ON THE ASYMPTOTIC OPTIMALITY OF UI-LMP RANK TESTS FOR RESTRICTED ALTERNATIVES *

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SUMMARY. For testing against restricted alternatives, the optimality of UI-LMP rank tests is characterized in the light of the asymptotic most stringency and somewhere most powerful character. Under additional regularity conditions, this characterization is also extended to the multivariate case. Some allied asymptotic optimality results on the UI-LMP rank tests are considered in the same vein.

1. INTRODUCTION

Host of nonparametric tests against ordered, orthant and other forms of restricted alternatives have mostly been considered on ad hoc basis. The union-intersection (UI-) principle of Roy (1953) has also been effectively employed by Chatterjee and De(1972,1974) and Chinchilli and Sen(1981a,b), among others. These authors have studied the (asymptotic) power superiority of the UI-rank tests to their global versions. However, these were done mostly under rather restrictive conditions on the associated covariance matrix and noncentrality vector, leaving open the possibility for relaxation of some of these conditions. Sen (1982) incorporated the theory of locally most powerful (LMP-) rank tests in the UI-setup . However, the resulting tests may not have the asymptotic optimality in a conventional sense. This naturally raises the question on the asymptotic optimality of UI-LMP (and other related UI-) rank tests in a well defined manner. Our main contention is to provide an affirmative answer in the light of asymptotic most stringency and somewhere most powerful character of these tests.

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Section 2 deals with the basic regularity conditions, preliminary notions and a general formulation of such restricted alternatives where UI-rank tests have been considered. Asymptotically optimal rank tests for restricted alternatives in the univariate case are derived in Section 3. This characterization is extended to the multivariate case in Section 4. Some further remarks on the asymptotic optimality of UI-rank tests are appended in the concluding section.

2. PRELIMINARY NOTIONS

Let $X_1,\ldots,X_N$ be independent random variables (r.v.) with continuous distribution functions (d.f.) $F_1,\ldots,F_N$, respectively, all defined on $E = (-\infty,\infty)$. Suppose that F_i has an absolutely continuous probability density function (pdf) f_i, where

$$f_i(x) = f(x;\underset{\sim}{\zeta}_{Ni}) \text{ where } \underset{\sim}{\zeta}_{Ni} = (\beta_1 c_{Ni1},\ldots,\beta_q c_{Niq}),\ i=1,\ldots,N, \tag{2.1}$$

the $\underset{\sim}{c}_{Ni} = (c_{Ni1},\ldots,c_{Niq})'$ are q-vectors of known constants, not all equal , and $\underset{\sim}{\beta} = (\beta_1,\ldots,\beta_q)'$ is a q-vector of unknown parameters; $q \geq 1$. [To be more precise, we should have started with a triangular scheme of r.v.'s and d.f.'s. However, for the sake of notational simplicity, this refinement will be suppressed.]

In a nonparametric testing problem, one is usually interested in the null hypothesis (H_0) of the homogeneity of the F_i, i.e.,

$$H_0 : F_1=\ldots=F_N = F \text{ (unknown) or, in (2.1), } \underset{\sim}{\beta} = \underset{\sim}{0} . \tag{2.2}$$

In testing for H_0 against a global alternative, we allow $\underset{\sim}{\beta}$ to be arbitrary. On the other hand, in testing for H_0 against a restricted alternative, we need to pose the parameter space (of $\underset{\sim}{\beta}$) in a more structured way. We assume that there is a positively homogeneous set Γ , such that under the alternative

$$H^*: \underset{\sim}{\beta} \in \Gamma = \{ \underset{\sim}{\beta} \in E^q: \underset{\sim}{\mu}^* = \underset{\sim}{A}\,\underset{\sim}{\beta} \geq \underset{\sim}{0} \text{ , where } \underset{\sim}{A} \text{ is of order } a_o \times q \text{ and is of full rank } a_o (1 \leq a_o \leq q) \}. \tag{2.3}$$

The simplest form of Γ relates to the orthant alternative: $\underset{\sim}{\beta} \geq \underset{\sim}{0}$ i.e., $\Gamma = E^{+q}$, the positive orthant in E^q. Other notable forms of Γ relate to the ordered alternative: $\beta_1 \leq \ldots \leq \beta_q$, with at least one strict inequality, umbrella alternatives, tree alternatives, loop alternatives etc. These alternatives are all special cases of Γ , defined by (2.3), and often, in practice, they appear to be more realistic than the global alternative that $\underset{\sim}{\beta} \neq \underset{\sim}{0}$. Hence, we confine ourselves to a general form of restricted alternatives Γ, and study the asymptotic optimality of UI-LMPR tests for this problem.

Generally, for the nonparametric hypothesis testing problem , rank tests are distribution-free (and hence, valid for a broad class of d.f.'s). However, they are based on the use of some scores , suitably chosen to maintain the high power property for an important subclass of such d.f.'s. In testing for H_0 against a global alternative, a convenient way to choose such scores is to maximize the local (asymptotic) power of the test for a specific family of d.f.'s. In the literature, this is referred to as the locally most powerful rank (LMPR-) test. A very elegant treatment of LMPR tests is given in Hájek and Šidák (1967). We shall find it convenient to adapt their notations for the restricted alternative case too.

First, note that for the orthant alternative problem, we have $\Gamma = \{\underset{\sim}{\beta}: \underset{\sim}{\beta} \geq \underset{\sim}{0}\}$.
For any given $\underset{\sim}{\gamma} = (\gamma_1,\ldots,\gamma_q)$ (with nonnegative elements), we denote by

$$H^*_{\underset{\sim}{\gamma}} : \underset{\sim}{\beta} = \delta \underset{\sim}{\gamma} \text{ , for some arbitrary and positive } \delta . \tag{2.4}$$

Then, we have

$$H^* = \bigcup_{\underset{\sim}{\gamma} \in \Gamma} H^*_{\underset{\sim}{\gamma}} . \tag{2.5}$$

Also, for every $\varepsilon > 0$, we let

$$H^{*\varepsilon}_{\underset{\sim}{\gamma}} = \{H^*_{\underset{\sim}{\gamma}} ; 0 < \delta \leq \varepsilon\} \quad \text{and} \quad H^{*\varepsilon} = \bigcup_{\underset{\sim}{\gamma} \in \Gamma} H^{*\varepsilon}_{\underset{\sim}{\gamma}} \tag{2.6}$$

The basic idea is to incorporate the LMPR test theory for a given $\underset{\sim}{\gamma}$, and then to use the UI-principle to have the test constructed for the entire class in (2.5). Towards the formulation of such UI-LMPR tests, we make the following assumptions:

[A1] For every N, we define $\bar{\underset{\sim}{c}}_N = N^{-1}\Sigma_{i=1}^N \underset{\sim}{c}_{Ni}$ and let

$$\underset{\sim}{C}_N = ((C_{Nkk'})) = \Sigma_{i=1}^N (\underset{\sim}{c}_{Ni} - \bar{\underset{\sim}{c}}_N)(\underset{\sim}{c}_{Ni} - \bar{\underset{\sim}{c}}_N)' . \tag{2.7}$$

We assume that there exists a positive definite (p.d.) matrix $\underset{\sim}{C}$, such that as N increases, $N^{-1}\underset{\sim}{C}_N$ converges to $\underset{\sim}{C}$, and further

$$\lim_{N\to\infty} \{ \max_{1\leq i\leq N} (\underset{\sim}{c}_{Ni} - \bar{\underset{\sim}{c}}_N)' \underset{\sim}{C}_N^{-1} (\underset{\sim}{c}_{Ni} - \bar{\underset{\sim}{c}}_N) \} = 0 . \tag{2.8}$$

[A2] Let us define $f_i(x;\delta\underset{\sim}{\gamma})$ as in (2.1) with $\underset{\sim}{\beta} = \delta\underset{\sim}{\gamma}$, $i \geq 1$. Then, (i) for every $i(\geq 1)$, $f_i(x;\delta\underset{\sim}{\gamma})$ is absolutely continuous for almost all x and $\underset{\sim}{\gamma} \in \Gamma$,so that if we let $\dot{f}_{i\theta}(x;\theta) = (\partial/\partial\theta) f_i(x;\theta)$ and $\dot{f}_{i\gamma}(x;\delta\gamma) = (\partial/\partial\delta) f_i(x;\delta\gamma)$, then, for $\theta = \delta\gamma$, $\dot{f}_{i\gamma}(x;\delta\gamma) = \gamma' \dot{f}_{i\theta}(x;\delta\gamma)$, $\gamma \in \Gamma$, $x \in E$. (ii) For almost all x and γ, the limit $\dot{f}_{i\gamma}(x;0) = \lim_{\delta\downarrow 0} \delta^{-1}[f_i(x;\delta\gamma) - f_i(x;0)]$ exists. (iii) For every $\gamma \in \Gamma$, $\lim_{\delta\downarrow 0} \int_{-\infty}^{\infty} |\dot{f}_{i\gamma}(x;\delta\gamma)|dx = \int_{-\infty}^{\infty} |\dot{f}_{i\gamma}(x;0)|dx$ is finite. (iv) The largest characteristic root of $I(f) = E_\theta[(\partial/\partial\theta)\log f(X;\theta)(\partial/\partial\theta')\log f(X;\theta)]$ is finite.

