ASYMPTOTIC DISTRIBUTION OF UI-LMPR TESTS FOR RESTRICTED ALTERNATIVES *

Ming-Tan M. Tsai and Pranab Kumar Sen

University of North Carolina at Chapel Hill

SUMMARY. For testing against some restricted alternatives, in the light of the most stringency and somewhere most powerful character, the UI-LMPR tests are asymptotically optimal. The asymptotic distribution theory of such UI-LMPR test statistics is developed , and the same is incorporated in the comparative study of the asymptotic powers of the restricted UI-LMPR tests and their unrestricted versions (over the restricted parameter space).

1. INTRODUCTION

Let $\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_N$ be N independent random vectors (i.r.v.) with continuous distribution functions (d.f.) $F_1,\ldots,F_N$, respectively, all defined on the p-dimensional Euclidean space E^p, for some $p \geq 1$. We assume that

$$F_i(\underset{\sim}{x}) = F(\underset{\sim}{x};\underset{\sim}{\zeta}_{Ni}) \ , \quad \underset{\sim}{\zeta}_{Ni} = ((\ \beta_{j\ell}c_{Nij\ell}\ ))_{j=1,\ldots p;\ell=1,\ldots q},\ i=1,\ldots,N, \quad (1.1)$$

where the $\underset{\sim}{C}_{Ni} = ((c_{Nij\ell}))$ are p×q matrices of known constants and $\underset{\sim}{\beta} = ((\beta_{j\ell}\ ))$ is a p×q matrix of unknown parameters. The traditional null hypothesis relates to

$$H_0:\ \underset{\sim}{\beta} = \underset{\sim}{0}\ ,\text{i.e.,}\quad F_1 = \ldots = F_N = F\ (\text{ unknown}), \quad (1.2)$$

against the global alternative that

$$H_1:\ \underset{\sim}{\beta} \neq \underset{\sim}{0}\ ,\ \text{i.e., the } F_i \text{ are not all the same.} \quad (1.3)$$

For this problem, granted the asymptotic optimality of the classical likelihood ratio test (LRT), several authors have cosidered the corresponding nonparametric versions and studied their asymptotic optimality [viz., Puri and Sen (1985)]. In practice, often, we may have a restricted alternative hypothesis which may be characterized by a suitable subset Γ of E^{pq} and framed as

---

AMS 1979 Subject Classification Nos : 62E20, 62F05, 62G10, 62G20

Key Words and Phrases: Bahadur-Cochran defficiency; Bahadur-efficiency; likelihood ratio test; mixture of chi square distributions; rate of convergence; shortcoming; union-intersection locally most powerful rank test.

* Work partially supported by the Office of Naval Research, Contract No. N00014-83-K0387.

$$H^* : \underset{\sim}{\beta} \in \Gamma \quad \text{, for some proper subset } \Gamma \text{ of } E^{pq} \text{ .} \qquad (1.4)$$

The orthant alternative, ordered alternatives and a broad class of others all belong to the type in (1.4). Since the classical LRT,for H_0 vs. H_1,has an exponential rate of convergence (to 0) for the error probability (under non-local alternatives) which does not improve for a restricted alternative, Brown (1971) advocated the use of the classical LRT for testing H_0 vs. H^*. However, some Monte Carlo studies [viz., Barlow et al. (1972)] exhibit the power superiority of the restricted LRT to their unrestricted counterparts. In the nonparametric setup under consideration, the current authors have incorporated the notion of locally most powerful rank tests (LMPR) [viz., Hoeffding (1951)] along with the union-intersection (UI-) principle of Roy (1953) in the construction of some UI-LMPR tests which are asymptotically power-equivalent to the corresponding LRT under restricted (contiguous) alternatives. Like in the parametric case, the asymptotic distribution theory of such restricted alternative tests becomes more involved. In fact, the simple chi square (central under H_0 and non-central under the global alternative) distributional approximations are generally not tenable for such restricted cases. Our main contention is to study the asymptotic distribution theory of the UI-LMPR test statistics and to incorporate the same in the study of the asymptotic power properties of these tests and their classical versions.

Along with the basic regularity conditions, these UI-LMPR tests are introduced in Section 2. The asymptotic null and non-null (under contiguous alternatives) distributions are then derived in Section 3. The rates of convergence of the actual distributions to their asymptotic forms are investigated in Section 4. Some asymptotic relative efficiency and asymptotic optimality results are presented in the concluding section.

2. PRELIMINARY NOTIONS

We assume that in (1.4), Γ is a positively homogeneous set. Without any loss of generality, we may take Γ to be of the form

$$\Gamma = \{ \underset{\sim}{\beta} \in E^{pq} : \underset{\sim}{A}\,\mathrm{vec}\underset{\sim}{\beta} \geq \underset{\sim}{0} \text{ where } \underset{\sim}{A} \text{ is an } a_0 \times pq \text{ matrix of rank } a_0 (\leq pq) \}, \quad (2.1)$$

and vec$\underset{\sim}{\beta}$ denotes the pq-vector obtained by stacking the rows of $\underset{\sim}{\beta}$ under each other. To construct the UI-LMPR test statistics, we make the following assumptions.

[A1] For every N , we let $\underset{\sim}{c}^*_{Ni} = N^{-\frac{1}{2}}\mathrm{vec}\underset{\sim}{C}_{Ni}$, $\bar{\underset{\sim}{c}}^*_N = N^{-1}\Sigma_{i=1}^N \underset{\sim}{c}^*_{Ni}$ and

$$\underset{\sim}{C}^*_N = ((\, c^*_{Nkk'}))_{k,k'=1,\ldots,pq} = \Sigma_{i=1}^N (\underset{\sim}{c}^*_{Ni} - \bar{\underset{\sim}{c}}^*_N)(\underset{\sim}{c}^*_{Ni} - \bar{\underset{\sim}{c}}^*_N)' . \quad (2.2)$$

