

A Nonparametric, Intersection-Union Test for Stochastic Order

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1. Introduction. A distribution function (cdf) $G(x)$ is said to be stochastically larger than another cdf $F(x)$ if $G(x) \leq F(x)$ for all x with strict inequality for some x . This relationship will be denoted by $G \succ^{st} F$. Stochastic ordering defines a situation in which, in a very strong sense, observations from a population with cdf G tend to be larger than observations from a population with cdf F .

To test if $G \succ^{st} F$, null and alternative hypotheses commonly considered are $H_0: G = F$ and $H_a: G \succ^{st} F$. The Mann-Whitney-Wilcoxon rank sum test (RST) is often recommended as a nonparametric test for testing H_0 versus H_a . See for example Lehmann (1975). Lee and Wolfe (1976) study a competitor to the RST, based on a restricted maximum likelihood estimate, for testing H_0 versus H_a .

Using the asymptotic distribution of the RST statistic, it can be shown that the power of the RST converges to one for any pair (F,G) for which $\theta = \int_{-\infty}^{\infty} F(x)dG(x) > \frac{1}{2}$. The same is true of the Lee and Wolfe (1976) test. Here $\theta = P(X \leq Y)$ if X and Y are independent random variables with cdfs F and G respectively. If $G \succ^{st} F$ then $\theta > \frac{1}{2}$. But there are many other pairs (F,G) for which G is not stochastically larger than F and still $\theta > \frac{1}{2}$. If the model includes such pairs, then rejection of the null hypothesis by the RST or Lee and Wolfe test cannot be interpreted as convincing evidence that $G \succ^{st} F$ but only that $\theta > \frac{1}{2}$. Indeed, the power of the RST converges to 0 and α (the size of the test) according as $\theta < \frac{1}{2}$ or $\theta = \frac{1}{2}$. Thus the RST might be more properly thought of as a test of $H_0': \theta \leq \frac{1}{2}$ versus $H_a': \theta > \frac{1}{2}$ rather than as a test of H_0 versus H_a .

To develop a nonparametric test for which "rejection of the null hypothesis" could be interpreted as convincing evidence that $G \succ^{st} F$, one might try to derive a size α test of $H_0^*: G$ is not stochastically larger than F

versus H_0^* : $G \succ^* F$. Unfortunately, as we shall show in Section 4, this is not practical. The uniformly most powerful level α test of H_0^* versus H_1^* is the "no data" test which rejects H_0^* with probability α , regardless of the data. The reason for this is that for any pair (F,G) satisfying H_0^* , there is a sequence of pairs (F,G_n) , $n = 1,2,\dots$, satisfying H_0^* which converge to (F,G) in such a way that the power of any test at (F,G) is the limit of the powers at (F,G_n) . Since (F,G_n) satisfies H_0^* , this limit must be at most α . The proof of this fact suggests that the difficulty may be that $G(x)$ may be less than $F(x)$ for most x 's but $G(x)$ may be slightly greater than $F(x)$ for some x in the tail of the distributions. Thus, although $P(Y \succ x) = 1 - G(x) \succ 1 - F(x) = P(X \succ x)$ for most x 's, G is not stochastically larger than F . In this paper we consider null and alternative hypotheses which are similar to H_0^* and H_1^* but which circumvent this troublesome tail behavior and allow the development of a reasonable level α test.

Specifically, in this paper we consider testing

$$H_0: F(x) = G(x) \text{ for all } x \in [a,b]$$

$$\text{or } F(x) < G(x) \text{ for at least one } x \in [a,b]$$

versus

$$H_1: F(x) \geq G(x) \text{ for all } x \in [a,b] \text{ with strict inequality for at least one } x \in [a,b].$$

Here $-\infty < a \leq b < \infty$ are fixed constants which define the interval of x values on which $F(x)$ and $G(x)$ are to be compared. The relationship described in H_0 is the same as that which defines stochastic ordering except that the inequality is required to hold only for $x \in [a,b]$ rather than for all x . If the interval $[a,b]$ is reasonably wide, H_0 and H_1 provide practical approximations to H_0^* and H_1^* . Rejection of H_0 by a level α test (α small) will provide convincing evidence that $F(x) \geq G(x)$ for all $x \in [a,b]$. In many situations H_0

and H_n provide more meaningful and easily understood approximations to H_0^* and H_n^* than do the hypotheses H_0' and H_n' associated with RST. In this paper we consider the one sample problem in which $F(x)$ is a known standard and a random sample is available from $G(x)$. Research is currently being completed on the two sample problem.

In Section 2, a level α intersection-union test of H_0 versus H_n is described. A table of small sample critical values and a large sample approximation for the critical values are given. Also a graphical description of the test is provided. In Section 3, the size of the test is shown to be the prescribed value. The test is shown to be the uniformly most powerful level α test among all monotone, nonrandomized, permutation invariant tests. The consistency class of the test is described. The problem of testing $H_0^*: G$ is not stochastically larger than F versus $H_n^*: G \succ^* F$ is considered in Section 4.

2. Derivation of the test. Let F be a known cdf. Let Y_1, \dots, Y_n be a random sample from an unknown cdf G . Note that neither F nor G are required to be continuous. We will note in later results when continuity simplifies something. Let $0 < \alpha < 1$ be the desired level of the test. Define the inverse of F in the usual way, namely, $F^{-1}(p) = \inf\{x: F(x) \geq p\}$. (If $F(x) < 1$ for all x , define $F^{-1}(1) = \infty$.) Let a and b be fixed values satisfying $F^{-1}(1 - \alpha^{1/n}) \leq a \leq b < F^{-1}(1)$. Then a level α test of H_0 versus H_n is given by the following.

First define constants p_0, \dots, p_{n-1} by the relationship

$$(2.1) \quad P_{p_i} (B \leq i) = \alpha, \quad i = 0, \dots, n-1,$$

where B is a binomial random variable with parameters p_i and n . The

constant p_i , also depends on n and α but that dependence will be suppressed in the notation. Table 1 provides values of p_i for $n = 1, \dots, 30$, $i = 0, \dots, n-1$, and $\alpha = .01, .05$, and $.10$. Next, for $i = 0, \dots, n-1$, define critical values $c_i = F^{-1}(p_i)$ and let $c_n = F^{-1}(1)$. Note that the restriction $F^{-1}(1 - \alpha^{1/n}) \leq a$ simply requires that $a \geq c_0$. Define integers I and J by the relationships $c_{I-1} \leq a < c_I$ and $c_{J-1} \leq b < c_J$. Note that $I \leq J$ since $a \leq b$. Let $Y_{(1)} \leq \dots \leq Y_{(n)}$ denote the order statistics from the sample Y_1, \dots, Y_n . Then the proposed test, that we will call test T , is defined as follows

$$(2.2) \quad \text{Reject } H_0 \text{ if and only if } Y_{(i)} \geq c_i, \quad i = I, \dots, J-1 \text{ and } Y_{(J)} \geq b.$$

Note that if $I = J$ then the set $i = I, \dots, J-1$ is empty and test T simply rejects H_0 if and only if $Y_{(J)} \geq b$. But this corresponds to the atypical situation in which the interval $[a, b]$ is very short and $c_{J-1} \leq a \leq b < c_J$. Typically the interval is wide so that H_0 and H_a are similar to the stochastic ordering hypotheses H_0^* and H_a^* from Section 1. In all the remaining discussion we will assume that $I < J$. The arguments can be easily modified to handle the $I = J$ case.