Consider now the score generating function (vector) :

$$f^*(x) = \{f(x;0)\}^{-1} \dot{f}_\theta(x;0) = (f_1^*(x),\ldots,f_q^*(x))', \quad x \in E, \tag{2.9}$$

and let $a_N(.) = (a_{N1}(.),\ldots,a_{Nq}(.))'$ with

$$a_{Nj}(i) = E\{f_j^*(U_{Ni}^*)\}, \quad i=1,\ldots,N, \quad j=1,\ldots,q, \tag{2.10}$$

where $U_{N1}^*,\ldots,U_{NN}^*$ are the ordered random variables of a sample of size N from the d.f. F (for which the p.d.f. is $f(x;0)$). Finally, let R_{Ni} be the rank of X_i among $X_1,\ldots,X_N$, for $i=1,\ldots,N$. Then, we define the vector of linear rank statistics

$$T_{Nj} = N^{-\frac{1}{2}} \Sigma_{i=1}^N c_{Nij}\, a_{Nj}(R_{Ni}) = \Sigma_{i=1}^N c_{Nij}^* a_{Nj}(R_{Ni}), \quad j=1,\ldots,q;$$
$$T_N = (T_{n1},\ldots,T_{Nq})' . \tag{2.11}$$

For every (fixed) $\gamma \in \Gamma$, on letting $T_N(\gamma) = \gamma' T_N$, and proceeding as in Hájek and Šidák (1967) and Sen (1982), it follows that the test with the critical region $W_N(\gamma) = \{(X_1,\ldots,X_N) : T_N(\gamma) \geq k_N(\alpha)\}$ (with suitable critical value $k_N(\alpha)$) is the LMPR test for H_0 against H_γ^*, at the level of significance α ($0 < \alpha < 1$). Note that for every $\gamma \in \Gamma$, $E[T_N(\gamma|H_0] = 0$ and $Var[T_N(\gamma)|H_0] = \gamma' M_N \gamma$, where $M_N = ((v_{Njj'} N^{-1} C_{Njj'})) = ((v_{Njj'} C_{Njj'}^*))$ with $V_N = ((v_{Njj'}))$ defined by

$$v_{Njj'} = (N-1)^{-1} \Sigma_{i=1}^N a_{Nj}(i) a_{Nj'}(i), \quad \text{for } j,j' = 1,\ldots,q. \tag{2.12}$$

Thus, if we let

$$T_N^*(\gamma) = \{\gamma' M_N \gamma\}^{-\frac{1}{2}} T_N(\gamma), \quad \text{for } \gamma \in \Gamma, \tag{2.13}$$

then, in accordance with the UI-principle of Roy (1953), as in Sen (1982), we may consider the following UI-LMP rank statistic

$$Q_N = \sup\{ T_N^*(\gamma) : \gamma \in \Gamma \}. \tag{2.14}$$

As has been mentioned earlier, we consider the case of Γ of the form in (2.3), so that for the computation of Q_n in (2.14), we need to maximize $\gamma' T_N$ subject to $\mu = A\gamma \geq 0$ and $\gamma' M_N \gamma = 1$. If we let $h(\gamma) = -\gamma' T_N$, $h_1(\gamma) = -A\gamma$ and $h_2(\gamma)$

$= \underset{\sim}{\gamma}'\underset{\sim}{M}_N\underset{\sim}{\gamma} - 1$, then incorporating the Kuhn-Tucker-Lagrange (K.T.L.) point formula theorem, we arrive at the following solution. Let

$$\underset{\sim}{U}_N = \underset{\sim}{A}\,\underset{\sim}{M}_N^{-1}\underset{\sim}{T}_N \quad \text{and} \quad \underset{\sim}{\Delta}_N = \underset{\sim}{A}\,\underset{\sim}{M}_N^{-1}\underset{\sim}{A}' \tag{2.15}$$

and also let $\mathcal{J}$ be any subset of $P = \{1,\ldots,a_o\}$ and $\mathcal{J}'$ be its complement. For each of the 2^{a_o} set $\mathcal{J}$, we partition (following reorganization, if necessary) $\underset{\sim}{U}_N$ and $\underset{\sim}{\Delta}_N$ as

$$\underset{\sim}{U}_N = \begin{pmatrix} \underset{\sim}{U}_{N(\mathcal{J})} \\ \underset{\sim}{U}_{N(\mathcal{J}')} \end{pmatrix} \begin{matrix} k(\mathcal{J}) \\ k(\mathcal{J}') \end{matrix} \quad \text{and} \quad \underset{\sim}{\Delta}_N = \begin{pmatrix} \underset{\sim}{\Delta}_{N(\mathcal{J}\mathcal{J})} & \underset{\sim}{\Delta}_{N(\mathcal{J}\mathcal{J}')} \\ \underset{\sim}{\Delta}_{N(\mathcal{J}'\mathcal{J})} & \underset{\sim}{\Delta}_{N(\mathcal{J}'\mathcal{J}')} \end{pmatrix} \tag{2.16}$$

Also, for each $\mathcal{J} : \emptyset \subseteq \mathcal{J} \subseteq P$, we let

$$\underset{\sim}{U}_{N(\mathcal{J}:\mathcal{J}')} = \underset{\sim}{U}_{N(\mathcal{J})} - \underset{\sim}{\Delta}_{N(\mathcal{J}\mathcal{J}')}\underset{\sim}{\Delta}^{-1}_{N(\mathcal{J}'\mathcal{J}')}\underset{\sim}{U}_{N(\mathcal{J}')} , \tag{2.17}$$

$$\underset{\sim}{\Delta}_{N(\mathcal{J}\mathcal{J}:\mathcal{J}')} = \underset{\sim}{\Delta}_{N(\mathcal{J}\mathcal{J})} - \underset{\sim}{\Delta}_{N(\mathcal{J}\mathcal{J}')}\underset{\sim}{\Delta}^{-1}_{N(\mathcal{J}'\mathcal{J}')}\underset{\sim}{\Delta}_{N(\mathcal{J}'\mathcal{J})} . \tag{2.18}$$

Then [viz., Sen (1982)], we have

$$Q_N^2 = \underset{\sim}{T}_N'\{\underset{\sim}{M}_N^{-1} - \underset{\sim}{M}_N^{-1}\underset{\sim}{A}'\underset{\sim}{\Delta}_N^{-1}\underset{\sim}{A}\,\underset{\sim}{M}_N^{-1}\}\,\underset{\sim}{T}_N \; + \sum_{\emptyset\subseteq\mathcal{J}\subseteq P} \{\underset{\sim}{U}_{N(\mathcal{J}:\mathcal{J}')}'\underset{\sim}{\Delta}^{-1}_{N(\mathcal{J}\mathcal{J}:\mathcal{J}')}\underset{\sim}{U}_{N(\mathcal{J}:\mathcal{J}')}\}\, I(\underset{\sim}{U}_{N(\mathcal{J}:\mathcal{J}')} > \underset{\sim}{0})\, I(\underset{\sim}{\Delta}^{-1}_{N(\mathcal{J}'\mathcal{J}')}\underset{\sim}{U}_{N(\mathcal{J}')} \underset{\sim}{\leq 0}), \tag{2.19}$$

where I(B) stands for the indicator function of the set B. Our main interest is to study the asymptotic optimality of this UI-LMPR test.

3. ASYMPTOTIC OPTIMALITY OF THE UI-LMPR TEST

As a first step, we shall study the asymptotic power equivalence of the UI-LMPR test and the UI-LR (likelihood ratio) test for local alternatives. Towards this, we define $\underset{\sim}{S}_N = (S_{N1},\ldots,S_{Nq})'$ with

$$\underset{\sim}{S}_{Nj} = \Sigma_{i=1}^N \, c^*_{Nij} \, f^*_j(X_i) , \text{ for } j = 1,\ldots,q . \tag{3.1}$$

Note that the summands in (3.1) are independent, so that the central limit theorem (in the multivariate case) can easily be invoked to conclude that for every $\underset{\sim}{\gamma} \in \Gamma$,

$$S_N^*(\underset{\sim}{\gamma}) = (\underset{\sim}{\gamma}'\tilde{\underset{\sim}{\Sigma}}_N\underset{\sim}{\gamma})^{-\frac{1}{2}}\underset{\sim}{\gamma}'\underset{\sim}{S}_N \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) \quad [\text{under } H_0] , \tag{3.2}$$

where $\tilde{\underset{\sim}{\Sigma}}_N = ((C^*_{Njj'}\varphi_{jj'}))$ with $\underset{\sim}{I}(f) = ((\varphi_{jj'}))$. If we let

$$Q_N^o = \sup\{ S_N^*(\underset{\sim}{\gamma}) : \underset{\sim}{\gamma} \in \Gamma \}, \tag{3.3}$$

then, by virtually repeating the proof of Sen(1982), we obtain that as $N \to \infty$,

$$Q_N - Q_N^o \xrightarrow{P} 0 , \text{ under } H_0 \text{ as well as a sequence of contiguous alternatives } \{ K_N : \underset{\sim}{\beta} = N^{-\frac{1}{2}}\underset{\sim}{\gamma} ; \underset{\sim}{\gamma} \in \Gamma \} . \tag{3.4}$$

Now, Q_N^o corresponds to the UI-version of the efficient scores statistic, and is asymptotically (power-) equivalent to the UI-LR test statistic (for restricted alternatives) , under H_0 as well as $\{K_N\}$. As such, we conclude that the UI-LMPR test based on the specific scores $a_N(.)$ in (2.10) is asymptotically power-equivalent (under $\{K_N\}$) to the UI-LR test, and thus the two tests share the same asymptotic optimality properties. Note that $C_N^* \to C^*$ (p.d.) as $N \to \infty$, so that there exist p.d. matrices M and Δ, such that $M_N \xrightarrow{p} M$ and $\Delta_N \xrightarrow{p} \Delta$,as $N \to \infty$. As such, defining $\mu_{J:J'}$ and $\Delta_{JJ:J'}$ as in (2.18) and (2.19), we may conclude that for every $\gamma \in \Gamma$, there is only one $J : \emptyset \subseteq J \subseteq P$, such that $\mu_{J:J'} > 0$ and $\Delta_{J'J'}^{-1}\mu_{J'} \leq 0$. Then the UI-LMPR test has asymptotically the best constant power over the ellipsoid in the parameter space specified by $\gamma'\{M^{-1}-M^{-1}A'\Delta^{-1}AM^{-1}\}\gamma + \mu'_{J:J'}\Delta_{JJ:J'}\mu_{J:J'}$ = constant, and is also the asymptotically most stringent test against the restricted domain Γ. However, it is generally not the asymptotically most powerful test against the entire domain of alternatives in Γ; in fact , there may not be any such optimal test for restricted alternatives. Towards our main objective of locating a subspace of Γ for which the UI-LMPR test has asymptotically the best power, we consider the following.