We assume that

$$\max_{1\leq j\leq p}\ \max_{1\leq \ell\leq q}\ \max_{1\leq i\leq N} |c^*_{Nij\ell}| = O((N^{-1}\log\log N)^{\frac{1}{2}}), \quad (2.3)$$

and there exists a positive definite (p.d.) matrix $\underset{\sim}{C}^* = ((c^*_{kk'}))$, such that $\underset{\sim}{C}^*_N$ converges to $\underset{\sim}{C}^*$ as N increases. Note that (2.3) and the above ensure that

$$\lim_{N\to\infty} \{ \max_{1\leq i\leq N} (\underset{\sim}{c}^*_{Ni} - \bar{\underset{\sim}{c}}^*_N)' \underset{\sim}{C}^{*-1}_N (\underset{\sim}{c}^*_{Ni} - \bar{\underset{\sim}{c}}^*) \} = 0.$$

[A2] F is absolutely continuous with continuous density function $f(\underset{\sim}{x}) = f(\underset{\sim}{x};\underset{\sim}{\Theta})$, $\underset{\sim}{\Theta} = (\underset{\sim}{\theta}_1,\ldots,\underset{\sim}{\theta}_p)'$ which satisfies the conditions: (a) For every i $(1 \leq i \leq N)$, $f_i(\underset{\sim}{x};\delta\underset{\sim}{\gamma})$ is absolutely continuous for almost all $\underset{\sim}{x} \in E^p$ and $\underset{\sim}{\gamma} \in \Gamma$, so that if we let $\nabla f_{i\underset{\sim}{\Theta}}(\underset{\sim}{x};\underset{\sim}{\Theta}) = (\frac{\partial}{\partial\,\mathrm{vec}\,\underset{\sim}{\Theta}}) f_i(\underset{\sim}{x};\underset{\sim}{\Theta})$ and $f_{i\underset{\sim}{\gamma}}(\underset{\sim}{x};\delta\underset{\sim}{\gamma}) = (\frac{\partial}{\partial\delta}) f_i(\underset{\sim}{x};\delta\underset{\sim}{\gamma})$, then for $\underset{\sim}{\Theta} = \delta\underset{\sim}{\gamma}$, $\dot{f}_{i\underset{\sim}{\gamma}}(\underset{\sim}{x};\delta\underset{\sim}{\gamma}) = (\mathrm{vec}\ \underset{\sim}{\gamma})' \cdot \nabla f_{i\underset{\sim}{\Theta}}(\underset{\sim}{x};\underset{\sim}{\theta})$, $\forall\ \underset{\sim}{x} \in E^p$ and $\underset{\sim}{\gamma} \in \Gamma$.

(b) For almost all $\underset{\sim}{x}$ and $\underset{\sim}{\gamma}$, the limit $\dot{f}_{i\underset{\sim}{\gamma}}(\underset{\sim}{x};\underset{\sim}{0}) = \lim_{\delta\to 0} \delta^{-1}[f_i(\underset{\sim}{x};\delta\underset{\sim}{\gamma}) - f_i(\underset{\sim}{x};\underset{\sim}{0})]$ exists.

(c) For every $\underset{\sim}{\gamma} \in \Gamma$, $\lim_{\delta\to 0} \int_{-\infty}^{\infty} |f_{i\underset{\sim}{\gamma}}(x;\delta\underset{\sim}{\gamma})| d\underset{\sim}{x} = \int_{-\infty}^{\infty} |\dot{f}_{i\underset{\sim}{\gamma}}(\underset{\sim}{x};\underset{\sim}{0})| d\underset{\sim}{x}$ is finite.

(d) The largest characteristic root of $\underset{\sim}{I}(f) = E_{\underset{\sim}{\Theta}}[(\frac{\partial}{\partial\,\mathrm{vec}\,\underset{\sim}{\Theta}}) \log f(\underset{\sim}{x};\underset{\sim}{\Theta}) (\frac{\partial}{\partial\ \mathrm{vec}\ \underset{\sim}{\Theta}}) \log f(\underset{\sim}{x};\underset{\sim}{\Theta})]$ is finite.

[A3] For the p-variate p.d.f. $f(\underset{\sim}{x};\underset{\sim}{\Theta})$, we denote the conditional p.d.f. of the j^{th} coordinate, given the others by $f_j(x_j;\underset{\sim}{\theta}_j|\underset{\sim}{x}^0_j)$ and let

$$g_j(x_j;\underset{\sim}{\theta}_j|\underset{\sim}{x}^0_j) = (\frac{\partial}{\partial\,\underset{\sim}{\theta}_j}) \log f(\underset{\sim}{x};\underset{\sim}{\Theta}) = (\frac{\partial}{\partial\underset{\sim}{\theta}_j}) \log f_j(x_j;\underset{\sim}{\theta}_j|\underset{\sim}{x}^0_j) \quad (2.5)$$

for j=1,...,p . Also let $f_{[j]}$ denote the marginal p.d.f. of the j^{th} coordinate, and

$$f^*_{[j]}(x_j;\underset{\sim}{\theta}_j) = (\frac{\partial}{\partial\underset{\sim}{\theta}_j})\log f_{[j]}(x_j;\underset{\sim}{\theta}_j) \quad j=1,2,\ldots,p\,. \tag{2.6}$$

We denote the corresponding pq-vectors by $\underset{\sim}{g}(\underset{\sim}{x};\underset{\sim}{\theta})$ and $\underset{\sim}{f}^*(\underset{\sim}{x};\underset{\sim}{\theta})$ respectively, and assume that there exists a p.d. matrix $\underset{\sim}{H}$ such that

$$\underset{\sim}{g}(\underset{\sim}{x};\underset{\sim}{0}) = \underset{\sim}{H}\underset{\sim}{f}^*(\underset{\sim}{x};\underset{\sim}{0}) \quad \text{for all } \underset{\sim}{x}\in E^p \tag{2.7}$$

[A4] Suppose for each $j\,(=1,\ldots,p)$ and $\ell\,(=1,\ldots,q)$, $f^*_{[j]\ell}(x_j;\underset{\sim}{0})$ is differentiable with respect to x_j on (a_j,b_j), where $-\infty < a_j \le -\sup\{x_j;F_{[j]}(x_j;\underset{\sim}{0})=0\}$, $\infty > b_j = \inf\{x_j;F_{[j]}(x_j;\underset{\sim}{0})=1\}$, and

(i) $f_{[j]}(x_j;\underset{\sim}{0}) > 0$ on its support (a_j,b_j).

