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by

M. Ahsanullah and Bikas K. Sinha

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M. Ahsanullah
Rider College
Lawrenceville, NJ 08648-3099

Bikas K. Sinha*
North Carolina State University
Raleigh, NC 27695-8203

## ABSTRACT

Consider $X = (X_0, X_1 \ldots, X_p)'$ and suppose $X_0|X_i = x_i(i=1,2,\ldots,p) \sim N(\alpha_0 + \Sigma_1^p \alpha_i x_i, \sigma^2)$. Assume further that $X_i$'s $(i=0,1,\ldots,p)$ are marginally identically distributed. Does this imply normality of X? Ahsanullah (*Metrika* (1985), 32, 215-218) raised this question and resolved it in the affirmative for p=1. This is, of course, not true for $p > 1$. We give a counter-example to that effect. Next we prove that exchangeability of the components $X_0, \ldots, X_p$ of X along with conditional normality of $X_0$ (as stated above) indeed ensure normality of X.

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* On leave from Indian Statistical Institute (Calcutta), India.

In this note we address a problem considered recently by Ahsanullah (1985) regarding characterization of normality via conditional normality (i.e., normality in a conditional distribution). Consider the following statements regarding the joint distribution of two random variables (X,Y):

S1 Conditionally given Y=y, $X \sim N(\alpha + \beta y, \sigma^2)$.

S2 The marginals are identical (up to a change of location and scale).

S3 (X,Y) have an exchangeable distribution (up to a change of location and scale).

Ahsanullah (1985) essentially demonstrated that S1 and S2 together imply joint normality of X and Y. It is clear that S3 implies S2.

A multivariate generalization of this result is developed below. Consider the following statements regarding the distribution of a (p+1)-component random vector $X = (X_0, X_1, \ldots, X_p)'$ (assumed to possess a non-singular distribution):

P1 Conditionally given $X_i = x_i (i=1,2,\ldots,p)$, $X_0 \sim N(\alpha_0 + \Sigma\alpha_i x_i, \sigma_0^2)$.

P2 The marginals are all identical (up to a change of location and scale).

P3 The components of X have an exchangeable distribution (up to a change of location and scale).

P4 The marginals of $X_0$ and $X_1$ are identical (up to a change of location and scale).

P5 Conditionally given $X_0 = x_0, X_1 = x_1, \ldots, X_k = x_k$,

$X_{k+1} \sim N(\mu_{k+1} + \Sigma_0^k \mu_{k+1,i}\, x_i, \sigma_{k+1}^2)$ simultaneously for k=0,1,2,...,p.

Ahsanullah (1985) conjectured that just P1 and P2 together imply normality of X. This is, however, *not* true. See counter-example at the end. Using the result for two variables, one can easily verify that P4 and P5 together imply normality of X. We show below that P1 and P3 together also imply normality of X. At this moment, this last assertion seems to be rather uninteresting. (The proof is quite straightforward as well.) However, we do <u>not</u> see how it could be strengthened. See remark at the end.

It is clear that whenever the components of X have an exchangeable distribution for themselves,

$$E(X) = (\Theta, \Theta, \ldots, \Theta)' \text{ and } D(X) = \delta^2[(1-\rho)I + \rho J] \text{ which}$$

is the well-known intra-class covariance structure. We take $\Theta=0$ and $\delta=1$ with no loss of generality. Note that $-1/p < \rho < 1$. According to P1, we have then that

$$X_0 | X_i = x_i (i=1,2,\ldots,p) \sim N(\alpha \Sigma_1^p x_i, \sigma_0^2)$$

where $\alpha = \rho/\{1+(p-1)\rho\}$ and $1 = \sigma_0^2 + \alpha^2 p\{1+(p-1)\rho\}$. This means that the joint density of $X_0$, $X_1$, ..., $X_p$ can be represented as

$$f(x_0,x_1,\ldots,x_p) = \text{constant}\bullet\exp\{-\frac{1}{2\sigma_0^2}(x_0 - \alpha \Sigma_1^p x_i)^2\}g(x_1,x_2,\ldots,x_p) \qquad (1)$$

where, by hypothesis, $f(\bullet)$ is exchangeable in $x_0$, $x_1$, ..., $x_p$ and hence, consequent to (1), $g(x_1,\ldots,x_p)$ is likewise exchangeable in $x_1$, $x_2$, ..., $x_p$.

$$\text{Set } g(x_1,\ldots,x_p) = \text{constant}\bullet\exp\{-\frac{1}{2\sigma_0^2}(\delta_1\Sigma_1^p x_i^2 + \delta_2 \Sigma_{i\neq j}^p \Sigma x_i x_j)\}, m(x_1,\ldots,x_p) \qquad (2)$$

where $m(\bullet)$ is exchangeable in its arguments.