THEOREM 3.1. For each J *(* $\emptyset \subseteq J \subseteq P$*), let* $\Gamma_J = \{ \mu = A\gamma \;\varepsilon\; E^{+a_o} , \mu_{J:J'} > 0 \}$ *and* $\Gamma_o = \bigcap_{\emptyset \subseteq J \subseteq P} \Gamma_J$ *, where* E^{+a_o} *is* a_o*-dimensional positive orthant space. For testing (2.2) against (2.3) (under local alternatives in (3.4)), the UI-LMPR test in (2. 19) is asymptotically most stringent for* Γ *and is asymptotically most powerful for* Γ_o *, at the respective significance level* α *.*

<u>Proof</u>. Consider a r.v. W having a q-variate normal d.f. with mean vector $M\gamma$ and a (known p.d.) dispersion matrix M . Let $Z = AM^{-1}W$, and define

$$Q^* = W'(M^{-1}-M^{-1}A'\Delta^{-1}AM^{-1})W + \sum_{\emptyset \subseteq J \subseteq P} \{Z'_{J:J'}\Delta_{JJ:J'}^{-1}Z_{J:J'}\} I(Z_{J:J'} > 0). I(\Delta_{J'J'}^{-1}Z_{J'} \leq 0), \qquad (3.5)$$

where the partitioned matrices and vectors are defined as in (2.16)-(2.19). Then, by virtue of (3.2)-(3.4), it suffices to show that for testing $H_0: \gamma = 0$ against $H_1: \mu \geq 0$, the test with the critical region $Q^* \geq k_\alpha$ is the most stringent one for Γ and most powerful for Γ_o, where Γ and Γ_o are defined in the theorem. For this, it suffices to show that the shortcoming of Q* is equal to 0 for some $\underset{\sim}{\gamma}$ inside the positive orthant Γ (excluding the null point). Towards this, we show that Q* is power equivalent to the most stringent test for testing H_0: $\underset{\sim}{\gamma} = \underset{\sim}{0}$ against H_1^*: $\underset{\sim}{B}\underset{\sim}{\mu} \geq \underset{\sim}{0}$, where $\underset{\sim}{B}$ is a given q×q matrix of real constants. Since $\underset{\sim}{\Delta}$ is p.d., there exists a upper triangular matrix $\underset{\sim}{B}$ such that $\underset{\sim}{B}\underset{\sim}{\Delta}\underset{\sim}{B}' = \underset{\sim}{D}$ (say) is a diagonal matrix. For each J ($\emptyset \subseteq J \subseteq P$), we take $\underset{\sim}{B} = \underset{\sim}{B}_J = \begin{pmatrix} \underset{\sim}{I} & -\underset{\sim}{\Delta}_{JJ'}\underset{\sim}{\Delta}^{-1}_{J'J'} \\ \underset{\sim}{0} & \underset{\sim}{I} \end{pmatrix}$, so that $\underset{\sim}{D} = \underset{\sim}{D}_J = \begin{pmatrix} \underset{\sim}{D}_{JJ} & \underset{\sim}{D}_{JJ'} \\ \underset{\sim}{D}_{J'J} & \underset{\sim}{D}_{J'J'} \end{pmatrix} = \begin{pmatrix} \underset{\sim}{\Delta}_{JJ:J'} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{\Delta}_{J'J'} \end{pmatrix}$. Then we have

$$\underset{\sim}{D}_{JJ:J'} = \underset{\sim}{D}_{JJ} = \underset{\sim}{\Delta}_{JJ:J'} \quad . \tag{3.6}$$

We also let $\underset{\sim}{Y} = \underset{\sim}{B}\underset{\sim}{Z}$ and partition $\underset{\sim}{Y}$ as in (2.15), so that $\underset{\sim}{Y}' = (\underset{\sim}{Y}'_J, \underset{\sim}{Y}'_{J'})$. Then we may rewrite (3.5) as

$$Q^* = \underset{\sim}{W}'(\underset{\sim}{M}^{-1}-\underset{\sim}{M}^{-1}\underset{\sim}{A}'\underset{\sim}{\Delta}^{-1}\underset{\sim}{A}\underset{\sim}{M}^{-1})\underset{\sim}{W} + \sum_{\emptyset\subseteq J\subseteq P} (\underset{\sim}{Y}'_{J:J'}\underset{\sim}{D}^{-1}_{JJ:J'}\underset{\sim}{Y}_{J:J'})\, I(\underset{\sim}{Y}_{J:J'} > \underset{\sim}{0})\, I(\underset{\sim}{D}^{-1}_{J'J'}\underset{\sim}{Y}_{J'} \leq \underset{\sim}{0}) . \tag{3.7}$$

Then the UI-efficient score test for H_0: $\underset{\sim}{\gamma} = \underset{\sim}{0}$ against H_J: $\underset{\sim}{B}_J\underset{\sim}{\mu} > \underset{\sim}{0}$ is given by (3.7). Thus, for each J($\emptyset \subseteq J \subseteq P$), the most powerful property holds for Q* whenever $\underset{\sim}{\gamma}$ lies in $\Gamma \cap \Gamma_J$, so that Q* is most powerful for the region $\bigcap_{\emptyset\subseteq J\subseteq P} \{ \Gamma \cap \Gamma_J \}$ (which is by definition, Γ_o), and hence, the desired result follows. Q.E.D.

Note that in the characterization of the asymptotic optimality of the UI-LMPR test, the UI-efficient score statistic and (3.4) play the basic role. We shall see in the next section that this feature remains true in the multivariate case also

4. MULTIVARIATE GENERALIZATIONS

Suppose that $\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_N$ are independent r.v.'s with continuous (p-variate) d.f.'s $F_1,\ldots,F_N$, respectively, where $p \geq 1$ and, as in (2.1), we assume that

$$F_i(\underset{\sim}{x}) = F(\underset{\sim}{x};\underset{\sim}{\zeta}_{Ni}),\ \underset{\sim}{x} \in E^p\ ;\quad \underset{\sim}{\zeta}_{Ni} = ((\beta_{j\ell}c_{Nij\ell}))_{1\leq j\leq p;1\leq \ell\leq q},\ i=1,\ldots,N, \tag{4.1}$$

where $\underset{\sim}{\beta} = ((\beta_{j\ell}))$ is a p×q matrix of unknown parameters and $\underset{\sim}{c}_{Ni} = ((c_{Nij\ell}))$, i=1,...,N are p×q matrices of known constants. The general multivariate linear model, treated in detail in Chinchilli and Sen (1981a,b) , is a special case of

(4.1) where $\zeta_{Ni} = \beta c_{Ni}$, the c_{Ni} are q-vectors and $F(x;\zeta_{Ni}) = F(x-\zeta_{Ni})$. For the general model in (4.1), the null hypothesis H_0 relates to the homogeneity of the F_i, i.e.,

$$H_0 : F_1 = \ldots = F_N = F \quad (\text{or } \beta = 0), \tag{4.2}$$

and we consider a restricted alternative of the form

$$H^* : \beta \in \Gamma = \{ \beta \in E^{pq} : \mu^* = A(\text{vec}\beta) \geq 0 \}. \tag{4.3}$$

where a_o is a positive integer ($\leq$ pq) and A is of oder $a_o \times pq$; $\text{vec}\beta$ denotes the pq-vector obtained by stacking the rows of β under each other.

For this generalized orthant alternative problem, the test procedure is similar to the one considered in the previous sections. First, we consider a simple null hypothesis H_0 against a simple(local) alternative for which for a fixed $\gamma \in \Gamma$, $\beta = \delta\gamma$, where $\delta > 0$. To obtain LMPR test for this specific case and to incorporate the same in the formulation of UI-LMPR tests, we make the following assumptions.

[A1] F is absolutely continuous with continuous density function $f(x) = f(x;\Theta)$, $\Theta = (\theta_1, \ldots, \theta_p)' \in \Gamma$, which satisfies the conditions : (a) For every i($1 \leq i \leq N$), $f_i(x;\delta\gamma)$ is absolutely continuous for almost all x and $\gamma \in \Gamma$, so that if we let $\dot{f}_{i\Theta}(x;\Theta) = (\partial/\partial\Theta) f_i(x;\Theta)$ and $\dot{f}_{i\gamma}(x;\delta\gamma) = (\partial/\partial\delta) f_i(x;\delta\gamma)$, then for $\Theta = \delta\gamma$, $\dot{f}_{i\gamma}(x;\delta\gamma) = (\text{vec}\gamma)'\text{vec}\, \dot{f}_{i\Theta}(x;\delta\gamma)$, $\forall\, x \in E^p$ and $\gamma \in \Gamma$. (b) Conditions (ii) and (iii) in [A2] of Section 2 hold. (c) Let

$$I(f) = E_0[(\partial/\partial\text{vec}\Theta)\log f(X;\Theta)(\partial/\partial(\text{vec}\Theta)')\log f(X;\Theta)] \tag{4.4}$$

where E_0 denotes the expectation under the null hypothesis. Then, $I(f)$ is p.d.

[A2] For the p-variate pdf $f(x;\Theta)$, we denote the conditional pdf of the jth coordinate , given the others, by $f_j(x_j;\Theta \mid x)$, and let

$$g_j(x_j;\Theta \mid x) = (\partial/\partial\theta_j)\log f(x;\Theta) = (\partial/\partial\theta_j)\log f_j(x_j;\Theta \mid x), \tag{4.5}$$

for j=1,...,p. Also, let $f_{[j]}$ denote the marginal pdf of the jth coordinate, and

$$f^*_{[\theta_j]}(x_j;\theta_j) = (\partial/\partial\theta_j)\log f_{[j]}(x_j;\theta_j) \text{ , } j=1,\ldots,p. \tag{4.6}$$

Note that both the g_j and $f^*_{[\theta_j]}$ are q-vectors. We denote the corresponding pq-vectors by $g(x;\Theta)$ and $f^*(x;\Theta)$, respectively, and assume that there exists a p.d. matrix L^*, such that

$$g(x;0) = L^* f^*(x;0) \text{ , for almost all } x \in E^p . \tag{4.7}$$

For multivariate linear models, (4.7) holds for elliptically symmetric d.f.'s.