(ii) Let $f^{*'}_{[j]\ell}(x_j;\underset{\sim}{0}) = (\frac{\partial}{\partial x_j})f^*_{[j]\ell}(x_j;\underset{\sim}{0})$, and assume, for some $r\in(0,\frac{1}{2})$

$$\sup_{1\le i\le N}\ \sup_{1\le j\le p} \left|f^{*'}_{[j]\ell}(X_j;\underset{\sim}{0})\right| \overset{a.s.}{=} o(N^r), \text{ as } N\to\infty. \tag{2.8}$$

Let R_{ij} be the rank of X_{ij} among $X_{1j},X_{2j},\ldots,X_{Nj}$, for $i=1,\ldots,N$ and $j=1,\ldots,p$; we denote the $p\times N$ rank collection matrix by $\underset{\sim}{R}_N$. Let $\underset{\sim}{R}^*_N$ be the $p\times N$ matrix obtained from $\underset{\sim}{R}_N$ by permuting the columns in such a way that the top row is in natural order, and for any given $\underset{\sim}{R}^*_N$, let $S(\underset{\sim}{R}^*_N)$ be the set of N! possible rank-collection matrices which can be reduced to $\underset{\sim}{R}^*_N$ by column permutations. Then the Chatterjee and Sen (1964) rank permutation principle applies to $\underset{\sim}{R}_N$ and under H_0, we have

$$P\{\underset{\sim}{R}_N = \underset{\sim}{r} \mid S(\underset{\sim}{R}^*_N),H_0\} = (N!)^{-1} \qquad \forall\ \underset{\sim}{r}\in S(\underset{\sim}{R}^*_N)\ ; \tag{2.9}$$

We denote by P_N the permutational probability measure in (2.9). For every j (=1,...,p), let $X_{j(1)} < \ldots < X_{j(N)}$ be the order statistics corresponding to $X_{1j},\ldots,X_{Nj}$; defining the $f^*_{[j]\ell}$ as in (2.6), we let

$$b_{Nj\ell}(r) = E\{\ f^*_{[j]\ell}(X_{j(r)};\underset{\sim}{0}) \mid H_0\ \},\ r=1,\ldots,N;\ j=1,\ldots,p;\ell=1,\ldots,q. \tag{2.10}$$

We define the p×q matrix $\underset{\sim}{T}^0_N = ((\,T^0_{Nj\ell}))$ of linear rank statistics by

$$T^{\theta}_{Nj\ell} = \Sigma^N_{i=1}\, c^*_{Nij\ell}\, b_{Nj\ell}(R_{ij}),\quad j=1,\ldots,p;\ \ell=1,\ldots,q; \tag{2.11}$$

$$\underset{\sim}{T}_N = \text{vec}\ \underset{\sim}{T}^0_N\,. \tag{2.12}$$

Also, let

$$\underset{\sim}{I}^{*} = ((\varphi^{*}_{kk'}))_{k,k'=1,\ldots,pq} = E\{ \underset{\sim}{f}^{*}(\underset{\sim}{X}_i;\underset{\sim}{0})[\underset{\sim}{f}^{*}(\underset{\sim}{X}_i;\underset{\sim}{0})]' \mid H_0\} , \tag{2.13}$$

$$\underset{\sim}{D}(\underset{\sim}{I}^{*}) = \mathrm{Diag}(E\{ \underset{\sim}{f}^{*}_{[j]}(X_{ij};\underset{\sim}{0})[\underset{\sim}{f}^{*}_{[j]}(X_{ij};\underset{\sim}{0})]' \mid H_0\} , j=1,\ldots,p) , \tag{2.14}$$

and, for every i (=1,...,N), let

$$B_{Ni} = (b_{N11}(R_{i1}),\ldots,b_{N1q}(R_{i1}),\ldots,b_{Np1}(R_{ip}),\ldots,b_{Npq}(R_{ip}))' . \tag{2.15}$$

Let then

$$\underset{\sim}{V}_N = (N-1)^{-1} \Sigma_{i=1}^{N} \underset{\sim}{B}_{Ni}\underset{\sim}{B}'_{Ni} = ((v_{Nkk'}))_{k,k'=1,\ldots,pq} , \tag{2.16}$$

$$\underset{\sim}{\Sigma}_N = ((c^{*}_{Nkk'}v_{Nkk'})) \quad \text{and} \quad \underset{\sim}{\Sigma} = ((c^{*}_{kk'}\varphi_{kk'})) . \tag{2.17}$$

Finally, consider a sequence of (restricted) alternatives $\{K_N\}$:

$$K_N : \underset{\sim}{\beta} = N^{-\frac{1}{2}} \underset{\sim}{\gamma} , \quad \underset{\sim}{\gamma} \in \Gamma . \tag{2.18}$$

Then, following the lines of Puri and Sen (1985), we have

$$E(\underset{\sim}{T}_N \mid \mathcal{P}_N) = \underset{\sim}{0} , \quad E(\underset{\sim}{T}_N\underset{\sim}{T}'_N \mid \mathcal{P}_N) = \underset{\sim}{\Sigma}_N , \tag{2.19}$$

$$\underset{\sim}{\Sigma}_N \xrightarrow{P} \underset{\sim}{\Sigma} , \text{ and } \underset{\sim}{T}_N \xrightarrow[\{K_N\}]{\mathcal{D}} \Phi_{pq}(\underset{\sim}{\Sigma}\underset{\sim}{\lambda} ; \underset{\sim}{\Sigma}), \underset{\sim}{\lambda} = \mathrm{vec}\underset{\sim}{\gamma} . \tag{2.20}$$

We adefine $\underset{\sim}{D}(\underset{\sim}{\Sigma}_N)$ as in (2.14) (with $\underset{\sim}{I}^*$ replaced by $\underset{\sim}{\Sigma}_N$), and let

$$\underset{\sim}{U}_N = \underset{\sim}{A}\, \underset{\sim}{D}^{-1}(\underset{\sim}{\Sigma}_N)\underset{\sim}{T}_N \quad \text{and} \quad \underset{\sim}{\Delta}_N = \underset{\sim}{A}\, \underset{\sim}{D}^{-1}(\underset{\sim}{\Sigma}_N)\, \underset{\sim}{\Sigma}_N \underset{\sim}{D}^{-1}(\underset{\sim}{\Sigma}_N)\underset{\sim}{A}' . \tag{2.21}$$