Values for p_i , that are used to define the critical values $c_i = F^{-1}(p_i)$, are given in Table 1 for $n = 1, \dots, 30$. For larger values of n , $p_0 = 1 - \alpha^{1/n}$ and $p_{n-1} = (1 - \alpha)^{1/n}$ are easily calculated. For other values of i , a normal approximation to the binomial distribution (with continuity correction) gives reasonable approximations of p_i , as follows. The approximation is based on

$$P_{p_i} \left[\frac{B - np_i}{\sqrt{np_i(1-p_i)}} \leq \frac{i + \frac{1}{2} - np_i}{\sqrt{np_i(1-p_i)}} \right] = \alpha \approx P_{p_i} \left[\frac{B - np_i}{\sqrt{np_i(1-p_i)}} \leq z_\alpha \right]$$

where z_α is the lower $(100\alpha)^{\text{th}}$ percentile of a standard normal distribution. So setting $z_\alpha = (i + \frac{1}{2} - np_i) / (np_i(1-p_i))^{1/2}$ and solving for p_i yields

$$p_i = \frac{i + \frac{1}{2} + \frac{1}{2}z_\alpha^2 + \sqrt{z_\alpha^2(- (i + \frac{1}{2})^2/n + i + \frac{1}{2} + \frac{1}{2}z_\alpha^2)}}{n + z_\alpha^2}$$

Table 2 gives the actual and approximate values of p_i , for $n = 30$ and $\alpha = .01$, $.05$, and $.10$. The approximation is quite good, especially for i near $\frac{1}{2}n$. The approximation is within $.005$ of the actual value for $1 \leq i \leq 28$ for $\alpha = .10$.

The rejection region for test T can be related to the empirical cdf, $G_n(x) = n^{-1} \sum_{i=1}^n I(Y_{(i)} \leq x)$, of the sample Y_1, \dots, Y_n . ($I(\cdot)$ is the indicator function.) The condition $Y_{(i)} \geq c_i$ is equivalent to $G_n(x) \leq (i-1)/n$ for all $x < c_i$. Similarly $Y_{(j)} \leq b$ is equivalent to $G_n(x) \leq (j-1)/n$ for all $x \leq c_j$. These conditions define a step-shaped region as shown in Figures 1a and 1b. Test T rejects H_0 if and only if the graph of the empirical cdf lies entirely in the shaded, step-shaped region. The rejection region extends to $-\infty$, as in Figure 1a, if $a \geq c_1$ and, hence, $I \geq 2$. In this case the rejection region puts no restriction on $Y_{(1)}$. But if $c_0 \leq a < c_1$, then the rejection region has a finite lower endpoint, as in Figure 1b.

Test T was originally derived using the intersection-union (IU) method of test construction. This method was used as early as 1952 by Lehmann. More recently the method has been used by a variety of authors including Gleser (1973), Sasabuchi (1980), Cohen and Marden (1983), Berger (1984a) and Berger (1984b). The method has not always been identified as the IU method.

The IU method of test construction can be used if the null hypothesis can be conveniently expressed as a union of sets. In our problem, we express H_0 as $(\bigcup_{a \leq x \leq b} G_x) \cap G_{\infty}$ where, for each $x \in [a, b]$, $G_x = \{G: G(x) \geq F(x)\}$ and $G_{\infty} = \{G: G(x) = F(x) \text{ for all } x \in [a, b]\}$. The IU method prescribes that each of the hypotheses $H_{0,x}: G \in G_x$ as well as $H_{0,\infty}: G \in G_{\infty}$ be tested with a level

α test and H_0 is rejected if and only if each of the individual tests rejects its hypothesis. That is the rejection region for the test of H_0 is the intersection of the rejection regions for each of the tests of H_{0x} and $H_{0=}$.

The usual test of H_{0x} is based on the statistic $B_x =$ the number of sample values Y_i that are less than or equal to x . The hypothesis H_{0x} is rejected if B_x is sufficiently small. In particular, if $c_{i-1} \leq x < c_i$, then $p_{i-1} \leq F(x) < p_i$. Thus, from (2.1) we see that the level α test of H_{0x} rejects H_{0x} if and only if $B_x \leq i - 1$. This is equivalent to $Y_{(i)} \geq x$. Now, if $a < c_i \leq b$, the requirement that $Y_{(i)} \geq x$ for all $x \in [c_{i-1}, c_i) \cap [a, b]$ is equivalent to $Y_{(i)} \geq c_i$. If $c_{i-1} \leq b < c_i$, the requirement that $Y_{(i)} \geq x$ for all $x \in [c_{i-1}, c_i) \cap [a, b]$ is equivalent to $Y_{(i)} \geq b$. By the definition of I and J , the only intervals of the form $[c_{i-1}, c_i)$ which intersect $[a, b]$ are $[c_{I-1}, c_I), \dots, [c_{J-1}, c_J)$. Thus the intersection over all $x \in [a, b]$ of the rejection regions of the tests of the hypotheses H_{0x} is the rejection region described in (2.2). According to the IU method, $H_{0=}$ must also be tested with a level α test. But note that for any $x \in [a, b]$, the test which rejects if $Y_{(i)} \geq x$ is also a level α test for $\bar{H}_{0x}: G \in \bar{G}_x$ where $\bar{G}_x = \{G: G(x) \geq F(x)\}$. Since $G_{0=} \subset \bar{G}_x$, this test which was already included as the test for H_{0x} is also a level α test of $H_{0=}$. Thus, no additional test for $H_{0=}$ need be included and, indeed, the inclusion of a different test for $H_{0=}$ would only serve to reduce the power of the resulting test.

Some properties of the test T that we have developed using the IU methodology will be discussed in the next section.

3. Power properties of test. In this section we will discuss three properties of the power function of test T . First we will show that test T is a level α test and give conditions under which the size is exactly α . Next we will show that test T is a uniformly most powerful (UMP) level α test in the

class of nonrandomized, monotone, permutation invariant tests. Finally we will describe the consistency class for test T .