[A3] For the $\underset{\sim}{c}_{Ni}$, [A1] in Section 2 holds (where we use the vec $\underset{\sim}{c}_{Ni}$, if needed) and assume

$$\sup_{1\le i\le N}\ \sup_{1\le j\le p}\ \sup_{1\le \ell\le q}\ |c_{Nij\ell}| = 0\left[\sqrt{\frac{\log\log N}{N}}\right] \tag{4.8}$$

[A4] For each j (j=1,...,p) and ℓ (ℓ=1,...,q), $f^*_{[j]\ell}(x;\underset{\sim}{0})$ is differentiable with respect to x on (a_j,b_j), where $-\infty \le a_j \le -\sup\{x;\ F_{[j]}(x;\underset{\sim}{0}) = 0\}$, $\infty \ge b_j = \inf\{x;\ F_{[j]}(x;\underset{\sim}{0}) = 1\}$, and (i) $f_{[j]}(x;\underset{\sim}{0}) > 0$ on its support (a_j,b_j), (ii) let $f^{*\prime}_{[j]\ell}(x;\underset{\sim}{0}) = \frac{\partial}{\partial x} f^*_{[j]\ell}(x;\underset{\sim}{0})$, then assume for some $r \in (0,\frac{1}{2})$

$$\sup_{1\le i\le N}\ \sup_{1\le j\le p}\ |f^{*\prime}_{[j]\ell}(X_j,\underset{\sim}{0})| \overset{a.s.}{=} o(N^r) \tag{4.9}$$

Further, let

$$E_0\{\underset{\sim}{f}^*(\underset{\sim}{X}_i;\underset{\sim}{\theta})(\underset{\sim}{f}^*(\underset{\sim}{X}_i;\underset{\sim}{\theta}))'\} = \underset{\sim}{I}^* = ((\varphi^*_{mm'}))_{1\le m;m'\le pq} \tag{4.10}$$

and define the bolck diagonal matrix of $\underset{\sim}{I}^*$ by

$$\underset{\sim}{B}(\underset{\sim}{I}^*) = \mathrm{Diag}\Big[E_0\{\underset{\sim}{f}^*_{[1]}(X_1;\underset{\sim}{\theta}_1)(\underset{\sim}{f}^*_{[1]}(X_1;\underset{\sim}{\theta}_1))'\},\ \ldots, E_0\{\underset{\sim}{f}^*_{[p]}(X_p;\underset{\sim}{\theta}_p)(\underset{\sim}{f}^*_{[p]}(X_p;\underset{\sim}{\theta}_p))'\}\Big] \tag{4.11}$$

THEOREM 4.1 For the family of distributions $f(x;\underset{\sim}{\theta})$, under the assumptions [A1] and [A2], then $\underset{\sim}{L} = \underset{\sim}{B}(\underset{\sim}{I}^*)\underset{\sim}{I}^{*-1}$.

<u>proof</u>. Let $\hat{\underset{\sim}{\theta}}{}^0_N = ((\hat{\theta}^0_{Nj\ell}))_{j=1,\ldots,p;\ell=1,\ldots,q}$ be the maximum likelihood estimator (M.L.E.) of $\underset{\sim}{\theta}$ (parameter of multivariate density function f) and $\hat{\underset{\sim}{\theta}}{}^{*0}_{Nj} = (\hat{\theta}^{*0}_{Nj1},\ldots,\hat{\theta}^{*0}_{Njq})'$, (j=1,...,p) be the M.L.E. of $\underset{\sim}{\theta}_j$ (parameter of the jth coordinate density function $f_{[j]}$). Under some regularity conditions, $\hat{\underset{\sim}{\theta}}{}^0_N$ and $\hat{\underset{\sim}{\theta}}{}^{*0}_N$ may then be solved by the likelihood equations $\Sigma_{i=1}^N \underset{\sim}{g}(\underset{\sim}{X}_i;\underset{\sim}{\theta}) = \underset{\sim}{0}$ and $\Sigma_{i=1}^N \underset{\sim}{f}^*(\underset{\sim}{X}_i;\underset{\sim}{\theta}) = \underset{\sim}{0}$, respectively. Note that $\hat{\underset{\sim}{\theta}}{}^{*0}_N = (\hat{\underset{\sim}{\theta}}{}^{*0\prime}_{N1},\ldots,\hat{\underset{\sim}{\theta}}{}^{*0\prime}_{Np})$. Since $\underset{\sim}{L}$ is assumed to be nonsingular, $\Sigma_{i=1}^N \underset{\sim}{g}(\underset{\sim}{X}_i;\underset{\sim}{\theta}) = \underset{\sim}{0}$ iff $\Sigma_{i=1}^N \underset{\sim}{f}^*(\underset{\sim}{X}_i;\underset{\sim}{\theta}) = \underset{\sim}{0}$. Hence

$$\hat{\underset{\sim}{\theta}}_N = \text{vec } \hat{\underset{\sim}{\theta}}{}^0_N = \hat{\underset{\sim}{\theta}}{}^*_N = \text{ vec } \hat{\underset{\sim}{\theta}}{}^{*0}_N . \tag{4.12}$$

Under the regularity conditions of Wald (1943), $\hat{\underset{\sim}{\theta}}{}^0_N$ is a consistent estimator, so that the usual expansion of the scores $\Sigma_{i=1}^N \underset{\sim}{g}(\underset{\sim}{X}_1;\ \hat{\underset{\sim}{\theta}}{}^0_N)$ and $\Sigma_{i=1}^N\ \underset{\sim}{f}^*(\underset{\sim}{X}_i;\ \hat{\underset{\sim}{\theta}}{}^{*0}_N)$ around $\underset{\sim}{\theta}$ [under H_0] and the use of the weak laws of large numbers lead to

$$N^{\frac{1}{2}}(\ \hat{\underset{\sim}{\theta}}_N - \underset{\sim}{0}\ )\{1 + \underset{\sim}{o}_p(1)\} = \underset{\sim}{I}^{-1}(f)\{\ N^{-\frac{1}{2}}\ \Sigma_{i=1}^N \underset{\sim}{g}(\underset{\sim}{X}_i;\underset{\sim}{0})\ \}\ , \tag{4.13}$$

$$N^{\frac{1}{2}}(\ \hat{\underset{\sim}{\theta}}{}^*_N - \underset{\sim}{0}\ )\{1 + \underset{\sim}{o}_p(1)\} = \underset{\sim}{B}^{-1}(\underset{\sim}{I}^*)\{\ N^{-\frac{1}{2}}\ \Sigma_{i=1}^N \underset{\sim}{f}^*(\underset{\sim}{X}_i;\underset{\sim}{0})\}\ , \tag{4.14}$$

and these two equations lead us to

$$\underset{\sim}{L}\underset{\sim}{B}(\underset{\sim}{I}^*) = \underset{\sim}{I}(f) = \underset{\sim}{L}\underset{\sim}{I}^*\underset{\sim}{L}' . \tag{4.15}$$

This completes the proof of the theorem.

Let R_{ij} be the rank of X_{ij} among the set $X_{1j},X_{2j},\ldots,X_{Nj}$, for i=1,...,N, and j = 1,...,p, and let $\underset{\sim}{R}_N$ be the p×N rank collection matrix and $\underset{\sim}{R}^*_N$ be the p×N matrix formed by permuting the columns of $\underset{\sim}{R}_N$ in such a way that the top row is in the natural order; it is termed the reduced rank collection matrix. For any given $\underset{\sim}{R}^*_N$, let $S(\underset{\sim}{R}^*_N)$ be the set of N! possible rank collection matrices which can be reduced to $\underset{\sim}{R}^*_N$ by column permutations. Then, by virture of the Chatterjee-Sen (1964) rank permutation principle, we have

$$P\{\ \underset{\sim}{R}_N = \underset{\sim}{r}\ |\ \ S(\underset{\sim}{R}^*_N)\ ,\ H_0\ \} = (N!)^{-1}\ ,\ \ \forall\ \underset{\sim}{r} \in S(\underset{\sim}{R}^*_N)\ . \tag{4.16}$$

We denote by P_N the permutational probability measure generated by (4.16). Also, for every j (=1,...,p), working with the $f^*_{[\underset{\sim}{\theta}_j]}(\cdot;\underset{\sim}{0})$, we define the scores $\underset{\sim}{b}_N(i)$, i=1,...,N, as in (2.10)[with F being replaced by $F_{[j]}$]; note that

$$\underset{\sim}{b}_{Nj}(k) = (b_{Nj1}(k),\ldots,b_{Njq}(k))',\ \text{for } k=1,\ldots,N \text{ and } j=1,\ldots,p. \tag{4.17}$$

Consider then the p×q matrix $\underset{\sim}{T}^0_N = ((\ T^0_{Nj\ell}\ ))$ of linear rank statistics

$$T^0_{Nj\ell} = N^{-\frac{1}{2}}\Sigma_{i=1}^N\ c_{Nij}\ b_{Nj\ell}(R_{ij}) = \Sigma_{i=1}^N c^*_{Nij\ell} b_{Nj\ell}(R_{ij}),\ j=1,\ldots,p;\ \ell=1,\ldots,q. \tag{4.18}$$

For later use, we write

$$\underset{\sim}{T}_N = \text{vec } \underset{\sim}{T}^0_N \text{ and } \underset{\sim}{C}^*_N = N^{-1}\underset{\sim}{C}_N = ((\ c^*_{Nmm'}))_{m,m'=1,\ldots,pq}\ , \tag{4.19}$$

$$\underset{\sim}{B}_{Ni} = (b_{N11}(R_{i1}),\ldots,b_{N1q}(R_{i1}),\ldots,b_{Np1}(R_{ip}),\ldots,b_{Npq}(R_{ip}))', \tag{4.20}$$

for i =1,...,N, and let

$$\underset{\sim}{V}_N = (N-1)^{-1}\ \Sigma_{i=1}^N\ \underset{\sim}{B}_{Ni}\underset{\sim}{B}'_{Ni} = ((\ v_{Nmm'}\ ))_{m,m'=1,\ldots,pq} \tag{4.21}$$

We assume that $\underset{\sim}{V}_N$ is p.d., in probability. Then, as in Puri and Sen(1969), we have

$$E(\underset{\sim}{T}_N \mid \mathcal{P}_N) = \underset{\sim}{0} \quad \text{and} \quad E(\underset{\sim}{T}_N \underset{\sim}{T}_N' \mid \mathcal{P}_N) = \underset{\sim}{\Sigma}_N , \tag{4.22}$$

where

$$\underset{\sim}{\Sigma}_N = ((c^*_{Nmm'} v_{Nmm'})) , \text{ and we let } \underset{\sim}{\Sigma} = \lim_{N\to\infty} E(\underset{\sim}{\Sigma}_N | H_0). \tag{4.23}$$

Further, under the Assumptions [A1] - [A4] ,

$$\underset{\sim}{\Sigma}_N \xrightarrow{P} \underset{\sim}{\Sigma} \quad \text{and} \quad \underset{\sim}{T}_N \xrightarrow[\{K_N\}]{\mathcal{D}} N_{pq}(\underset{\sim}{\Sigma}\underset{\sim}{\lambda} , \underset{\sim}{\Sigma}) , \ \underset{\sim}{\lambda} = \text{vec } \underset{\sim}{\gamma} \tag{4.24}$$

where $\{K_N\}$ is a sequence of contiguous alternatives, defined as in Section 3.