Also let J be any subset of $A_0 = \{1,\ldots,a_0\}$, and J' be its complement. For each of 2^{a_0} set J, we partition $\underset{\sim}{U}_N$ and $\underset{\sim}{\Delta}_N$ as

$$\underset{\sim}{U}_N = \begin{pmatrix} \underset{\sim}{U}_{N(J)} \\ \underset{\sim}{U}_{N(J')} \end{pmatrix} , \quad \underset{\sim}{\Delta}_N = \begin{pmatrix} \underset{\sim}{\Delta}_{N(JJ)} & \underset{\sim}{\Delta}_{N(JJ')} \\ \underset{\sim}{\Delta}_{N(J'J)} & \underset{\sim}{\Delta}_{N(J'J')} \end{pmatrix} \tag{2.22}$$

For each J ($\phi \subseteq J \subseteq A_0$), we define

$$\underset{\sim}{U}_{N(J:J')} = \underset{\sim}{U}_{N(J)} - \underset{\sim}{\Delta}_{N(JJ')}\underset{\sim}{\Delta}^{-1}_{N(J'J')} \underset{\sim}{U}_{N(J')} \tag{2.23}$$

$$\underset{\sim}{\Delta}_{N(JJ:J')} = \underset{\sim}{\Delta}_{N(JJ)} - \underset{\sim}{\Delta}_{N(JJ')}\underset{\sim}{\Delta}^{-1}_{N(J'J')}\underset{\sim}{\Delta}_{N(J'J)} \tag{2.24}$$

Thus, the UI-LMPR test statistics for testing $H_0 : \underset{\sim}{\beta} = \underset{\sim}{0}$ against $H_* : \underset{\sim}{\beta} \in \Gamma$ (defined in (2.1)) and for testing $H_0 : \underset{\sim}{\beta} = \underset{\sim}{0}$ versus $H_1 : \underset{\sim}{\beta} \neq \underset{\sim}{0}$ are of the forms :

$$Q_N^2 = \underset{\sim}{T}'_N(\underset{\sim}{\Sigma}_N^{-1} - \underset{\sim}{D}^{-1}(\underset{\sim}{\Sigma}_N)\underset{\sim}{A}'\underset{\sim}{\Delta}_N^{-1}\underset{\sim}{A}\underset{\sim}{D}^{-1}(\underset{\sim}{\Sigma}_N))\underset{\sim}{T}_N + \tag{2.25}$$

$$+ \sum_{\phi\subseteq J\subseteq A_0} \{\underset{\sim}{U}'_{N(J:J')}\underset{\sim}{\Delta}^{-1}_{N(JJ:J')}\underset{\sim}{U}_{N(J:J')}\} 1\{\underset{\sim}{U}_{N(J:J')} > \underset{\sim}{0}\} 1\{\underset{\sim}{\Delta}^{-1}_{N(J'J')}\underset{\sim}{U}_{N(J')} \le 0\}$$

and

$$R_N^2 = \underset{\sim}{T}_N' \, \underset{\sim}{\Sigma}_N^{-1} \, \underset{\sim}{T}_N , \tag{2.26}$$

respectively, where 1(B) stands for the indicator function of the set B.

3. UI-LMPR TEST STATISTICS : ASYMPTOTIC DISTRIBUTION THEORY

The task of studying the exact distribution theory (even under H_0) becomes prohibitively laborious as N increases. For this reason, we take recourse to the asymptotic distribution theory. Let

$$Q^{*2} = \underset{\sim}{W}'(\underset{\sim}{\Sigma}^{-1} - \underset{\sim}{D}^{-1}(\underset{\sim}{\Sigma})\underset{\sim}{A}'\underset{\sim}{\Delta}^{-1}\underset{\sim}{A}\underset{\sim}{D}^{-1}(\underset{\sim}{\Sigma}))\underset{\sim}{W} + + \sum_{\emptyset \subseteq J \subseteq A_0} \{\underset{\sim}{Z}'_{J:J'}\underset{\sim}{\Delta}^{-1}_{J:J'}\underset{\sim}{Z}_{J:J'}\} \, 1\{\underset{\sim}{Z}_{J:J'} > \underset{\sim}{0}\} 1\{\underset{\sim}{\Delta}^{-1}_{J'J'}\underset{\sim}{Z}_{J'} \le \underset{\sim}{0}\} \tag{3.1}$$

where $\underset{\sim}{W}$ is the random vector having a pq-variate normal d.f with mean vector $\underset{\sim}{\Sigma}\underset{\sim}{\gamma}$ and covariance matrix $\underset{\sim}{\Sigma}$ (i.e., $\underset{\sim}{W} \sim \Phi_{pq}(\cdot;\underset{\sim}{\Sigma}\underset{\sim}{\gamma},\underset{\sim}{\Sigma})$), $\underset{\sim}{Z} = \underset{\sim}{A}\underset{\sim}{D}^{-1}(\underset{\sim}{\Sigma})\underset{\sim}{W}$, $\underset{\sim}{\Delta} = \lim_{N\to\infty} E(\underset{\sim}{\Delta}_N|H_0)$ and the partitioned vectors and matrices are defined as in (2.22)-(2.24).

THEOREM 3.1. Under the Assumptions [A1]- [A4] ,

$$\lim_{N\to\infty} P\{ Q_N^2 \le x \mid H_0\} = P\{Q^{*2} \le x \mid \underset{\sim}{\gamma} = \underset{\sim}{0}\} = \sum_{k(J)=0}^{a_0} e_{k(J)} P\{\chi^2_{pq-a_0+k(J)} \le x\}, \tag{3.2}$$

where $\chi^2_{pq-a_0+k(J)}$ represents a Chi-square random variable with $pq-a_0+k(J)$ degrees of freedom, and $e_{k(J)} = \sum^*_{k(J)} P_{H_0}\{\underset{\sim}{Z}_{J:J'}>\underset{\sim}{0},\ \underset{\sim}{\Delta}^{-1}_{J'J'}\underset{\sim}{Z}_{J'}\le\underset{\sim}{0}\}$ with $\sum^*_{k(J)}$ denoting the sum over all set $J(\emptyset \subseteq J \subseteq A_0)$ such that the cardinality of J is k(J).