3.1 *Size of test.* Test T is a level α test of H_0 versus H_a . In Theorem 3.1, a condition is given under which the size of the test is exactly the specified value α . This property is important because sometimes the use of level α tests in constructing a test using the IU method will yield an overall test with size much less than α . Thus the constructed test will be overly conservative. Note, in particular, that the condition in Theorem 3.1 that ensures a size α test is satisfied if F is a continuous cdf and $[a,b]$ is wide enough that $I < J$.

THEOREM 3.1. Test T is a level α test of H_0 . That is,

$$(3.1) \quad \sup_{G \in H_0} P_G(\text{reject } H_0) \leq \alpha .$$

If in addition, for some j satisfying $I \leq j < J$, there exists an x such that $F(x) = p_j$, then test T has size exactly α , that is (3.1) is true with equality.

PROOF. Fix $G \in H_0$. Either $G \in G_x$ for some $x \in [a,b]$ or $G \in G_-$. If $G \in G_x$ then, as explained in Section 2, the test will reject only if $Y_{(i)} > x$ where i satisfies $c_{i-1} \leq x < c_i$. Thus, $P_G(\text{reject } H_0) \leq P_G(Y_{(i)} > x) \leq \alpha$. If $G \in G_-$, then, as also explained in Section 2, for any $x \in [a,b]$ the test for $H_{0,x}$ is also a level α test for $H_{0,-}$. Thus, $P_G(\text{reject } H_0) \leq P_G(Y_{(i)} > x) \leq \alpha$ and (3.1) is verified.

To prove the equality in (3.1) note that the definition of c_j as $F^{-1}(p_j)$, the right continuity of F , and the existence of an x with $F(x) = p_j$ together imply that $F(c_j) = p_j$ and $c_{j+1} > c_j$. The definitions of I and J imply that $a < c_j \leq b$. Fix an $x^* > b$ and for any ν satisfying $0 < \nu < 1 - p_j$ define $G_\nu(x)$

as the cdf of the probability measure which puts mass $p_j + \nu$ on the point c_j and mass $1 - p_j - \nu$ on the point x^* . For every ν , $0 < \nu < 1 - p_j$, $G_\nu \in H_0$ because $G_\nu(c_j) = p_j + \nu > p_j = F(c_j)$ and $c_j \in [a, b]$. Consider the inequalities which define the rejection region in (2.2). If G_ν is the true distribution, the inequalities corresponding to subscripts i satisfying $I \leq i \leq j$ are true with probability one. The inequalities for subscripts i satisfying $j + 1 \leq i \leq J$ are true if and only if $Y_{(j+1)} = x^*$ (recall $c_{j+1} > c_j$), that is, if at most j of the Y_i 's equal c_j . But B , the random variable that counts the number of Y_i 's equal to c_j , has a binomial distribution with success probability $p_j + \nu$. Combining (2.1), (3.1), and the fact that $G_\nu \in H_0$ we have

$$\begin{aligned} \alpha &\geq \sup_{G \in H_0} P_G(\text{reject } H_0) \geq \overline{\lim}_{\nu \downarrow 0} P_{G_\nu}(\text{reject } H_0) \\ &= \overline{\lim}_{\nu \downarrow 0} P_{p_j + \nu}(B \leq j) = P_{p_j}(B \leq j) = \alpha. \end{aligned}$$

Thus the equality in (3.1) is proved. \square

3.2 Optimality property. Test T is a UMP level α test of H_0 in the class of nonrandomized, monotone, permutation invariant tests. A nonrandomized test is called *monotone* if (y_1, \dots, y_n) is in rejection region of the test and $y_i^* \geq y_i$, $i = 1, \dots, n$, imply that (y_1^*, \dots, y_n^*) is in the rejection region of the test. Monotone tests have monotone power functions in that if $G \succ^{st} G^*$ then $P_G(\text{reject } H_0) \geq P_{G^*}(\text{reject } H_0)$ for any monotone test. Optimality properties such as this for IU tests in other problems have been discussed by Lehmann (1952), Berger (1982), and Cohen and Marden (1983). It is interesting to note that the following results does not require test T to have size exactly α . The distribution F may be discrete and the test may have size less than α and still the test will be UMP in this class of tests.

THEOREM 3.2. Test T is a UMP level α test of H_0 in the class of nonrandomized, monotone, permutation invariant tests.

PROOF: We will show that any monotone, permutation invariant, level α rejection region is a subset of the rejection region for test T defined in (2.2). Hence test T is UMP. We will show this by showing that any monotone, permutation invariant rejection region which is not a subset of the region in (2.2) has size greater than α .

Consider a monotone, permutation invariant rejection region which is not a subset of the region in (2.2). Let (y_1, \dots, y_n) be a point in this rejection region which does not satisfy all the inequalities in (2.2). Fix j , $I \leq j \leq J$ such that the inequality in (2.2) involving $y_{(j)}$ is not satisfied. That is $y_{(j)} < c_j$, if $j < J$, or $y_{(j)} \leq b$ if $j = J$. Set $y^* = \max(y_{(j)}, a)$. Recall that $a < c_I$ and $b < c_J$ so that, regardless of which subscript j we are considering, $F(y^*) < p_j$. Choose $\nu > 0$ so that $p_j - \nu > F(y^*)$. Fix $x^* > \max(y_{(n)}, b)$. Note that $x^* > y^*$. Let G_ν denote the cdf of the probability distribution that puts mass $p_j - \nu$ on y^* and $1 - p_j + \nu$ on x^* . Then $G_\nu \in H_0$ since $a \leq y^* \leq b$ and $G_\nu(y^*) = p_j - \nu > F(y^*)$. Let B be the random variable which counts the number of Y_i 's equal to y^* . Under G_ν , B has a binomial distribution with success probability $p_j - \nu < p_j$.

Now consider sample points (n -vectors) consisting only of x^* 's and y^* 's. Since $x^* > \max(y_{(n)}, b) \geq y^* \geq y_{(j)}$, by the monotonicity and permutation invariance of the rejection region, all such sample points with at most j y^* 's are in this rejection region. Thus for this rejection region, by (2.1) we have

$$\begin{aligned} \sup_{G \in H_0} P_G(\text{reject } H_0) &\geq P_{G_\nu}(\text{reject } H_0) \geq P_{G_\nu}(B \leq j) \\ &= P_{p_j - \nu}(B \leq j) > P_{p_j}(B \leq j) = \alpha. \end{aligned}$$

This rejection region has size greater than α as was to be shown. \square

A nonparametric upper confidence bound for the cdf G might be used to construct a level α test of H_0 . A test which rejected H_0 only if the $100(1-\alpha)\%$ upper confidence bound was less than or equal to $F(x)$ for all $x \in [a,b]$ would be a level α test of H_0 . But the usual upper confidence bounds, such as those described by Sandford (1985), would produce a nonrandomized, monotone, permutation invariant test. Thus, by Theorem 3.2, test T is a more powerful test than a test constructed in this way.