Now we choose $\delta_1$ and $\delta_2$ in such a way that

$$\{x_0 - \alpha(\Sigma_1^p x_i)\}^2 + \delta_1\Sigma_1^p x_i^2 + \delta_2 \Sigma_{i\neq j}\Sigma x_i x_j = A\,\Sigma_0^p x_i^2 + B\,\Sigma_{\substack{i\neq j\\0}}^p \Sigma x_i x_j$$

for some A and B with $A > B$ and $A + pB > 0$. Equating both sides, $A = 1 = \alpha^2 + \delta_1$ and $B = -\alpha = \alpha^2 + \delta_2$. This gives $\delta_1 = 1 - \alpha^2$ and $\delta_2 = -\alpha - \alpha^2$ where $\alpha = \rho/\{1+(p-1)\rho\}$. Now $-1/p < \rho < 1 \Rightarrow -1 < \alpha < 1/p$ and, hence, $\delta_1 > 0$, $\delta_1 - \delta_2 = 1-\alpha > 0$, $\delta_1 + (p-1)\delta_2 = 1 - (p-1)\alpha - p\,\alpha^2 > 0$. Further, $A - B = 1+\alpha>0$, $A+pB = 1-p\alpha = 1 - \frac{p\rho}{1+(p-1)\rho} = (1-\rho)/\{1+(p-1)\rho\} > 0$. Thus we have observed that f has a representation

f = (a (p+1)-variate normal density with p.d. intra-class covariance structure).

$$m(x_1, x_2, \ldots, x_p) .$$

In view of exchangeability of f, it now follows that $m(\cdot)$ is necessarily a constant. Thus we have established that P1 and P3 together imply normality of X.

Counter example

Let U be an uniformly distributed rv in (0,1), and

$$Y_i \overset{iid}{\sim} N(0,1) , \quad i=1,2,3,4,5 .$$

Suppose
$$X_1 = \sqrt{U}\,Y_1 + \sqrt{1-U}\,Y_2$$
$$X_2 = \sqrt{U}\,Y_3 + \sqrt{1-U}\,Y_2$$
$$X_3 = \sqrt{U}\,Y_4 + \sqrt{1-U}\,Y_5 .$$

So that $X_i \overset{d}{=} N(0,1)$, $i=1,2,3$.

Now $\phi_{X_3|X_1,X_2}(t) = E(e^{itX_3}|X_1,X_2) = E\,E(e^{itX_3}|X_1, X_2, U) = e^{-t^2/2}$

implying thereby that $X_3|X_1, X_2 \sim N(0,1)$.

However,

$$\phi_{X_1,X_2}(t_1,t_2) = E\,E(e^{it_1(\sqrt{U}\,Y_1 + \sqrt{1-U}\,Y_2) + it_2(\sqrt{U}\,Y_3 + \sqrt{1-U}\,Y_2)}|U)$$

$$= E(e^{-t_1^2U/2 - t_2^2U/2 - (t_1+t_2)^2(1-U)/2})$$

$$= e^{-(t_1+t_2)^2/2}\;E(e^{t_1t_2U})$$

$$= \frac{e^{t_1t_2} - 1}{t_1t_2}\;e^{-(t_1+t_2)^2/2} .$$

Thus $X_1$, $X_2 \not\sim$ BVN and hence $X_1$, $X_2$, $X_3 \not\sim$ MVN .

Remark: The above counter-example also shows that exchangeability in the joint marginal distribution of $(X_1,X_2)$ is not helpful again in settling the normality of $(X_1, X_2, X_3)$.

## REFERENCE

Ahsanullah, M. (1985). Some characterizations of the bivariate normal distribution. *Metrika*, **32**, 215-218.