Proof. Since the first term and the second term of r.h.s. of (3.1) are independent, so

$$E_{H_0}\left\{e^{itQ^{*2}}\right\} = E_{H_0}\left\{e^{it\underset{\sim}{W}'(\underset{\sim}{\Sigma}^{-1}-\underset{\sim}{D}^{-1}(\underset{\sim}{\Sigma})\underset{\sim}{A}'\underset{\sim}{\Delta}^{-1}\underset{\sim}{A}\underset{\sim}{D}^{-1}(\underset{\sim}{\Sigma}))\underset{\sim}{W}}\right\} E_{H_0}\left\{e^{\left(it\sum_{\emptyset\subseteq J\subseteq A_0} Q^*(J)a^*(J)\right)}\right\} \tag{3.4}$$

where

$$Q^*(J) = \underset{\sim}{Z}'_{J:J'}\underset{\sim}{\Delta}^{-1}_{JJ:J'}\underset{\sim}{Z}_{J:J'} \quad \text{and} \quad a^*(J) = 1\{\underset{\sim}{Z}_{J:J'} > \underset{\sim}{0}, \underset{\sim}{\Delta}^{-1}_{J'J'}\underset{\sim}{Z}_{J'} \le \underset{\sim}{0}\} \tag{3.5}$$

Note that the matrix $\underset{\sim}{\Sigma}(\underset{\sim}{\Sigma}^{-1}-\underset{\sim}{D}^{-1}(\underset{\sim}{\Sigma})\underset{\sim}{A}'\underset{\sim}{\Delta}^{-1}\underset{\sim}{A}\underset{\sim}{D}^{-1}(\underset{\sim}{\Sigma}))$ is idempotent, and thus

$$\text{rank}[\underset{\sim}{\Sigma}(\underset{\sim}{\Sigma}^{-1}-\underset{\sim}{D}^{-1}(\underset{\sim}{\Sigma})\underset{\sim}{A}'\underset{\sim}{\Delta}^{-1}(\underset{\sim}{\Sigma}))] = \text{tr}[\underset{\sim}{\Sigma}(\underset{\sim}{\Sigma}^{-1}-\underset{\sim}{D}^{-1}(\underset{\sim}{\Sigma})\underset{\sim}{A}'\underset{\sim}{\Delta}^{-1}\underset{\sim}{A}\underset{\sim}{D}^{-1}(\underset{\sim}{\Sigma}))] = pq-a_0 \tag{3.6}$$

Hence we have

$$E_{H_0}\left\{e^{it\underset{\sim}{W}'(\underset{\sim}{\Sigma}^{-1}-\underset{\sim}{D}^{-1}(\underset{\sim}{\Sigma})\underset{\sim}{A}'\underset{\sim}{\Delta}^{-1}\underset{\sim}{A}\underset{\sim}{D}^{-1}(\underset{\sim}{\Sigma}))\underset{\sim}{W}}\right\} = (1-2it)^{\frac{-(pq-a_0)}{2}} \tag{3.7}$$

If we write

$$R(J) = \left\{ \underset{\sim}{Z} \in E^{a_0};\ \underset{\sim}{Z}_{J:J'} > \underset{\sim}{0},\ \underset{\sim}{\Delta}^{-1}_{J'J'}\underset{\sim}{Z}_{J'} \leq \underset{\sim}{0} \right\} \tag{3.8}$$

then the collection of all 2^{a_0} set R(J) is a disjoint and exhaustive partitioning of E^{a_0}. Thus we get

$$E_{H_0}\left\{ e^{it \sum_{\emptyset \subseteq J \subseteq A_0} Q^*(J)a^*(J)} \right\} = \int_{E^{a_0}} \cdots \int e^{it \sum_{\emptyset \subseteq J \subseteq A_0} Q^*(J)a^*(J)} d\Phi_{a_0}(\underset{\sim}{Z};\underset{\sim}{0},\underset{\sim}{\Delta})$$

$$= \sum_{\emptyset \subseteq J \subseteq A_0} \int_{R(J)} \cdots \int e^{itQ^*(J)} d\Phi_{a_0}(\underset{\sim}{Z};\underset{\sim}{0},\underset{\sim}{\Delta}) \tag{3.9}$$

Kudo (1963) showed that, for each J ($\emptyset \subseteq J \subseteq A_0$), Q*(J) and R(J) are independent under H_0. Thus (3.9) can be rewritten as

$$\sum_{\emptyset \subseteq J \subseteq A_0} \left[\int_{E^{k(J)}} \cdots \int e^{itQ^*(J)} d\Phi_{k(J)}(\underset{\sim}{Z}_{J:J'};\underset{\sim}{0},\underset{\sim}{\Delta}_{JJ:J'}) \right] \left[\int_{R(J)} \cdots \int d\Phi_{a_0}(\underset{\sim}{Z};\underset{\sim}{0},\underset{\sim}{\Delta}) \right]$$

$$= \sum_{k(J)=0}^{a_0} e_{k(J)} (1-2it)^{-\frac{k(J)}{2}}, \tag{3.10}$$

so

$$E_{H_0}\left\{ e^{itQ^{*2}} \right\} = \sum_{k(J)=0}^{a_0} e_{k(J)} (1-2it)^{-\frac{(pq-a_0+k(J))}{2}}. \tag{3.11}$$

Therefore, using the Fourier inversion formula on (3.11), we obtain that

$$\lim_{N\to\infty} P\{ Q_N^2 \leq x \mid H_0 \} = \sum_{k(J)}^{a_0} e_{k(J)}\ P\{ \chi^2_{pq-a_0+k(J)} \leq x \}, \tag{3.12}$$

for every real x, and this completes the proof of (3.2).

Next, we consider the non-null case and restrict ourselves to a sequence of restricted (contiguous) alternatives [as in (2.18)]. Note that under $\{K_N\}$ in (2.18), for any J ($\emptyset \subseteq J \subseteq A_0$), Q*(J) and R(J) are no longer stochastically independent, and this introduces complications in the form of the asymptotic distributions under consideration. We have the following.