Test T is not UMP among all level α tests. Test T is biased in that $P_G(\text{reject } H_0) < \alpha$ if $G(x) = F(x)$ for a nondegenerate interval of x values, even if $G \in H_0$. On the other hand, the randomized "no data" test that simply rejects H_0 with probability α , regardless of the data, has $P_G(\text{reject } H_0) = \alpha$ for all G . Thus the "no data" test has higher power than test T for some $G \in H_0$.

3.3 Consistency. In this section, consistency properties of test T are described. Let $\gamma = \inf_{a \leq x \leq b} [F(x) - G(x)]$. We show that the asymptotic value of $P_G(\text{reject } H_0)$ depends on the value of γ . An interpretation of the parameter γ is given at the end of this section. The following two theorems relate the power of the test to γ .

THEOREM 3.3. For test T , if $\gamma \leq 0$ then $P_G(\text{reject } H_0) \leq \alpha$, for every sample size n , and hence, $\lim_{n \rightarrow \infty} P_G(\text{reject } H_0) \leq \alpha$.

THEOREM 3.4. For test T , if $\gamma > 0$ then $\lim_{n \rightarrow \infty} P_G(\text{reject } H_0) = 1$.

PROOF OF THEOREM 3.3. If $\gamma < 0$, $G(x) > F(x)$ for some $x \in [a,b]$. Thus $G \in H_0$, and by Theorem 3.1, $P_G(\text{reject } H_0) \leq \alpha$. If $\gamma = 0$ and $G(x) = F(x)$ for some $x \in [a,b]$, then as explained in Section 2, the test which rejects if $Y_{(i)} > x$ (where $c_{i-1} \leq x < c_i$) is a level α test of $\tilde{H}_{0,x}: G \in \tilde{G}_x = \{G: G(x) \geq F(x)\}$.

So also for this type of G we have $P_G(\text{reject } H_0) \leq P_G(Y_{(1)} > x) \leq \alpha$. The following argument handles the remaining case, $\gamma = 0$ and $G(x) < F(x)$ for all $x \in [a, b]$.

Since $[a, b]$ is compact, there exist $x_0 \in [a, b]$ and a sequence of values $x_m \in [a, b]$ such that $\lim_{m \rightarrow \infty} x_m = x_0$ and $\lim_{m \rightarrow \infty} [F(x_m) - G(x_m)] = 0$. Since $F(x_0) > G(x_0)$, the right continuity of F and G imply that x_m must approach x_0 from below and

$$(3.2) \quad G(x_0^-) = \lim_{x \uparrow x_0} G(x) = \lim_{x \uparrow x_0} F(x) = F(x_0^-).$$

Note that

$$(3.3) \quad 0 < F(a) \leq F(x_0^-) = G(x_0^-) \leq G(x_0) < F(x_0) \leq F(b) < 1.$$

Define a cdf $H(x)$ by

$$(3.4) \quad H(x) = \begin{cases} G(x)/G(x_0^-) & x < x_0 \\ 1 & x \geq x_0 \end{cases}.$$

For any $\nu \in (0, 1)$ define $G_\nu(x) = (1 - \nu)G(x) + \nu H(x)$.

First we will show that for any $\nu \in (0, 1)$, $G_\nu \in H_0$ and hence, by Theorem 3.1,

$$(3.5) \quad P_{G_\nu}(\text{reject } H_0) \leq \alpha.$$

Recall that $F(x_0^-) = G(x_0^-) < 1 = H(x_0^-)$. Thus

$$\begin{aligned} \lim_{m \rightarrow \infty} [F(x_m) - G_\nu(x_m)] &= F(x_0^-) - (1 - \nu)G(x_0^-) - \nu H(x_0^-) \\ &= \nu (G(x_0^-) - 1) < 0. \end{aligned}$$

Thus $G_\nu(x_m) > F(x_m)$ for some x_m , verifying that $G_\nu \in H_0$ and (3.5).

Now we will show that

$$(3.6) \quad \lim_{\nu \downarrow 0} P_{G_\nu}(\text{reject } H_0) \geq P_G(\text{reject } H_0)$$

which with (3.5) will complete the proof that $P_G(\text{reject } H_0) \leq \alpha$. Let $Y = (Y_1, \dots, Y_n)$ be a random sample from G . Let Z_1, \dots, Z_n be a random sample from H . Fix $\nu \in (0, 1)$. Let I_1, \dots, I_n be independent Bernoulli random variables with $P(I_i = 1) = \nu$. All the Y 's, Z 's, and I 's are mutually independent. Define $Y_i^* = Z_i I_i + Y_i(1 - I_i)$. That is $Y_i^* = Z_i$ if $I_i = 1$ and $Y_i^* = Y_i$ if $I_i = 0$. Then $Y^* = (Y_1^*, \dots, Y_n^*)$ is a random sample from the cdf G_ν . Let $A =$ the rejection region for test T and $B = I_1 + \dots + I_n$. Note that B and Y are independent and $Y = Y^*$ if $B = 0$. Using Theorem 3.1 and the fact that $G_\nu \in H_0$ we have

$$\begin{aligned} \alpha &\geq P_{G_\nu}(\text{reject } H_0) = P(Y^* \in A) \geq P(Y^* \in A, B = 0) \\ &= P(Y \in A, B = 0) = P(Y \in A)P(B = 0) = P_G(\text{reject } H_0) \cdot (1 - \nu)^n. \end{aligned}$$

Taking the limit as $\nu \downarrow 0$ yields (3.6). \square

SKETCH OF PROOF OF THEOREM 3.4. A proof of Theorem 3.4 can be based on the graphical representation of the rejection region described in Section 2.

Let $[a]$ denote the greatest integer less than or equal to a . Then the Weak Law of Large Numbers can be used to show that for any $\nu \in (0, 1)$, $\lim_{n \rightarrow \infty} p_{[n\nu]} = \nu$. This convergence is uniform for $\nu \in [F(a), F(b)]$. These facts can be used to show that the step-shaped boundary, with vertices at the points $(F^{-1}(p_i), i/n)$, converges to $F(x)$ uniformly on the interval $[a, b]$. On the other hand, by the Glivenko-Cantelli Theorem, the empirical cdf converges to $G(x)$ uniformly on $[a, b]$ with probability one. Thus, if $\gamma = \inf_{a \leq x \leq b} [F(x) - G(x)] > 0$, for large n the boundary is greater than $F(x) - \gamma/2 \geq G(x) + \gamma/2$ for all $x \in [a, b]$. Test T will reject H_0 if

$G_n(x) < G(x) + \gamma/2$ for all $x \in [a,b]$ and this event has probability approaching one as $n \rightarrow \infty$. \square

Theorems 3.3 and 3.4 say that test T is a level α test of $H_0^+ : \gamma \leq 0$ versus $H_a^+ : \gamma > 0$ and the test is consistent against all points in H_a^+ . Thus the parameter γ plays the same role for test T that the parameter $\theta = P(X \leq Y)$ plays for the RST. The parameter $\gamma = \inf_{a \leq x \leq b} [F(x) - G(x)]$ may be interpreted as follows:

$\gamma > 0$ if and only if $P(X \leq x) > P(Y \leq x)$ for all $x \in [a,b]$
and $P(X < x) > P(Y < x)$ for all $x \in [a,b]$

$\gamma \leq 0$ if and only if $P(X \leq x) \leq P(Y \leq x)$ for some $x \in [a,b]$
or $P(X < x) \leq P(Y < x)$ for some $x \in [a,b]$.

