(p) = n \int_0^1 \int_0^{t_1} \cdots \int_0^{t_1} p(\underset{\sim}{t})dt_2 \ldots dt_n dt_1 . \tag{3.6}$$

Replacing t_i by $t_1 t_i$, $i=2,\ldots,n$, (3.6) becomes

$$c(p) = n \int_0^1 \cdots \int_0^1 t_1^{n-1} \, p(t_1, t_1 t_2, \ldots, t_1 t_n) dt_1 \ldots dt_n$$

$$= n \int_0^1 \cdots \int_0^1 t_1^{n+k-1} \, p(1, t_2, \ldots, t_n) dt_1 \ldots dt_n$$

$$= \frac{n}{n+k} \int_0^1 \cdots \int_0^1 p(1, t_2, \ldots, t_n) dt_2 \ldots dt_n .$$

Substituting the expansion (3.2) into the above integral and integrating termwise, we obtain (3.4).

3.2. Remarks. (i) As we noted earlier, the proof of (3.3) abstracts an argument due to Selberg [9] for the polynomial $\Delta(\underset{\sim}{t})$.

(ii) Since products of monomially bounded polynomials are also monomially bounded, (3.3) carries over to integral powers of $p(\underset{\sim}{t})$ and even (via Carlson's theorem) to complex powers of $p(\underset{\sim}{t})$. However, the evaluation of $c(p)$ may become far more difficult.

Next, we derive the appropriate extension of (1.2).

3.3. Theorem (Mehta-Dyson Integral). Let $p(\underset{\sim}{t})$ be a monomially bounded Selberg polynomial. Then,

$$(2\pi)^{-\frac{1}{2}n} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \exp\left(-\tfrac{1}{2}\sum_{i=1}^{n} t_i^2\right) p(\underset{\sim}{t})\,dt_1\ldots dt_n =$$

$$= c(p) \prod_{j=1}^{n} \frac{\Gamma(\beta_j+2)}{\Gamma^2(\alpha_j+1)} . \qquad (3.7)$$

Proof. We use the standard procedure [2; p. 938] for deriving Mehta's formula from Selberg's integral. Set $x=y$ and $2t_i=1+(2x)^{-\frac{1}{2}}\, s_i (i=1,\ldots,n)$ in (3.3); using the homogeneity and translationality of $p(\underset{\sim}{t})$, we obtain

$$\int_{-(2x)^{\frac{1}{2}}}^{(2x)^{\frac{1}{2}}}\cdots\int_{-(2x)^{\frac{1}{2}}}^{(2x)^{\frac{1}{2}}} p(\underset{\sim}{s}) \prod_{i=1}^{n}\left(1-\frac{s_i^2}{2x}\right)^{x-1} ds_i$$

$$= x^{\frac{1}{2}(k+n)}\, 2^{2xn+\frac{1}{2}(3k-n)}\, c(p) \prod_{j=1}^{n} \frac{\Gamma^2(x+\alpha_i)\,\Gamma(\beta_i+2)}{\Gamma(2x+\beta_i)\,\Gamma^2(\alpha_i+1)} . \qquad (3.8)$$

Now we let $x \to \infty$; then, the left-hand-side of (3.8) converges to

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp(-\tfrac{1}{2} \sum_{i=1}^{n} s_i^2)\, p(\underset{\sim}{s}) ds_1 \ldots ds_n \ .$$

Denoting the right-hand-side of (3.8) by $r(x)$, Stirling's formula shows that as $x \to \infty$,

$$r(x) \sim (2\pi)^{n/2}\, 2^{\delta(p)}\, c(p) x^{-\delta(p)} \prod_{i=1}^{n} \frac{\Gamma(\beta_i+2)}{\Gamma^2(\alpha_i+1)} \ . \qquad (3.9)$$

where $\delta(p) = \tfrac{1}{2}k - \sum_{1}^{n} \alpha_i$. Necessarily, $\delta(p) \geq 0$ since (3.9) remains finite as $x \to \infty$. If $\delta(p) > 0$, then $\delta(p^2) = 2\,\delta(p) > 0$ and (3.9) entails

$$(2\pi)^{-\frac{1}{2}n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp(-\tfrac{1}{2} \sum_{i=1}^{n} s_i^2)\, (p(\underset{\sim}{s}))^2 ds_1 \ldots ds_n = 0 \ ,$$

which is absurd. Therefore, $\delta(p) = 0$ and (3.7) follows from (3.9).

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REFERENCES

1. ANDREWS, G.E., Notes on the Dyson conjecture, SIAM J. Math. Anal., 11 (1980), 787-792.

2. ASKEY, R., Some basic hypergeometric extensions of integrals of Selberg and Andrews, SIAM J. Math. Anal., 11 (1980), 938-951.

3. ASKEY, R., Computer algebra and definite integrals, preprint.

4. HEARN, A.C. REDUCE 2 User's Manual, Second Edition, University of Utah Computational Physics Group, Report No. UCP-19, March 1973.

5. MACDONALD, I.G., Symmetric Functions and Hall Polynomials, Clarendon Press, Oxford, 1979.

6. MACDONALD, I.G., Some conjectures for root systems and finite reflection groups, SIAM J. Math. Anal., 13 (1982), 988-1007.

7. MENA, R., BRIDGES, W., ISSACSON, E. and RICHARDS, D., Selberg polynomials, in preparation.

8. MEHTA, M.L., Random Matrices and the Statistical Theory of Energy Levels, Academic Press, New York, 1967.

9. MORRIS, W.G. II, Constant term identities for finite and affine root systems: conjectures and theorems, Ph.D. thesis, University of Wisconsin-Madison, 1982.

10. SELBERG, A. Bemerkninger om et multipelt integral, Norsk. Mat. Tidsskr., 26 (1944), 71-78.