THEOREM 4.2. For the family of distributions in (4.1), under the assumptions [A1] through [A4], the test with the critical region $(\text{vec}\underset{\sim}{\gamma})'\underset{\sim}{L}\underset{\sim}{T}_N \geq k_N(\alpha)$ is locally most powerful rank (LMPR) test for testing H_0 in (4.2) against H_1^*: $\underset{\sim}{\beta} = \delta\underset{\sim}{\gamma}$, $\underset{\sim}{\gamma} \in \Gamma$ and fixed, at the significance level α .

Proof. Note that

$$P\{\underset{\sim}{R}_N = \underset{\sim}{r} | H^*_{\gamma}\} = \int_{\underset{\sim}{R}_N = \underset{\sim}{r}} \cdots \int \prod_{i=1}^{N} f(\underset{\sim}{x}_i ; \underset{\sim}{\zeta}_{Ni}) \, d\underset{\sim}{x}_1 \, d\underset{\sim}{x}_2 \cdots d\underset{\sim}{x}_N$$

$$= \int_{\underset{\sim}{R}_N = \underset{\sim}{r}} \cdots \int \prod_{i=1}^{N} f(\underset{\sim}{x}_i ; \underset{\sim}{0}) d\underset{\sim}{x}_1 \cdots d\underset{\sim}{x}_N$$

$$+ \int_{\underset{\sim}{R}_N = \underset{\sim}{r}} \cdots \int \left[\prod_{i=1}^{N} f(\underset{\sim}{x}_i ; \underset{\sim}{\zeta}_{Ni}) - \prod_{i=1}^{N} f(\underset{\sim}{x}_i ; \underset{\sim}{0}) \right] d\underset{\sim}{x}_1 \cdots d\underset{\sim}{x}_N \tag{4.25}$$

Then, virtually repeating the elegant proof (for the uniparameter case) of Hájek and Šidák (1967,pp.71-72) and extending it to the multiparameter case, we obtain that under [A1] through [A4], as $\delta \downarrow 0$,

$$\int_{\underset{\sim}{R}_N = \underset{\sim}{r}} \cdots \int \prod_{i=1}^{N} f(\underset{\sim}{x}_i ; \underset{\sim}{0}) d\underset{\sim}{x}_1 \cdots d\underset{\sim}{x}_N$$

$$+ \delta \sum_{i=1}^{N} \sum_{j=1}^{p} \sum_{\ell=1}^{q} c_{Nij\ell} \, \gamma_{j\ell} \int_{\underset{\sim}{R}_N = \underset{\sim}{r}} \cdots \int g_{j\ell}(x_{ij} ; \underset{\sim}{0} | \underset{\sim}{x}) \prod_{i=1}^{N} f(\underset{\sim}{x}_i ; \underset{\sim}{0})$$

$$d\underset{\sim}{x}_1 \cdots d\underset{\sim}{x}_N + o(\delta) \tag{4.26}$$

Noting that the gradients in (4.5) have all null expectations, we obtain on summing over all $\underset{\sim}{r} \in S(\underset{\sim}{R}^*_N)$ that as $\delta \downarrow 0$,

$$\sum_{\underset{\sim}{r} \in S(\underset{\sim}{R}_N^*)} P\{\underset{\sim}{R}_N = \underset{\sim}{r} | H_{\gamma}^*\} = N! \int \ldots \int_{\underset{\sim}{R}_N = \underset{\sim}{R}_N^*} \prod_{i=1}^{N} f(\underset{\sim}{x}_i; \underset{\sim}{0}) dx_1 \ldots d\underset{\sim}{x}_N + o(\delta) \quad (4.27)$$

From (4.26) and (4.27), we have

$$P\{\underset{\sim}{R}_N = \underset{\sim}{r} | H_{\gamma}^*\} \Big/ \Big[\sum_{\underset{\sim}{r} \in S(\underset{\sim}{R}_N^*)} P\{\underset{\sim}{R}_N = \underset{\sim}{r} | H_{\gamma}^*\} \Big] \quad (4.28)$$

$$= \frac{1}{N!} \left[1 + \delta \left\{ \frac{\sum_{i=1}^{N} \sum_{j=1}^{p} \sum_{\ell=1}^{q} \int \ldots \int_{\underset{\sim}{R}_N = \underset{\sim}{r}} c_{Nij\ell}\, \gamma_{j\ell}\, g_{j\ell}(\underset{\sim}{x}_{ij}; \underset{\sim}{0} | \underset{\sim}{x}) \prod_{i=1}^{N} dF(\underset{\sim}{x}_i; \underset{\sim}{0})}{\int \ldots \int_{\underset{\sim}{R}_N = \underset{\sim}{R}_N^*} \prod_{i=1}^{N} dF(\underset{\sim}{x}_i; \underset{\sim}{0})} \right\} + o(\delta) \right]$$

where $(N!)^{-1}$ represents the probability ratio under H_0. Hence, using the Neyman-Pearson Lemma on this conditional probability setup, we obtain that a LMPR test statistic for testing H_0 vs. $H_{\underset{\sim}{\gamma}}^*$ is given by the coefficient of in the right hand side of (4.28); using (4.7), (4.16) and (4.17), we may further simplify this as

$$\frac{(\text{vec } \gamma)' \underset{\sim}{L} \sum_{i=1}^{N} \int_{\underset{\sim}{R}_N = \underset{\sim}{r}} \int [\underset{\sim}{f}^*(\underset{\sim}{x}_i; \underset{\sim}{0}) \times \underset{\sim}{c}_{Ni}] \prod_{i=1}^{N} dF(\underset{\sim}{x}_i; \underset{\sim}{0})}{N! \int \ldots \int_{\underset{\sim}{R}_N = \underset{\sim}{R}_N^*} \prod_{i=1}^{N} dF(\underset{\sim}{x}_i; \underset{\sim}{0})} \quad (4.29)$$

where for every $k(=1,\ldots,p)$ and $i(=1,\ldots,N)$,

$$\underset{\sim}{f}^*(\underset{\sim}{x}_i; \underset{\sim}{0}) \times \underset{\sim}{c}_{Ni} = ((\underset{\sim}{f}_{[1]}^*(x_{i1}; \underset{\sim}{0}) \times \underset{\sim}{c}_{Ni1})', \ldots, (f_{[p]}^*(x_{ip}; \underset{\sim}{0}) \times \underset{\sim}{c}_{Nip})');$$

$$f_{[k]}^*(x_{ik}; \underset{\sim}{0}) \times \underset{\sim}{c}_{Nik} = (f_{[k]1}^*(x_{ik}; \underset{\sim}{0}) c_{Nik1}, \ldots, f_{[k]q}^*(x_{ik}; \underset{\sim}{0}) c_{Nikq}) \quad (4.30)$$

Now, for every $k(=1,\ldots,p)$, let $U_{ik}, i=1,\ldots,N$ be i.i.d.r.v.'s having the uniform (0,1)d.f., and define

$$F_{[k]}^{-1}(t) = \inf\{ x : F_{[k]}(x; \underset{\sim}{0}) \geq t \}, \text{ for } t \in (0,1). \quad (4.31)$$

Also let

$$G_{Nk}(t) = N^{-1} \Sigma_{i=1}^{N} 1(U_{ik} \leq t) \text{ and } Z_{Nk}(t) = N^{\frac{1}{2}}[G_{Nk}^{-1}(t) - t],\ t \in (0,1), \quad (4.32)$$

where the quantile function $G_{Nk}^{-1}(.)$ is defined as in (4.31). Then, for $(N+1)^{-1} R_{ik} < t \leq (N+1)^{-1}(R_{ik}+1)$, a heuristic Taylor series expansion of $F_{[k]}^{-1}(G_{NK}^{-1}(t))$ yields that

$$f^*_{[k]\ell}(X_{ik};\underset{\sim}{0}) = f^*_{[k]\ell}(F^{-1}_{[k]}(t);\underset{\sim}{0}) + [N^{-\frac{1}{2}} Z_{Nk}(t)][f^{*'}_{[k]\ell}(F^{-1}_{[k]}(\xi);\underset{\sim}{0})] \ , \qquad (4.33)$$

for some ξ such that $|t-\xi| \le N^{-\frac{1}{2}}|Z_{Nk}(t)|$. Let $\varepsilon_N = (25 \text{ loglog } N)/N = (25 \log_2 N)/N$. Then, by [A4], we have for some $r \in (0, \frac{1}{2})$,

$$\sup_{\varepsilon_N \le \xi \le 1-\varepsilon_N} |f^{*'}_{[k]\ell}(F^{-1}_{[k]}(\xi);\underset{\sim}{0}) \ / \ f_{[k]}(F^{-1}_{[k]}(\xi);\underset{\sim}{0})| \overset{a.s.}{=} o(N^r) \ . \qquad (4.33a)$$

Further, by Theorem 4.5.5 of Csörgő and Révész (1981), the first factor of the second term on the right hand side of (4.33) satisfies

$$\overline{\lim_{N\to\infty}} \sup_{\varepsilon_N \le t \le 1-\varepsilon_N} N^{-\frac{1}{2}}|Z_{Nk}(t)|/\{t(1-t)\}^{\frac{1}{2}} / (N^{-1}\log_2 N)^{\frac{1}{2}} \le 4 \text{ a.s. .} \qquad (4.33b)$$

Thus, combining (4.33) through (4.33b) and letting $W_{k\ell}(R_{ik}) = f^*_{[k]\ell}(X_{ik};\underset{\sim}{0}) - f^*_{[k]\ell}(F^{-1}_{[k]}(R_{ik}/(N+1));\underset{\sim}{0})$, we obtain that for every $\ell : 1 \le \ell \le q$, as $N \to \infty$,

$$\sup_{\varepsilon_N < R_{ik}/(N+1) \le 1-\varepsilon_N} \ \max_{1 \le k \le p} | W_{k\ell}(R_{ik}) | \overset{a.s.}{=} o(((\log_2 N)/N^{1-r})^{\frac{1}{2}}). \qquad (4.34)$$