If F is continuous, then each of the inequalities involving $<$ can be dropped in the above interpretation. As mentioned in Section 1, we believe that in many situations the hypothesis $H_a^+ : \gamma > 0$ provides a more meaningful approximation to $H_a^* : G \succ^{st} F$ than does $H_a' : \theta > \frac{1}{2}$, the alternative hypothesis associated with the RST.

4. UMP test of H_0^* versus H_a^* . There is no satisfactory level α test for testing $H_0^* : G$ is not stochastically larger than F versus $H_a^* : G \succ^{st} F$. For any level α test of H_0^* , $P_G(\text{reject } H_0^*) \leq \alpha$ for every cdf G including all $G \in H_a^*$. This means that the randomized "no data" test that rejects H_0^* with probability α , regardless of the data, and has $P_G(\text{reject } H_0^*) = \alpha$ for all G is the UMP level α test of H_0^* versus H_a^* . This fact is proved in the following theorem.

THEOREM 4.1. Consider testing H_0^* versus H_a^* . Let G be any cdf. Then there exists a sequence of cdf's G_m , $m = 1, 2, \dots$, such that $G_m \in H_a^*$ for every m

and, for any (possibly randomized) level α test of H_0^* ,

$$(4.1) \quad \alpha \geq \lim_{m \rightarrow \infty} P_{G_m}(\text{reject } H_0^*) \geq P_G(\text{reject } H_0^*) .$$

PROOF: The proof is similar to the proof of Theorem 3.3. For each $m = 1, 2, \dots$, fix a value x_m such that $F(x_m) < 1/m$ and let H_m be a cdf such that $H_m(x) = 1$ for $x \geq x_m$. Let $G_m = (1 - 1/m)G + (1/m)H_m$. Then $G_m(x_m) = (1 - 1/m)G(x_m) + 1/m \geq 1/m > F(x_m)$. So G_m is not stochastically larger than F . Let $\mathbf{y} = (y_1, \dots, y_n)$ denote a sample point and let $\phi(\mathbf{y})$ denote the test function for any level α test. That is $\phi(\mathbf{y})$ is the probability H_0^* is rejected given that \mathbf{y} is observed. Let $\mathbf{Y} = (Y_1, \dots, Y_n)$ be a random sample from G , let $\mathbf{Z}_m = (Z_{1,m}, \dots, Z_{n,m})$ be a random sample from H_m and let $I_{1,m}, \dots, I_{n,m}$ be Bernoulli random variables with success probability $1/m$. Assume all the Y 's, Z 's and I 's are mutually independent. Let $\mathbf{Y}_m^* = (Y_{1,m}^*, \dots, Y_{n,m}^*)$ where $Y_{i,m}^* = Z_{i,m}I_{i,m} + Y_i(1 - I_{i,m})$. Note that \mathbf{Y}_m^* is a random sample from G_m . Let $B_m = I_{1,m} + \dots + I_{n,m}$ and note that \mathbf{Y} and B_m are independent and $\mathbf{Y} = \mathbf{Y}_m^*$ if $B_m = 0$. Thus the conditional distribution of \mathbf{Y}_m^* given $B_m = 0$ is the same as the conditional distribution of \mathbf{Y} given $B_m = 0$ and this is just the unconditional distribution of \mathbf{Y} . Thus we have for every $m = 1, 2, \dots$,

$$(4.2) \quad P_{G_m}(\text{reject } H_0^*) = E \phi(\mathbf{Y}_m^*) = E(\phi(\mathbf{Y}_m^*) | B_m = 0)P(B_m = 0) + E(\phi(\mathbf{Y}_m^*) | B_m > 0)P(B_m > 0)$$

$$\geq E(\phi(\mathbf{Y}_m^*) | B_m = 0)P(B_m = 0) = E\phi(\mathbf{Y})(1-1/m)^n = P_G(\text{reject } H_0^*)(1-1/m)^n .$$

Since $G_m \varepsilon H_0^*$ for every $m = 1, 2, \dots$, $P_{G_m}(\text{reject } H_0^*) \leq \alpha$ for every m . Thus taking limits as $m \rightarrow \infty$ in (4.2) yields (4.1). \square

In the above proof, since $F(x_m) < 1/m$, we must have $\overline{\lim}_{m \rightarrow \infty} x_m \in \inf\{x: F(x) > 0\}$. That is, the x_m are in the left tail of F . This suggests why we were able to develop a reasonable test of H_0 versus H_a . These hypotheses ignore the tails of F .

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TABLE 1

Critical values p_i for $\alpha = .10$

		n														
i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	
0	.9000	.6838	.5358	.4377	.3690	.3187	.2803	.2501	.2257	.2057	.1889	.1746	.1623	.1517	.1423	
1		.9487	.8042	.6795	.5839	.5103	.4526	.4062	.3684	.3368	.3102	.2875	.2678	.2507	.2356	
2			.9655	.8574	.7534	.6668	.5962	.5382	.4901	.4496	.4152	.3855	.3598	.3372	.3173	
3				.9740	.8878	.7991	.7214	.6554	.5994	.5517	.5108	.4753	.4443	.4170	.3928	
4					.9791	.9074	.8304	.7603	.6990	.6458	.5995	.5590	.5234	.4920	.4640	
5						.9826	.9212	.8531	.7896	.7327	.6823	.6377	.5982	.5631	.5317	
6							.9851	.9314	.8705	.8124	.7595	.7118	.6691	.6309	.5965	
7								.9869	.9392	.8842	.8308	.7813	.7363	.6954	.6585	
8									.9884	.9455	.8952	.8458	.7995	.7568	.7178	
9										.9895	.9505	.9043	.8584	.8149	.7744	
10											.9905	.9548	.9120	.8691	.8280	
11												.9913	.9583	.9185	.8782	
12													.9919	.9613	.9241	
13														.9925	.9640	
14															.9930	