THEOREM 3.2. Under $\{K_N\}$ and the Assumptions [A1] through [A4],

$$\lim_{N\to\infty} P\{Q_N^2 \le x \mid K_N\} = \sum_{h=0}^{\infty} d_h^* \, P\{\chi^2_{pq-a_0+h} \le x\} \qquad (3.13)$$

where

$$d_h^* = \begin{cases} e^{-\frac{1}{2}\xi} \; e^*_{a_0}(\underset{\sim}{\eta}) & \text{if } h=0 . \\ e^{-\frac{1}{2}\xi} \sum_{j+k(J)=h} d(k(J),j;\underset{\sim}{\eta},\underset{\sim}{\Delta}) & \text{if } h \text{ is odd, } k(J)=1,\ldots,a_0 . \\ e^{-\frac{1}{2}\xi}\left\{e^*_{a_0}(\underset{\sim}{\eta})(\xi/2)^{h/2}[(h/2)!]^{-1} + \sum_{k(J)+j=h} d(k(J);j;\underset{\sim}{\eta},\underset{\sim}{\Delta})\right\} & \text{if } h \text{ is even, } k(J)=1,\ldots,a_0 . \end{cases} \qquad (3.14)$$

$$\text{with } \xi = \underset{\sim}{\gamma}'[\underset{\sim}{\Sigma}-\underset{\sim}{\Sigma}\underset{\sim}{D}^{-1}(\underset{\sim}{\Sigma})\underset{\sim}{A}'\underset{\sim}{\Delta}^{-1}\underset{\sim}{A}\underset{\sim}{D}^{-1}(\underset{\sim}{\Sigma})\underset{\sim}{\Sigma}]\underset{\sim}{\lambda}, \quad \underset{\sim}{\eta} = \underset{\sim}{A}\underset{\sim}{D}^{-1}(\underset{\sim}{\Sigma})\underset{\sim}{\Sigma}\underset{\sim}{\lambda}, \qquad (3.15)$$

$$e^*_{k(J')}(\underset{\sim}{\eta}_{J'}) = \int_{\underset{\sim}{\Delta}^{-1}_{J'J'}\underset{\sim}{Z}_{J'}\le \underset{\sim}{0}} d\Phi_{k(J')}(\underset{\sim}{Z}_{J'};\underset{\sim}{\eta}_{J'},\underset{\sim}{\Delta}_{J'J'}) \quad \forall\ k(J')=0,1,\ldots,a_0 . \qquad (3.16)$$

and

$$d(k(J),j;\underset{\sim}{\eta},\underset{\sim}{\Delta}) = \sum_{\substack{\ell,m>0 \\ \ell+2m=j}} (\xi/2)^m (m!)^{-1}\, C(k(J),\ell;\underset{\sim}{\eta},\underset{\sim}{\Delta}) \qquad (3.17)$$

where

$$C(1,\ell;\underset{\sim}{\eta},\underset{\sim}{\Delta}) = \Gamma((\ell+1)/2)\,2^{\frac{\ell}{2}-1}(\ell!\sqrt{\pi})^{-1} \sum_{k(J)=1} e^*_{a_0-1}(\underset{\sim}{\eta}_{J'})\; e^{-\frac{1}{2}\underset{\sim}{\eta}'_{J:J'}\underset{\sim}{\Delta}^{-1}_{JJ:J'}\underset{\sim}{\eta}_{J:J'}} (\underset{\sim}{\eta}'_{J:J'}\underset{\sim}{\Delta}^{-\frac{1}{2}}_{JJ:J'})^{\ell} \qquad (3.18)$$

$$C(k(J),\ell;\underset{\sim}{\eta},\underset{\sim}{\Delta}) = \Gamma\left[\frac{\ell+k(J)}{2}\right]2^{\frac{\ell}{2}-1}\pi^{-\frac{k(J)}{2}}(\ell!)^{-1}\sum_{k(J)}^{*}\Bigg[e^*_{k(J')}(\underset{\sim}{\eta}_{J'})e^{-\frac{1}{2}\underset{\sim}{\eta}'_{J:J'}\underset{\sim}{\Delta}^{-1}_{JJ:J'}\underset{\sim}{\eta}_{J:J'}} \int_{\underset{\sim}{\Delta}^{\frac{1}{2}}_{JJ:J'}\underset{\sim}{\tau}_J\ge \underset{\sim}{0}} (\underset{\sim}{\eta}'_{J:J'}\underset{\sim}{\Delta}^{-\frac{1}{2}}_{JJ:J'}\underset{\sim}{\tau}_J)^{\ell} \prod_{i=1}^{k(J)-2}\cos^{k(J)-i-1}\theta_{Ji}\, d\theta_{J1}d\theta_{J2}\cdots d\theta_{Jk(J)-1}\Bigg], \qquad (3.19)$$

for all $k(J) = 2,\ldots,a_0$, $\Sigma^*_{k(J)}$ denotes the sum over all subsets J ($\emptyset \subseteq J \subseteq A_0$) such that the cardinality of J is $k(J)$, and for each J ($\emptyset \subseteq J \subseteq A_0$) , the vector $\underset{\sim}{\tau}_J$ is defined as

$$\tau_J = \begin{cases} \sin\Theta_{J1} \\ \cos\Theta_{J1}\sin\Theta_{J2} \\ \vdots \\ \cos\Theta_{J1}\cdots\cdots\cos\Theta_{Jk(J)-1}\sin\Theta_{Jk(J)-1} \\ \cos\Theta_{J1}\cdots\cdots\cos\Theta_{Jk(J)-1} \end{cases} \quad \begin{array}{l} -\frac{\pi}{2} \leq \Theta_{Jj} \leq \frac{\pi}{2},\ \forall\ j=1,\ldots,k(J)-2 \\ -\pi \leq \Theta_{Jk(J)-1} \leq \pi \end{array} \tag{3.20}$$

<u>Proof</u>. Note that the characteristic function of Q^{*2} under $\{K_N\}$ is

$$E_{K_N}\left\{e^{itQ^{*2}}\right\} = E_{K_N}\left\{e^{itW'(\Sigma^{-1}-D^{-1}(\Sigma)A'\Delta^{-1}AD^{-1}(\Sigma))W}\right\} E_{K_N}\left\{e^{it\sum_{\emptyset\subseteq J\subseteq A_0} Q^*(J)a^*(J')}\right\} \tag{3.21}$$

where

$$E_{K_N}\left\{e^{itW'(\Sigma^{-1}-D^{-1}(\Sigma)A'\Delta^{-1}AD^{-1}(\Sigma))W}\right\} = e^{-\frac{1}{2}\xi}\sum_{m=0}^{\infty} 2^{-m}(m!)^{-1}\xi^m(1-2it)^{\frac{-pq-a_0+2m}{2}} \tag{3.22}$$

and

$$E_{K_N}\left\{e^{it\sum_{\emptyset\subseteq J\subseteq A_0} Q^*(J)a^*(J)}\right\} = \int\cdots\int_{E^{a_0}} e^{it\sum_{\emptyset\subseteq J\subseteq A_0} Q^*(J)a^*(J)}\, d\Phi_{a_0}(z;\eta,\Delta)$$

$$= \sum_{\emptyset\subseteq J\subseteq A_0} e^*_{k(J')}(\eta_{J'})\int_{Z_{J:J'}>0} e^{itQ^*(J)}d\Phi_{k(J)}(Z_{J:J'};\eta_{J:J'},\Delta_{JJ:J'}) \,. \tag{3.23}$$