Next, we claim that for every $k (1 \le k \le p)$ and ℓ $(1 \le \ell \le q)$, as $N \to \infty$,

$$\sup_{(N+1)^{-1} R_{ik} \notin (\varepsilon_N, 1-\varepsilon_N)} |W_{k\ell}(R_{ik})| = o((N^{r-1}\log_2 N)^{\frac{1}{2}}) \text{ a.s..} \qquad (4.35)$$

We verify (4.35) only for $R_{ik} \le (N+1)\varepsilon_N$, as a similar proof holds for the upper tail. Let $0 < t \le \varepsilon_N$. For N sufficiently large, if $U_{ik} \le t$, then

$$| W_{k\ell}(t) | = | f^*_{[k]\ell}(x_{ik} \ ; \ \underset{\sim}{0}) - f^*_{[k]\ell}(F^{-1}_{[k]}(t);\underset{\sim}{0}) |$$

$$= \int_{U_{ik}}^{t} \left[f^{*'}_{[k]\ell}(F^{-1}_{[k]}(s);\underset{\sim}{0}) \ / \ f_{[k]}(F^{-1}_{[k]}(s);\underset{\sim}{0}) \right] ds |$$

$$\le o(N^r)t \le o(N^r)\epsilon_N = o\left[\frac{\log_2 N}{N^{1-r}}\right] \quad \text{a.s.} \qquad (4.35.a)$$

If $U_{ik} \ge t$, then

$$| W_{k\ell}(t) | \le o(N^r)U_{ik} = o(N^r)G^{-1}_{Nk}(t)$$

$$\le o(N^r)(|G^{-1}_{Nk}(\epsilon_N) - \epsilon_N| + \epsilon_N) \le O(N^r) \left[4\sqrt{\frac{\log_2 N}{N}} \sqrt{\epsilon_N(1-\epsilon_N)} + \epsilon_N\right]$$

$$= o(\log_2 N \ / \ N^{1-r}) \qquad \text{a.s.} \qquad (4.35.b)$$

This proves (4.35). By (4.34) and (4.35), we obtain that for every $\ell : 1 \le \ell \le q$,

$$\sup_{1\le i\le N}\ \sup_{1\le k\le p}\left| f^{*}_{[k]\ell}(x_{ik}\ ;\ \underset{\sim}{0}) - f^{*}_{[k]\ell}\left(F^{-1}_{[k]}\left[\frac{R_{ik}}{N+1}\right];\underset{\sim}{0}\right)\right| \overset{a.s.}{=} o\left[\frac{\log_2 N}{N^{1-r}}\right] \qquad (4.36)$$

Using (4.8) and (4.36), we may reduce (4.29) (a.s.) to $(\text{vec}\underset{\sim}{\gamma})'\underset{\sim}{L}\underset{\sim}{T}_N$, and this completes the proof of Theorem 4.2.

We may remark that for the coordinatewise independendence case, we do not need (4.9) as the Hájek-Šidák (1967) arguments directly apply to (4.27)-(4.29). In general, the LMPR test statistic based on $(\text{vec}\underset{\sim}{\gamma})'\underset{\sim}{L}\underset{\sim}{T}_N$ depends not only on $\underset{\sim}{\gamma}$ but also on the unknown matrix $\underset{\sim}{L}$. For testing (4.2) against (4.3), according to Roy's (1953) UI-principle, the overall test statistic is given by

$$Q_N = \sup\{\underset{\sim}{\lambda}'\underset{\sim}{L}\ \underset{\sim}{T}_N \,/\, \sqrt{\underset{\sim}{\lambda}'\underset{\sim}{L}\underset{\sim}{\Sigma}_N\ \underset{\sim}{L}'\underset{\sim}{\lambda}}\ ,\ \underset{\sim}{\lambda} = \text{vec}\ \underset{\sim}{\gamma},\ \underset{\sim}{\gamma}\in\Gamma\} \qquad (4.37)$$

Thus we let $h(\underset{\sim}{\lambda}) = -\underset{\sim}{\lambda}'\underset{\sim}{L}\underset{\sim}{T}_N$, $\underset{\sim}{h}_1(\underset{\sim}{\lambda}) = -\underset{\sim}{\mu} = -\underset{\sim}{A}\underset{\sim}{\lambda}$ and $h_2(\underset{\sim}{\lambda}) = \underset{\sim}{\lambda}'\underset{\sim}{L}\underset{\sim}{\Sigma}_N\underset{\sim}{L}'\underset{\sim}{\lambda} - 1$, then for this non-linear programming problem, the K.T.L. point formula theorem can be used again. For the lagrange function

$$\mathcal{L}(\underset{\sim}{\lambda},\underset{\sim}{t}_1,t_2) -\underset{\sim}{\lambda}'\underset{\sim}{L}\ \underset{\sim}{T}_N - \underset{\sim}{t}_1'\underset{\sim}{A}\underset{\sim}{\lambda} + t_2(\underset{\sim}{\lambda}'\underset{\sim}{L}\underset{\sim}{\Sigma}_N\underset{\sim}{L}'\underset{\sim}{\lambda}-1)$$

the point $(\underset{\sim}{\lambda}^*,\underset{\sim}{t}_1^*,t_2^*)$ is a K.T.L. point if it satisfies the system

$$\left.\begin{aligned}
&(a)\quad \underset{\sim}{t}_1^* \ge \underset{\sim}{0}\\
&(b)\quad \underset{\sim}{A}\ \underset{\sim}{\lambda}^* \ge \underset{\sim}{0}\\
&(c)\quad \underset{\sim}{\lambda}^{*\prime}\underset{\sim}{L}\ \underset{\sim}{\Sigma}_N\underset{\sim}{L}'\underset{\sim}{\lambda}^* - 1 = 0\\
&(d)\quad \underset{\sim}{t}_1^{*\prime}\underset{\sim}{A}\ \underset{\sim}{\lambda}^* = 0\\
&(e)\quad \partial\,\mathcal{L}(\underset{\sim}{\lambda},\underset{\sim}{t}_1,t_2)/\partial\underset{\sim}{\lambda}\Big|_{(\underset{\sim}{\lambda},\underset{\sim}{t}_1,t_2) = (\underset{\sim}{\lambda}^*,\underset{\sim}{t}_1^*,t_2^*)} = \underset{\sim}{0}
\end{aligned}\right\} \qquad (4.38)$$

From (4.38)-(e), we have

$$2\ t_2^*\ \underset{\sim}{\lambda}^* = (\underset{\sim}{L}\ \underset{\sim}{\Sigma}_N\ \underset{\sim}{L}')^{-1}\ \underset{\sim}{L}\ \underset{\sim}{T}_N + (\underset{\sim}{L}\ \underset{\sim}{\Sigma}_N\ \underset{\sim}{L}')^{-1}\ \underset{\sim}{A}'\underset{\sim}{t}_1^* \qquad (4.39)$$

After some manipulations, consequently we arrive

$$2\ t_2^*\ \underset{\sim}{t}_1^{*\prime}\{\underset{\sim}{A}(\underset{\sim}{L}\ \underset{\sim}{\Sigma}_N\ \underset{\sim}{L}')^{-1}\ \underset{\sim}{L}\ \underset{\sim}{T}_N + \underset{\sim}{A}(\underset{\sim}{L}\ \underset{\sim}{\Sigma}_N\ \underset{\sim}{L}')^{-1}\ \underset{\sim}{A}'\underset{\sim}{t}_1^*\} = 0 \qquad (4.40)$$

Define

$$\underset{\sim}{M}_N^* = \underset{\sim}{A}(\underset{\sim}{L}\ \underset{\sim}{\Sigma}_N\ \underset{\sim}{L}')^{-1}\ \underset{\sim}{L}\ \underset{\sim}{T}_N + \underset{\sim}{A}(\underset{\sim}{L}\ \underset{\sim}{\Sigma}_N\ \underset{\sim}{L}')^{-1}\underset{\sim}{A}'\underset{\sim}{t}_1^* \qquad (4.41)$$

and

$$\underset{\sim}{M}_N^0 = \underset{\sim}{A}\ \underset{\sim}{B}^{-1}(\underset{\sim}{\Sigma}_N)\ \underset{\sim}{T}_N + \underset{\sim}{A}\ \underset{\sim}{B}^{-1}(\underset{\sim}{\Sigma}_N)\ \underset{\sim}{\Sigma}_N\ \underset{\sim}{B}^{-1}(\underset{\sim}{\Sigma}_N)\underset{\sim}{A}'\underset{\sim}{t}_1^* \tag{4.42}$$

where $\underset{\sim}{B}(\underset{\sim}{\Sigma}_N)$ is defined as in (4.11). Then by Theorem 4.1 and (4.23) we have

$$\underset{\sim}{M}_N^* - \underset{\sim}{M}_N^0 \overset{P}{\to} 0 \quad \text{as} \quad N \to \infty \tag{4.43}$$

Further let

$$\underset{\sim}{U}_N = \underset{\sim}{A}\ \underset{\sim}{B}^{-1}(\underset{\sim}{\Sigma}_N)\underset{\sim}{T}_N \quad \text{and} \quad \underset{\sim}{\Delta}_N = \underset{\sim}{A}\underset{\sim}{B}^{-1}(\underset{\sim}{\Sigma}_N)\ \underset{\sim}{\Sigma}_N\ \underset{\sim}{B}^{-1}(\underset{\sim}{\Sigma}_N)\underset{\sim}{A}' \tag{4.44}$$

Then from (4.40) and (4.43) we have

$$\underset{\sim}{t}_1^{*'}(\underset{\sim}{U}_N + \underset{\sim}{\Delta}_N\ \underset{\sim}{t}_1^*) = 0 \quad \text{in probability as } N \to \infty \tag{4.45}$$