		n														
i	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	
0	.1340	.1267	.1201	.1141	.1087	.1038	.0994	.0953	.0915	.0880	.0848	.0817	.0789	.0763	.0739	
1	.2222	.2102	.1995	.1898	.1810	.1729	.1656	.1588	.1526	.1469	.1415	.1366	.1319	.1276	.1236	
2	.2996	.2837	.2694	.2565	.2448	.2340	.2242	.2152	.2069	.1991	.1920	.1853	.1791	.1733	.1678	
3	.3712	.3519	.3344	.3186	.3042	.2910	.2789	.2678	.2575	.2480	.2392	.2309	.2232	.2160	.2093	
4	.4389	.4164	.3960	.3775	.3607	.3452	.3310	.3180	.3059	.2947	.2842	.2745	.2655	.2570	.2490	
5	.5035	.4781	.4550	.4340	.4149	.3973	.3812	.3663	.3525	.3397	.3277	.3166	.3062	.2965	.2874	
6	.5654	.5374	.5118	.4886	.4673	.4477	.4297	.4131	.3976	.3833	.3700	.3575	.3459	.3349	.3247	
7	.6250	.5945	.5667	.5413	.5180	.4966	.4768	.4586	.4416	.4258	.4111	.3974	.3845	.3725	.3611	
8	.6822	.6496	.6198	.5925	.5673	.5442	.5228	.5029	.4845	.4673	.4513	.4364	.4224	.4092	.3968	
9	.7371	.7027	.6712	.6421	.6152	.5905	.5675	.5462	.5264	.5080	.4907	.4746	.4594	.4452	.4319	
10	.7896	.7539	.7208	.6902	.6618	.6356	.6112	.5885	.5674	.5477	.5293	.5120	.4958	.4806	.4663	
11	.8394	.8028	.7686	.7367	.7071	.6795	.6538	.6299	.6076	.5867	.5671	.5488	.5316	.5154	.5001	
12	.8862	.8494	.8145	.7817	.7509	.7222	.6954	.6703	.6468	.6249	.6043	.5849	.5667	.5496	.5334	
13	.9290	.8932	.8582	.8249	.7933	.7637	.7358	.7097	.6852	.6623	.6407	.6204	.6013	.5832	.5662	
14	.9663	.9333	.8994	.8661	.8341	.8038	.7752	.7482	.7228	.6989	.6764	.6552	.6352	.6163	.5985	
15	.9934	.9683	.9371	.9049	.8731	.8425	.8133	.7856	.7594	.7347	.7114	.6894	.6686	.6489	.6303	
16		.9938	.9701	.9405	.9098	.8794	.8500	.8218	.7951	.7697	.7456	.7229	.7013	.6809	.6616	
17			.9942	.9717	.9436	.9142	.8851	.8568	.8297	.8038	.7791	.7557	.7335	.7124	.6924	
18				.9945	.9731	.9463	.9183	.8903	.8631	.8368	.8117	.7878	.7650	.7433	.7227	
19					.9947	.9744	.9488	.9219	.8950	.8688	.8434	.8191	.7958	.7736	.7524	
20						.9950	.9756	.9511	.9253	.8994	.8740	.8495	.8259	.8032	.7816	
21							.9952	.9766	.9532	.9283	.9034	.8789	.8551	.8322	.8101	
22								.9954	.9776	.9551	.9312	.9071	.8834	.8603	.8380	
23									.9956	.9785	.9568	.9338	.9105	.8875	.8652	
24										.9958	.9794	.9585	.9362	.9137	.8914	
25											.9960	.9801	.9600	.9385	.9166	
26												.9961	.9808	.9614	.9406	
27													.9962	.9815	.9627	
28														.9964	.9821	
29															.9965	

TABLE 1 - Continued

Critical values p_i for $\alpha = .05$

		n													
i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	.9500	.7764	.6316	.5271	.4507	.3930	.3482	.3123	.2831	.2589	.2384	.2209	.2058	.1926	.1810
1		.9747	.8647	.7514	.6574	.5818	.5207	.4707	.4291	.3942	.3644	.3387	.3163	.2967	.2794
2			.9830	.9024	.8107	.7287	.6587	.5997	.5496	.5069	.4701	.4381	.4101	.3854	.3634
3				.9873	.9236	.8468	.7747	.7108	.6551	.6066	.5644	.5273	.4947	.4657	.4398
4					.9898	.9372	.8712	.8071	.7486	.6965	.6502	.6091	.5726	.5400	.5108
5						.9915	.9466	.8889	.8313	.7776	.7288	.6848	.6452	.6096	.5774
6							.9927	.9536	.9023	.8500	.8004	.7547	.7130	.6750	.6404
7								.9936	.9590	.9127	.8649	.8190	.7760	.7364	.7000
8									.9943	.9632	.9212	.8772	.8343	.7939	.7563
9										.9949	.9667	.9281	.8873	.8473	.8091
10											.9953	.9695	.9340	.8960	.8583
11												.9957	.9719	.9389	.9033
12													.9961	.9740	.9432
13														.9963	.9758
14															.9966

		n													
i	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	.1707	.1616	.1533	.1459	.1391	.1329	.1273	.1221	.1173	.1129	.1088	.1050	.1015	.0981	.0950
1	.2640	.2501	.2377	.2264	.2161	.2067	.1981	.1902	.1829	.1761	.1698	.1640	.1585	.1534	.1486
2	.3438	.3262	.3103	.2958	.2826	.2706	.2595	.2493	.2398	.2310	.2229	.2153	.2082	.2016	.1953
3	.4166	.3956	.3767	.3594	.3437	.3292	.3159	.3036	.2923	.2817	.2719	.2627	.2542	.2461	.2386
4	.4844	.4605	.4389	.4191	.4010	.3844	.3691	.3549	.3418	.3296	.3182	.3076	.2977	.2884	.2796
5	.5483	.5219	.4978	.4758	.4556	.4370	.4198	.4039	.3891	.3754	.3626	.3506	.3394	.3289	.3190
6	.6090	.5803	.5540	.5300	.5078	.4874	.4685	.4510	.4347	.4195	.4054	.3921	.3797	.3680	.3570
7	.6666	.6360	.6078	.5819	.5580	.5359	.5155	.4964	.4787	.4622	.4468	.4323	.4187	.4060	.3939
8	.7214	.6892	.6594	.6319	.6064	.5828	.5609	.5405	.5214	.5036	.4870	.4714	.4567	.4429	.4299
9	.7733	.7399	.7088	.6799	.6531	.6281	.6048	.5832	.5629	.5439	.5262	.5095	.4938	.4790	.4651
10	.8222	.7881	.7560	.7261	.6980	.6719	.6475	.6246	.6032	.5832	.5643	.5466	.5300	.5143	.4994
11	.8679	.8336	.8010	.7703	.7414	.7142	.6887	.6649	.6424	.6214	.6016	.5829	.5654	.5488	.5331
12	.9097	.8762	.8437	.8125	.7829	.7550	.7287	.7039	.6806	.6586	.6379	.6184	.6000	.5825	.5661
13	.9469	.9154	.8836	.8525	.8227	.7943	.7673	.7418	.7176	.6949	.6734	.6530	.6338	.6156	.5984
14	.9773	.9501	.9203	.8901	.8604	.8318	.8044	.7784	.7536	.7301	.7079	.6869	.6669	.6480	.6301
15	.9968	.9787	.9530	.9247	.8959	.8676	.8401	.8137	.7884	.7644	.7416	.7199	.6993	.6797	.6611
16		.9970	.9799	.9555	.9286	.9012	.8740	.8475	.8220	.7976	.7743	.7521	.7309	.7107	.6915
17			.9972	.9810	.9578	.9322	.9059	.8798	.8543	.8297	.8060	.7834	.7617	.7411	.7213
18				.9973	.9819	.9599	.9354	.9102	.8851	.8605	.8367	.8138	.7918	.7707	.7505
19					.9974	.9828	.9618	.9383	.9141	.8899	.8662	.8432	.8209	.7995	.7789
20						.9976	.9836	.9635	.9410	.9177	.8944	.8715	.8492	.8275	.8067
21							.9977	.9843	.9650	.9434	.9210	.8985	.8763	.8547	.8337
22								.9978	.9850	.9665	.9457	.9241	.9023	.8808	.8598
23									.9979	.9856	.9678	.9478	.9269	.9058	.8850
24										.9980	.9862	.9690	.9497	.9295	.9091
25											.9980	.9867	.9702	.9515	.9319
26												.9981	.9872	.9712	.9531
27													.9982	.9876	.9722
28														.9982	.9880
29															.9983