Let $\hat{Q}_J(t)$ be the integral of the r.h.s of (3.23). Then

$$\hat{Q}_J(t) = e^{-\frac{1}{2}\eta'_{J:J'}\Delta^{-1}_{JJ:J'}\eta_{J:J'}}\sum_{\ell=0}^{\infty}(\ell!)^{-1}$$

$$\int_{\Delta^{\frac{1}{2}}_{JJ:J'}\, Z_{J:J'}>0}(2\pi)^{\frac{-k(J)}{2}} e^{-\frac{1}{2}(1-2it)Q^*(J)}(\eta'_{J:J'}\Delta^{-\frac{1}{2}}_{JJ:J'}Z_{J:J'})^{\ell}dZ_{J:J'} \tag{3.24}$$

Therefore we have the following:

(a) If $k(J)=0$, then $Q_J(t)=1$. (b) If $k(J)=1$, then

$$\hat{Q}_J(t) = \sum_{\ell=0}^{\infty}(\ell!)^{-1}\Gamma\left(\frac{\ell+1}{2}\right)2^{\frac{\ell}{2}-1}\pi^{-\frac{1}{2}}e^{-\frac{1}{2}\eta'_{J:J'}\Delta^{-1}_{JJ:J'}\eta_{J:J'}}(\eta'_{J:J'}\Delta^{-\frac{1}{2}}_{JJ:J'})^{\ell}(1-2it)^{-\left(\frac{\ell+1}{2}\right)}. \tag{3.25}$$

And (c) if $k(J)=2,\ldots,a_0$, then make a transformation to the polar coordinates $(\rho_J,\theta_{J1},\ldots,\theta_{Jk(J)-1})$. Let

$$Z_{J:J'} = \rho_J\tau_J;\ \ \rho_J \geq 0 \ , \tag{3.26}$$

where $-\frac{\pi}{2} \le \theta_{Ji} \le \frac{\pi}{2}$ and $-\pi \le \theta_{Jk(J)-1} \le \pi \;\forall\; i=1,2,\ldots,k(J)-2$. Then the Jacobian of this polar transformation is $\rho_J^{k(J)-1} \prod_{i=1}^{k(J)-2} \cos^{k(J)-i-1}\theta_{Ji}$. We also let $R^*(J)$ be a set of restrictions on ρ_J and θ_J's of the form

$$\tilde{\Delta}^{\frac{1}{2}}_{J:J'}\,\tilde{\tau}_J > \tilde{0}\;, \quad \text{and} \quad \rho_J \ge 0 \tag{3.27}$$

Then, for every $k(J)$: $2 \le k(J) \le a_0$,

$$\hat{Q}_J(t) = \sum_{\ell=0} (\ell!)^{-1} e^{-\frac{1}{2}\tilde{\eta}'_{J:J'}\tilde{\Delta}^{-1}_{JJ:J'}\tilde{\eta}_{J:J'}}$$

$$\int\cdots\int_{R^*(J)} (2\pi)^{-k(J)} \rho_J^{\ell+k(J)-1} e^{-\frac{1}{2}(1-2it)\rho_J^2} \left(\tilde{\eta}'_{J:J'}\tilde{\Delta}^{-\frac{1}{2}}_{JJ:J'}\tilde{\tau}_J\right)^{\ell}$$

$$\prod_{i=1}^{k(J)-2} \cos^{k(J)-i-1}\theta_{Ji}\; d\tilde{\theta}_J d\rho_J \tag{3.28}$$

where $d\tilde{\theta}_J$ denotes $d\theta_{J1}d\theta_{J2}\cdots d\theta_{Jk(J)-1}$. After some simplication, we have

$$\hat{Q}_J(t) = \sum_{\ell=0}^{\infty} c(k(J),\ell;\tilde{\eta},\tilde{\Delta})\,(1-2it)^{-(\ell+k(J))/2} \quad . \tag{3.29}$$

Using (3.21) through (3.29), we conclude that

$$E_{K_N}\left\{e^{itQ^{*2}}\right\} = \left[e^{-\frac{1}{2}\xi} \sum_{m=0}^{\infty} 2^{-m}(m!)^{-1}\xi^m (1-2it)^{-\frac{(pq-a_0+2m)}{2}}\right]$$

$$\left[e^*_{a_0}(\tilde{\eta}) + \sum_{k(J)=1}^{a_0} \sum_{\ell=0}^{\infty} c(k(J),\ell;\tilde{\eta},\tilde{\Delta})(1-2it)^{-\frac{(\ell+k(J))}{2}}\right]$$

$$= (1-2it)^{-\frac{1}{2}(pq-a_0)} e^{-\frac{1}{2}\xi}\left[e^*_{a_0}(\tilde{\eta}) \sum_{m=0}^{\infty} (m!)^{-1}2^{-m}\xi^m + \right.$$

$$\left. + \sum_{k(J)=1}^{a_0} \sum_{j=0}^{\infty} d(k(J),j;\tilde{\eta},\tilde{\Delta})(1-2it)^{-\frac{(k(J)+j)}{2}}\right]$$

$$= \sum_{h=0}^{\infty} d^*_h (1-2it)^{-\frac{1}{2}(pq-a_0+h)} \quad . \tag{3.30}$$

Thus, we obtain that

$$\lim_{N\to\infty} P\left\{Q_N^2 \le x \mid K_N\right\} = \sum_{h=0}^{\infty} d_h^* \, P\left\{\chi^2_{pq-a_0+h} \le x \right\} \tag{3.31}$$

This completes the proof of Theorem 3.2.

Note that in the unrestricted alternative case, we have a central χ^2 distribution under the null hypothesis and a non-central χ^2 under the (local) alternative. However, the picture is different here. It is a mixture of central chi square d.f.'s under the null hypothesis, but the non-null case may not be reducible to the same mixture of non-central chi square d.f.'s.