If we denote J be any subset of $A_0 = \{1, 2, \ldots, a_0\}$ and J' be its complement, thus (4.45) implies

$$\underset{\sim}{t}_{1(J)}^* = \underset{\sim}{0},\ \underset{\sim}{t}_{1(J')}^* = -\underset{\sim}{\Delta}_{N(J'J')}^{-1}\ \underset{\sim}{U}_{N(J')} \geq \underset{\sim}{0} \tag{4.46}$$

and

$$\underset{\sim}{U}_{N(J:J')} > \underset{\sim}{0} \tag{4.47}$$

Moreover, Kudo (1963) showed that the collection of all 2^{a_0} sets

$$R_N(J) = \{\underset{\sim}{U}_N \in E^{a_0};\ \underset{\sim}{U}_{N(J:J')} > \underset{\sim}{0},\ \underset{\sim}{\Delta}_{N(J'J')}^{-1}\ \underset{\sim}{U}_{N(J')} \leq \underset{\sim}{0}\} \tag{4.48}$$

is a disjoint and exhaustive partition of E^{a_0}. Therefore we have

$$\underset{\approx}{\lambda}^{*'}\ \underset{\sim}{L}\underset{\sim}{T}_N = \{\underset{\sim}{T}_N'\ \underset{\sim}{\Sigma}_N^{-1}\ \underset{\sim}{T}_N - \underset{\sim}{U}_{N(J')}'\ \underset{\sim}{\Delta}_{N(J'J')}^{-1}\ \underset{\sim}{U}_{N(J')}\}^{\frac{1}{2}} \tag{4.49}$$

where J is the set such that conditions (4.46) and (4.47) hold. Therefore the UI-LMPR statistic in (4.37) is given by

$$Q_N^2 = \underset{\sim}{T}_N'\left[\underset{\sim}{\Sigma}_N^{-1} - \underset{\sim}{B}^{-1}(\underset{\sim}{\Sigma}_N)\underset{\sim}{A}'\underset{\sim}{\Delta}_N^{-1}\underset{\sim}{A}\ \underset{\sim}{B}^{-1}(\underset{\sim}{\Sigma}_N)\right]\underset{\sim}{T}_N + \sum_{\phi \subseteq J \subseteq A_0} \{\underset{\sim}{U}_{N(J:J')}'\ \underset{\sim}{\Delta}_{N(JJ:J')}^{-1}\underset{\sim}{U}_{N(J:J')}\}$$

$$1\{\underset{\sim}{U}_{N(J:J')} > \underset{\sim}{0}\}\ 1\{\underset{\sim}{\Delta}_{N(J'J')}^{-1}\ \underset{\sim}{U}_{N(J')} \leq \underset{\sim}{0}\} \tag{4.50}$$

in probability, where the partitioned matrices and vectors are defined as in (2.16)-(2.19).

THEOREM 4.3. For testing (4.2) against (4.3) (under the local alternative $\{K_N\}$), if [A1]- [A4] hold, then the UI-LMPR test statistic Q_N^2 and the corresponding UI-LR test statistic are asymptotically power-equivalent.

<u>Proof</u>. Define

$$\underset{\sim}{S}_N^0 = \left[\left[\sum_{i=1}^{N} c_{Nij\ell}^* \, g_{j\ell}(X_{ij};\underset{\sim}{0}|\underset{\sim}{X})\right]\right]_{j=1,\ldots,p;\ell=1,\ldots,q} \tag{4.51}$$

$$\underset{\sim}{S}_N^{*0} = \left[\left[\sum_{i=1}^{N} c_{Nij\ell}^* \, f_{[j]\ell}^*(x_{ij};\underset{\sim}{0})\right]\right]_{j=1,\ldots,p;\ell=1,\ldots,q}, \tag{4.52}$$

and write

$$\underset{\sim}{S}_N = \text{vec } \underset{\sim}{S}_N^0 \quad\text{and}\quad \underset{\sim}{S}_N^* = \text{vec } \underset{\sim}{S}_N^{*0}$$

then it is easy to show that

$$E\{\underset{\sim}{S}_N \underset{\sim}{S}_N' | H_0\} = \overset{\wedge}{\underset{\sim}{\Sigma}}_N \quad\text{and}\quad E\{\underset{\sim}{S}_N^* \underset{\sim}{S}_N^{*'} | H_0\} = \overset{\wedge}{\underset{\sim}{\Sigma}}{}_N^* \tag{4.53}$$

where

$$\hat{\underset{\sim}{\Sigma}}_N = \left[\left[\varphi_{mm} \, c_{Nmm'}^*\right]\right]_{m;m'=1,\ldots,pq} \quad \text{with } \underset{\sim}{I}(f) = ((\varphi_{mm})),$$

and

$$\hat{\underset{\sim}{\Sigma}}{}_N^* = \left[\left[\varphi_{mm}^* \, c_{Nmm'}^*\right]\right]_{m;m'=1,\ldots,pq} \tag{4.54}$$

Also, let $\underset{\sim}{Z}_N = \underset{\sim}{A} \, \hat{\underset{\sim}{\Sigma}}{}_N^{-1} \, \underset{\sim}{S}_N$ $(\underset{\sim}{Z}_N^* = \underset{\sim}{A} \, \underset{\sim}{B}^{-1}(\hat{\underset{\sim}{\Sigma}}{}_N^*)\underset{\sim}{S}_N)$ and $\underset{\sim}{\Delta}_N = \underset{\sim}{A} \, \hat{\underset{\sim}{\Sigma}}{}_N^{-1} \underset{\sim}{A}'$ $(\hat{\underset{\sim}{\Delta}}{}_N^* = \underset{\sim}{A} \, \underset{\sim}{B}^{-1}(\hat{\underset{\sim}{\Sigma}}{}_N^*) \, \hat{\underset{\sim}{\Sigma}}{}_N^* \, \underset{\sim}{B}^{-1}(\hat{\underset{\sim}{\Sigma}}{}_N^*)\underset{\sim}{A}')$. Then the UI-score statistic for testing (4.2) vs. (4.3) turns out to be of the form

$$Q_{N0}^2 = \underset{\sim}{S}_N'\left[\underset{\sim}{\Sigma}_N^{-1} - \hat{\underset{\sim}{\Sigma}}{}_N^{-1} \underset{\sim}{A}' \, \hat{\underset{\sim}{\Delta}}{}_N^{-1} \, \underset{\sim}{A} \, \hat{\underset{\sim}{\Sigma}}{}_N^{-1}\right]\underset{\sim}{S}_N + \sum_{\phi \subseteq J \subseteq A_0} \left\{\underset{\sim}{Z}_{N(J:J')}' \, \hat{\underset{\sim}{\Delta}}{}_{N(JJ:J')}^{-1} \, \underset{\sim}{Z}_{N(J:J')}\right\} \cdot$$

$$\cdot 1\{\underset{\sim}{Z}_{N(J:J')} > \underset{\sim}{0}\} \; 1\{\hat{\underset{\sim}{\Delta}}{}_{N(J'J')}^{-1} \, \underset{\sim}{Z}_{N(J')} \le \underset{\sim}{0}\} \quad . \tag{4.55}$$

We also define

$$Q_N^{*2} = \underset{\sim}{S}_N^{*'}\left[\hat{\underset{\sim}{\Sigma}}{}_N^{*-1} - \underset{\sim}{B}^{-1}(\hat{\underset{\sim}{\Sigma}}{}_N^*)\underset{\sim}{A}' \, \hat{\underset{\sim}{\Delta}}{}_N^{*-1} \underset{\sim}{A} \, \underset{\sim}{B}^{-1}(\hat{\underset{\sim}{\Sigma}}{}_N^*)\right]\underset{\sim}{S}_N^* \tag{4.56}$$

$$+ \sum_{\phi \subseteq J \subseteq A_0} \left\{\underset{\sim}{Z}_{N\,(J:J')}^{*'} \, \hat{\underset{\sim}{\Delta}}{}_{N\,(JJ:J')}^{*-1} \, \underset{\sim}{Z}_{N(J:J')}^*\right\} 1\{\underset{\sim}{Z}_{N(J:J')}^* > \underset{\sim}{0}\} \; 1\{\hat{\underset{\sim}{\Delta}}{}_{N(J'J')}^{-1} \, \underset{\sim}{Z}_{N(J')}^* \le \underset{\sim}{0}\},$$

Note that $\hat{\Sigma}_N = L\hat{\Sigma}^*_N L'$ (as $S_N = LS^*_N$), and hence, by Theorem 4.1 and the implications of contiguity, we have

$$\hat{\Sigma}_N - B(\hat{\Sigma}^*_N)\hat{\Sigma}^{*-1}_N B(\hat{\Sigma}^*_N) \xrightarrow{P} 0 \quad (\text{as } N \to \infty), \text{ under } H_0 \text{ as well as } \{K_N\} . \tag{4.57}$$

Thus, by (4.48) and (4.12)-(4.14), Q^2_{N0} and Q^{*2}_N are asymptotically equivalent in distribution under H_0 as well as $\{K_N\}$. Furthermore, under some regularity conditons the UI-score statistic Q^2_{N0} and the corresponding UI-LR statistic are asymptotically equivalent in distribution under $\{K_N\}$ .For this we may refer to Chapter 4 of Puri and Sen (1985) where the asymptotic quadratic mean equivalence of $T^{\theta}_{Nk\ell}$ and $\Sigma^N_{i=1} c^*_{Nik\ell} f^*_{[k]\ell}(X_{ik};0)$ (for k=1,...,p; ℓ=1,...,q) has been established. Thus, the test statistic Q^2_N and the corresponding UI-LR statistic both have the same asymptotic power function under $\{K_N\}$. This completes the proof of the theorem.

THEOREM 4.4. Let $\eta = A B^{-1}(\Sigma)\Sigma\lambda$ and $\Delta = \lim_{N\to\infty} E(\Delta_N|H_0)$. For each J ($\emptyset \subseteq J \subseteq A_0$), define $\Gamma_J = \{\eta \in E^{+a_0}: \eta_{J:J'} = \eta_J - \Delta_{JJ'}\Delta^{-1}_{J'J'}\eta_{J'} \geq 0\}$ and $\Gamma_0 = \bigcap_{\phi \subseteq J \subseteq A_0} \Gamma_J$.

Then for testing (4.2) vs. (4.3) (under $\{K_N\}$), the UI-LMPR test statistic with critical region $W_N(\gamma) = \{X: Q^2_N \geq k_N(\alpha)\}$ is asymptotically most stringent for Γ and is asymptotically most powerful for Γ_0 at the respective level of significance α.

Proof. The proof follows directly from Theorem 3.1 and Theorem 4.2, and hence is omitted.

5. GENERAL REMARKS

We consider here some specific problems and append some useful discussions.