TABLE 1 - Continued

Critical values p_i for $\alpha = .01$

		n														
i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	
0	.9900	.9000	.7846	.6838	.6019	.5358	.4821	.4377	.4005	.3690	.3421	.3187	.2983	.2803	.2644	
1		.9950	.9411	.8591	.7779	.7057	.6434	.5899	.5440	.5044	.4698	.4395	.4128	.3891	.3679	
2			.9967	.9580	.8944	.8269	.7637	.7068	.6563	.6117	.5723	.5373	.5062	.4783	.4532	
3				.9975	.9673	.9153	.8577	.8018	.7500	.7029	.6604	.6222	.5878	.5567	.5285	
4					.9980	.9732	.9292	.8791	.8290	.7817	.7378	.6976	.6609	.6274	.5969	
5						.9983	.9773	.9392	.8947	.8496	.8060	.7651	.7271	.6920	.6597	
6							.9986	.9803	.9467	.9068	.8656	.8254	.7871	.7512	.7177	
7								.9987	.9826	.9525	.9163	.8785	.8412	.8053	.7713	
8									.9989	.9845	.9572	.9241	.8892	.8543	.8205	
9										.9990	.9859	.9610	.9305	.8981	.8654	
10											.9991	.9872	.9642	.9360	.9056	
11												.9992	.9882	.9669	.9406	
12													.9992	.9890	.9693	
13														.9993	.9898	
14															.9993	

		n														
i	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	
0	.2501	.2373	.2257	.2152	.2057	.1969	.1889	.1815	.1746	.1682	.1623	.1568	.1517	.1468	.1423	
1	.3488	.3316	.3160	.3018	.2888	.2768	.2658	.2557	.2462	.2375	.2293	.2217	.2146	.2079	.2016	
2	.4305	.4099	.3912	.3741	.3583	.3439	.3305	.3181	.3066	.2959	.2859	.2766	.2679	.2596	.2519	
3	.5029	.4796	.4583	.4387	.4207	.4041	.3887	.3745	.3612	.3488	.3372	.3264	.3162	.3066	.2976	
4	.5690	.5434	.5199	.4983	.4783	.4598	.4426	.4267	.4118	.3979	.3849	.3727	.3613	.3505	.3403	
5	.6299	.6025	.5772	.5538	.5321	.5120	.4933	.4758	.4595	.4443	.4300	.4166	.4039	.3920	.3808	
6	.6866	.6577	.6309	.6060	.5829	.5613	.5412	.5224	.5048	.4884	.4729	.4584	.4447	.4317	.4195	
7	.7393	.7094	.6814	.6553	.6309	.6082	.5868	.5669	.5482	.5306	.5140	.4984	.4837	.4699	.4567	
8	.7883	.7578	.7290	.7020	.6766	.6528	.6304	.6094	.5897	.5711	.5535	.5370	.5214	.5066	.4927	
9	.8335	.8029	.7737	.7460	.7199	.6953	.6721	.6502	.6295	.6100	.5916	.5742	.5578	.5422	.5274	
10	.8749	.8448	.8156	.7876	.7610	.7358	.7119	.6892	.6678	.6476	.6284	.6102	.5930	.5767	.5612	
11	.9122	.8832	.8546	.8267	.7999	.7743	.7499	.7267	.7047	.6837	.6639	.6450	.6271	.6101	.5939	
12	.9446	.9178	.8904	.8632	.8366	.8109	.7862	.7626	.7401	.7186	.6982	.6787	.6602	.6426	.6258	
13	.9713	.9481	.9228	.8968	.8708	.8454	.8207	.7969	.7740	.7521	.7312	.7113	.6922	.6741	.6568	
14	.9905	.9731	.9512	.9272	.9025	.8777	.8532	.8295	.8065	.7844	.7631	.7428	.7233	.7047	.6868	
15	.9994	.9910	.9746	.9539	.9312	.9075	.8838	.8603	.8375	.8152	.7938	.7732	.7533	.7343	.7161	
16		.9994	.9915	.9760	.9564	.9347	.9121	.8893	.8668	.8447	.8232	.8024	.7824	.7631	.7445	
17			.9994	.9920	.9773	.9586	.9379	.9162	.8944	.8727	.8513	.8306	.8104	.7909	.7720	
18				.9995	.9924	.9784	.9606	.9408	.9200	.8990	.8780	.8574	.8373	.8177	.7987	
19					.9995	.9928	.9794	.9624	.9434	.9235	.9032	.8830	.8630	.8435	.8245	
20						.9995	.9931	.9804	.9640	.9458	.9266	.9071	.8875	.8682	.8492	
21							.9995	.9934	.9812	.9655	.9480	.9295	.9107	.8917	.8730	
22								.9996	.9937	.9820	.9669	.9500	.9322	.9140	.8956	
23									.9996	.9940	.9827	.9682	.9519	.9347	.9170	
24										.9996	.9942	.9834	.9694	.9537	.9370	
25											.9996	.9944	.9840	.9705	.9553	
26												.9996	.9946	.9845	.9715	
27													.9996	.9948	.9851	
28														.9997	.9950	
29															.9997	

TABLE 2

Comparison of actual and approximate
critical values p_i for $n = 30$

i	$\alpha = .10$		$\alpha = .05$		$\alpha = .01$	
	actual	approx.	actual	approx.	actual	approx.
0	.0739	.0802	.0950	.1110	.1423	.1798
1	.1236	.1282	.1486	.1601	.2016	.2283
2	.1678	.1715	.1953	.2044	.2519	.2724
3	.2093	.2124	.2386	.2460	.2976	.3137
4	.2490	.2515	.2796	.2857	.3403	.3530
5	.2874	.2894	.3190	.3239	.3808	.3905
6	.3247	.3264	.3570	.3609	.4195	.4267
7	.3611	.3625	.3939	.3969	.4567	.4617
8	.3968	.3979	.4299	.4321	.4927	.4957
9	.4319	.4326	.4651	.4665	.5274	.5287
10	.4663	.4667	.4994	.5002	.5612	.5608
11	.5001	.5003	.5331	.5332	.5939	.5921
12	.5334	.5334	.5661	.5655	.6258	.6226
13	.5662	.5660	.5984	.5973	.6568	.6523
14	.5985	.5981	.6301	.6284	.6868	.6813
15	.6303	.6297	.6611	.6590	.7161	.7095
16	.6616	.6608	.6915	.6890	.7445	.7370
17	.6924	.6914	.7213	.7184	.7720	.7637
18	.7227	.7215	.7505	.7472	.7987	.7897
19	.7524	.7512	.7789	.7753	.8245	.8149
20	.7816	.7802	.8067	.8028	.8492	.8393
21	.8101	.8087	.8337	.8296	.8730	.8628
22	.8380	.8365	.8598	.8556	.8956	.8853
23	.8652	.8636	.8850	.8807	.9170	.9068
24	.8914	.8899	.9091	.9048	.9370	.9271
25	.9166	.9152	.9319	.9278	.9553	.9460
26	.9406	.9392	.9531	.9492	.9715	.9632
27	.9627	.9616	.9722	.9688	.9851	.9784
28	.9821	.9815	.9880	.9857	.9950	.9907
29	.9965	.9967	.9983	.9977	.9997	.9987

REJECTION REGION
IN TERMS OF EMPIRICAL CDF

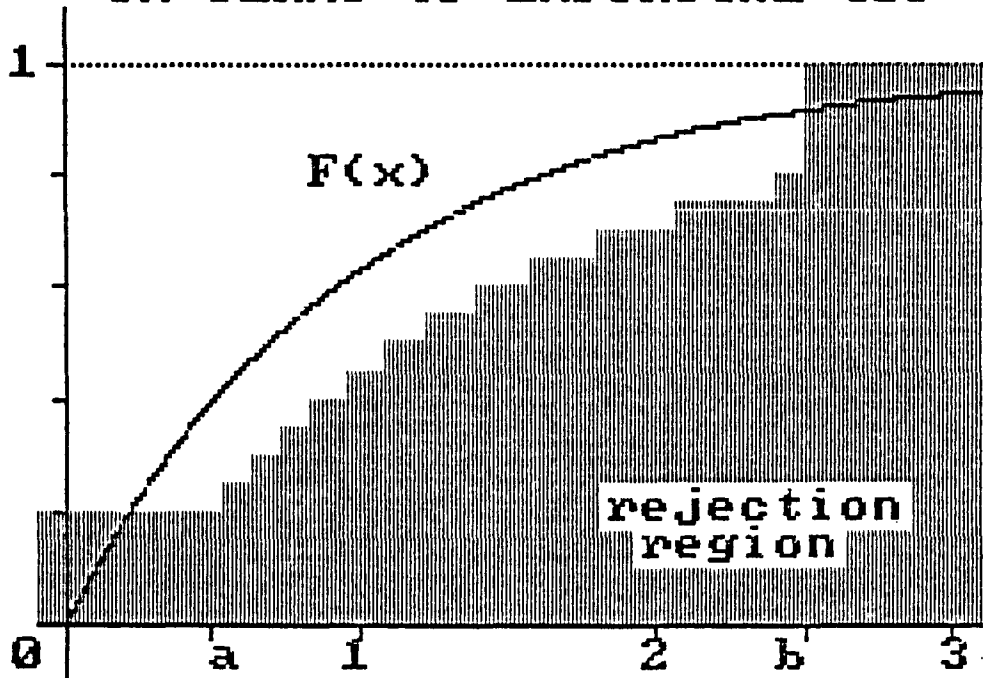


Figure 1a

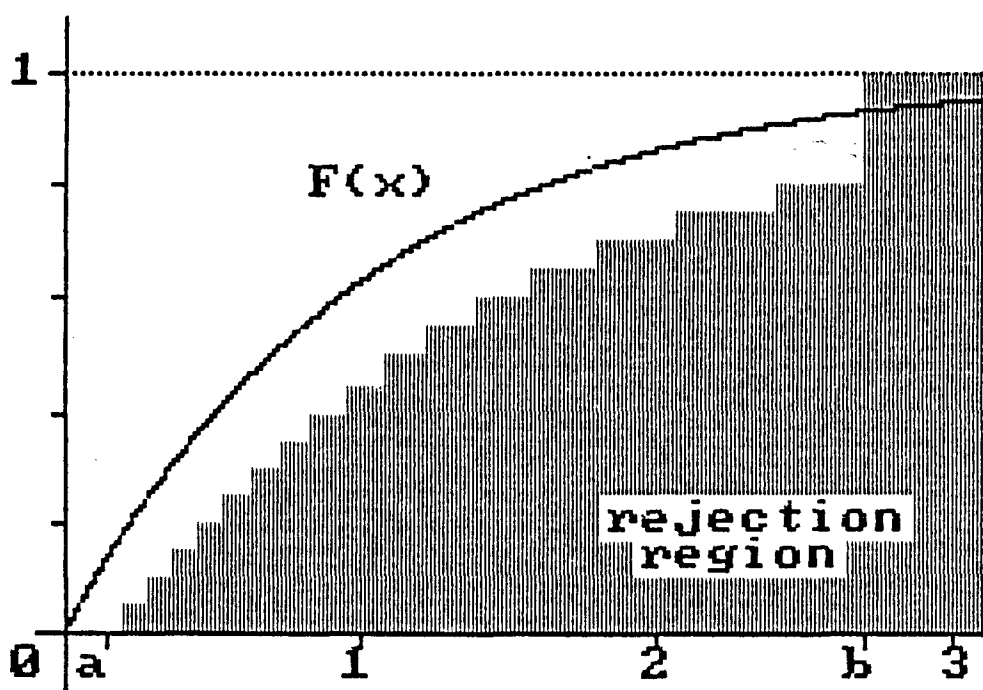


Figure 1b

$F(x)=1-\exp(-x)$ alpha=.1 n=20