(I) Univariate k-sample location/scale alternative problems. Let X_{ij}, j=1,...,n, i=1,...,k be independent r.v.'s with continuous d.f.'s $F_{ij}(x)$,$x \varepsilon R$, where

$$F_{ij}(x) = F_i(x) = F((x-\beta_i)/\delta_i),\ j=1,\ldots,n;i=1,\ldots,k. \tag{5.1}$$

In the conventional case, we take the scale parameters $\delta_1,\ldots,\delta_k$ to be all equal (and equal to 1, without any loss of generality), so that (5.1) corresponds to a special case of (2.1) where $c_{N1} = \ldots = c_{Nn} = (1,0,\ldots,0)'$, $c_{Nn+1} = \ldots = c_{N2n} = (0,1,\ldots,0)',\ldots, c_{N((k-1)n+1)} = \ldots = c_{NN} = (0,\ldots,0,1)'$ and N = kn. Here, we are interested in testing for

$$H_0^{(1)} : \beta_1 = \ldots = \beta_k \quad \text{(i.e., the homogeneity of the } F_i\text{)} , \tag{5.2}$$

against the simple ordered alternative

$$K^{(1)}: \underset{\sim}{\beta} \in \Gamma = \{\underset{\sim}{\beta} \in E^k: \underset{\sim}{A}\underset{\sim}{\beta} \geq \underset{\sim}{0} \text{ with } \underset{\sim}{A} = \begin{pmatrix} -1 & 1 & 0 & \ldots & 0 & 0 \\ 0 & -1 & 1 & \ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots & -1 & 1 \end{pmatrix} \} . \tag{5.3}$$

Let R_{ij} be the rank of X_{ij} among the N observations of the combined sample, for $j=1,\ldots,n$, $i=1,\ldots,k$. We also denote the corresponding order statistics by $X_{N(1)} < \ldots < X_{N(N)}$, respectively. Thus, as in (2.7), we have the scores

$$a_N(r) = E\ \{-\dot{f}(X_{N(r)})/f(X_{N(r)})\ \} , \text{ for } r =1,\ldots,N, \tag{5.4}$$

and the corresponding linear rank statistics are given by

$$T_{Ni} = n^{-1} \Sigma_{j=1}^{n} a_N(R_{ij}),\ i=1,\ldots,k; \quad \underset{\sim}{T}_N = (T_{N1},\ldots,T_{Nk})' . \tag{5.5}$$

We let

$$\sigma_N^2 = (N-1)^{-1} \Sigma_{r=1}^{N}[\ a_N(r) - \bar{a}_N\ ]^2 \ ; \ \bar{a}_N = N^{-1}\Sigma_{i=1}^{N} a_N(i) . \tag{5.6}$$

Then the dispersion matrix of $\underset{\sim}{T}_N$ (under H_0) is given by

$$\underset{\sim}{M}_N = \sigma_N^2 \ (\ k\underset{\sim}{I}_k - \underset{\sim}{1}\underset{\sim}{1}'\ )\ ,\ \underset{\sim}{1} = (1,\ldots,1)' , \tag{5.7}$$

Also, note that under K_N , $\underset{\sim}{\beta} = N^{-\frac{1}{2}}\underset{\sim}{\gamma}$ $\underset{\sim}{\mu} = \underset{\sim}{A}\underset{\sim}{\gamma}$ and $U_N = \underset{\sim}{A}\underset{\sim}{M}_N^{-1}\underset{\sim}{T}_N$, so that

$$\underset{\sim}{\Delta}_N = \underset{\sim}{A}\underset{\sim}{M}_N^{-1}\underset{\sim}{A}' = \sigma_N^{-2}\underset{\sim}{\Delta}^* \quad \text{where} \quad \underset{\sim}{\Delta}^* = ((\delta_{ij}^*)) \text{ with } \quad \delta_{ij}^* = \begin{cases} 2 , & i = j , \\ -1 , & |i-j| = 1, \\ 0 , & |i-j| \geq 2 . \end{cases} \tag{5.8}$$

Partitioning $\underset{\sim}{\mu}$ and $\underset{\sim}{\Delta}^*$ as in (2.16) - (2.18), it is easy to verify that here

$$\underset{\sim}{\mu}_{J:J'} \geq \underset{\sim}{0} ,\ \forall\ J : \emptyset \subseteq J \subseteq \mathcal{A} = \{1,\ldots,k-1\} \ , \text{ if } \underset{\sim}{\mu} \geq \underset{\sim}{0} , \tag{5.9}$$

and hence, by Theorem 3.1, we may conclude that for testing (5.2) against (5.3) (under local contiguous alternatives), the UI-LMPR test based on the score function in (5.4) is asymptotically uniformly most powerful for all alternatives in the order-restricted parameter space.

Consider next the scale problem (with homogeneous locations), where

$$F_i(x) = F(\ \delta_i^{-1}x)\ , \text{ for } i =1,\ldots,k\ ,\ x \in E. \tag{5.10}$$

Here the null hypothesis relates to the homogeneity of the δ_i while we have an ordered alternative as in (5.3) with the vector $\underset{\sim}{\beta}$ replaced by $\underset{\sim}{\delta} = (\ \delta_1,\ldots,\delta_k)'$.

For this k-sample scale ordered alternative problem, using the scores

$$b_N(r) = E\{ -X_{N(r)}\dot{f}(X_{N(r)})/f(X_{N(r)})\} \text{ , } r = 1,\ldots,N, \qquad (5.11)$$

it can be shown that the corresponding UI-LMPR test is asymptotically UMP for the entire (restricted) parameter space under the ordered alternative.

(II) Bivariate two-sample ordered location problem. Chatterjee and De (1972, 1974) developed an UI-rank statistic for this problem and showed that their test statistic is better than the unrestricted one . A generalization of this result to the general multivariate case is still an open problem. However, instead of the maximization of the Bahadur-efficiency [as has been considered in Chatterjee and De (1974)], we may maximize the Pitman-efficiency to determine the UI-LMPR test for the general restricted alternative problem under more stringent assumptions on the underlying density function. For the particular problem considered by Chatterjee and De (1972,1974), our UI-LMPR test is asymptotically uniformly most powerful for the restricted parameter space under the alternative hypotheses.

(III) Other models. The asymptotic UMP property of the UI-LMPR test holds for other problems as well : (i) a positive orthant alternative when the limiting dispersion matrix is diagonal, (ii) an ordered alternative problem when the limiting covariance matrix is of the form $\underset{\sim}{I} \otimes \underset{\sim}{\Sigma}_1 + (\underset{\sim}{1}\underset{\sim}{1}') \otimes \underset{\sim}{\Sigma}_2$, and this is typically the case with randomized block designs and with multivariate multi-sample problems, (iii) profile analysis and (iv) in alternatives which put constraints on the parameters in the form of lower dimensional hyperspaces. Chinchilli and Sen (1981b) have considered the power superiority of the UI-rank test in the general multivariate problem under certain restraints on the covariance matrix. Under the same condition on the covariance matrix, by our Theorem 4.4, we are able to obtain the asymptotic UMP character of the UI-LMPR test for a wider class of alternatives. In passing, we may remark that the UI-LMPR test is invariant under any permutation of the p-coordinates, while the corresponding rank test based on the step-down procedure is not so, and the latter may not be asymptotically UMP.

(IV) A nonparametric version of Schaafsma and Smid (1966) procedure. For simplicity, we consider only the case of the positive orthant alternative where $\underset{\sim}{A} = \underset{\sim}{I}_{pq}$ and $\Gamma = E^{+pq}$. Under assumptions [A1] and [A2] in Section 4, the use of the Neyman-Pearson Lemma (on the distribution of the ranks) leads us to the LMPR test of size α which rejects H_0 for

$$Q_N(\underset{\sim}{\gamma}) = \underset{\sim}{\lambda}' \underset{\sim}{T}_N \Big/ \sqrt{\underset{\sim}{\lambda}' \underset{\sim}{\Sigma}_N \underset{\sim}{\lambda}} \geq k_N(\alpha) \tag{5.12}$$

where $\underset{\sim}{T}_N$ and $\underset{\sim}{\Sigma}_N$ are defined as in (4.19) and (4.22) respectively.

Assume $\underset{\sim}{\ell}$ is the half-line of points $\delta \underset{\sim}{\lambda}$, $\delta > 0$ and for any two such vectors $\underset{\sim}{\ell}$ and $\underset{\sim}{m}$, let $\varphi(\underset{\sim}{\ell},\underset{\sim}{m}) = \cos^{-1}\left\{ \langle \underset{\sim}{\ell}, \underset{\sim}{m} \rangle_{\underset{\sim}{\Sigma}_N} \Big/ \sqrt{\|\underset{\sim}{\ell}\|_{\underset{\sim}{\Sigma}_N} \|\underset{\sim}{m}\|_{\underset{\sim}{\Sigma}_N}} \right\}$ (5.13)

where $\langle \cdot \rangle_{\underset{\sim}{\Sigma}_N}$ denotes the inner product with respect to $\underset{\sim}{\Sigma}_N$ and $\|\cdot\|_{\underset{\sim}{\Sigma}_N}$ is the Euclidean norm with respect to $\underset{\sim}{\Sigma}_N$. Following Schaafsma and Smid (1966), the test in (5.12) is asymptotically most strigent somewhere most powerful if $\underset{\sim}{\ell}_0 \in \Gamma$ satisfies

$$\varphi_0 = \sup_{\underset{\sim}{m} \in \Gamma} \varphi(\underset{\sim}{\ell}_0, \underset{\sim}{m}) = \inf_{\underset{\sim}{\ell} \in \Gamma} \sup_{\underset{\sim}{m} \in \Gamma} \varphi(\underset{\sim}{\ell}, \underset{\sim}{m}) \tag{5.14}$$

where the desired half-line $\underset{\sim}{\ell}_0$ can be determined by applying the method of Abelson and Tukey (1963). Namely

$$\varphi(\underset{\sim}{\ell}, \underset{\sim}{e}_1^0) = \varphi(\underset{\sim}{\ell}, \underset{\sim}{e}_2^0) = \dots = \varphi(\underset{\sim}{\ell}, \underset{\sim}{e}_{pq}^0) \tag{5.15}$$

where the edges $\underset{\sim}{e}_j^0$ $(j=1,\dots,pq)$ is defined by

$$\underset{\sim}{e}_j^0 : \lambda_j > 0 \text{ and } \lambda_k = 0 \quad \text{if } k \neq j \quad \forall\ j,k=1,\dots,pq \tag{5.16}$$

Since the maximum shortcoming of this test is an increasing function of $\varphi(\underset{\sim}{\ell}_0, \underset{\sim}{m})$, it becomes unsatisfactory for large values of φ_0.

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