4. RATES OF CONVERGENCE FOR DISTRIBUTIONS OF UI-LMPR TEST STATISTICS

Let G_N be the distribution function $\underset{\sim}{T}_N$ defined in (2.12). We suppose M is a measure on the Borel σ-field on E^{pq}, and denote $B(\underset{\sim}{w},\varepsilon)$ the open ball with center $\underset{\sim}{w}$ and radius ε, i.e., $\varepsilon > 0$,

$$B(\underset{\sim}{w},\varepsilon) = \{\underset{\sim}{y} \in E^{pq}; \|\underset{\sim}{w}-\underset{\sim}{y}\| < \varepsilon\} \quad \forall\, \underset{\sim}{w} \in E^{pq} \tag{4.1}$$

where $\|\cdot\|$ denotes the Euclidean norm. For any J ($\phi \subseteq J \subseteq A_0$) and any real-valued Borel measurable function g^J on $R(J)$ defined in (3.9), we define, $\forall\, \varepsilon > 0$, the oscillation function $S_{g^J}(B(\underset{\sim}{w},\varepsilon))$ as

$$S_{g^J}(\cdot;\varepsilon) = S_{g^J}(B(\underset{\sim}{w},\varepsilon))$$

$$= \sup_{\underset{\sim}{z}\in R(J)} \{|g(\underset{\sim}{z})-g(\underset{\sim}{y})| \,;\underset{\sim}{z},\underset{\sim}{y} \in B(\underset{\sim}{w},\varepsilon),\ \underset{\sim}{y} \in R(J)\} \tag{4.2}$$

and also define $\tilde{S}_{g^J}$ as the supremum of the average modulus of oscillation of g^J with respect to a finite measure M over all translate of g^J by

$$\tilde{S}_{g^J}(\varepsilon,M) = \sup\left\{\int_{R(J)} S_{g^J_{\underset{\sim}{y}}}(\underset{\sim}{w},\varepsilon)(M d\underset{\sim}{w}), \underset{\sim}{y} \in R(J)\right\} \tag{4.3}$$

where

$$g^J_{\underset{\sim}{y}}(\underset{\sim}{w}) = g^J(\underset{\sim}{y}+\underset{\sim}{w}). \tag{4.4}$$

For simplicity, we assume that

[B1] $\sum_{i=1}^{N} c^*_{Nij\ell} = 0$ and $\sum_{i=1}^{N} c^{*2}_{Nij\ell} = 1$, $\forall\, j = 1,\dots,p;\ \ell = 1,\dots,q.$ (4.5)

LEMMA 4.1. Let h be a bounded Borel measurable function defined on E^{pq}, then under the assumptions [A1]-[A4] and [B1], there exist constants d_1, d_2 and d_3 (not depending on N, $c^*_{Nij\ell}$, h) such that

$$|\int_{E^{pq}} h(\underset{\sim}{w})d(G_N(\underset{\sim}{w})-\Phi_{pq}(\underset{\sim}{w};\underset{\sim}{0},\underset{\sim}{\Sigma}_N)| \le d_1 \sup_{\underset{\sim}{w}\, E^{pq}} |h(\underset{\sim}{w})| \sum_{i=1}^{N}\sum_{j=1}^{p}\sum_{\ell=1}^{q} |c^*_{Nij\ell}|^{3+\delta}N^{\delta}$$

$$+ d_2 \tilde{S}_h\left(d_3 \sum_{i=1}^{N}\sum_{j=1}^{p}\sum_{=1}^{q} |c^*_{Nij\ell}|^{3+\delta}N^{\delta}, \Phi_{pq}(\cdot;\underset{\sim}{0},\underset{\sim}{\Sigma}_N)\right) \tag{4.6}$$

Proof. It follows along the lines of the main theorem of Hušková (1980), and hence is omitted.

THEOREM 4.2. Let $\underset{\sim}{X}_i$, $1 \le i \le N$, be N independent p-dimensional stochastic vectors having continuous c.d.f. $F(\underset{\sim}{x}, \underset{\sim}{\Sigma}_{Ni})$, where $\underset{\sim}{\Sigma}_{Ni}$ defined in (1.1). Consider the UI-LMPR test statistic Q_N^2 given in (2.25), then under the assumption [A1]-[A4] and [B1], there exists a constant d_4 such that

$$\sup_{x \in E} \left|P_{H_0}\left\{Q_N^2 \le x\right\} - P_{H_0}\left\{Q^{*2} \le x\right\}\right| \le d_4 \sum_{i=1}^{N}\sum_{j=1}^{p}\sum_{=1}^{q} |c^*_{Nij\ell}|^{3+\delta}N^{\delta} \tag{4.7}$$

Proof. For every J ($\emptyset \subseteq J \subseteq A_0$), defining Q*(J) as in (3.5), we let

$$Q(J) = \underset{\sim}{W}'(\underset{\sim}{\Sigma}^{-1} - \underset{\sim}{D}^{-1}(\Sigma)\underset{\sim}{A}'\underset{\sim}{\Delta}^{-1}\underset{\sim}{A}\underset{\sim}{D}^{-1}(\underset{\sim}{\Sigma}))\underset{\sim}{W} + Q^*(J). \tag{4.8}$$

Then, we have for every $x \ge 0$,

$$| P_{H_0}\{ Q_N^2 \le x \} - P_{H_0}\{ Q^{*2} \le x \} |$$

$$= \left|P_{H_0}\left\{\sum_{\phi\subseteq J\subseteq A_0} Q_N(J)a_N^*(J) \le x\right\} - P_{H_0}\left\{\sum_{\phi\subseteq J\subseteq A_0} Q(J)a^*(J) \le x\right\}\right|$$

$$= \left|\sum_{\phi\subseteq J\subseteq A_0} \left[P_{H_0}\left\{Q_N(J)a_N^*(J) \le x\right\} - P_{H_0}\left\{Q(J)a^*(J) \le x\right\}\right]\right|$$

$$\le \sum_{\phi\subseteq J\subseteq A_0} \left|\int_{\substack{Q(J)\le x\\ R(J)}} d\left(P_{H_0}\left\{\underset{\sim}{T}_N \le \underset{\sim}{w}\right\} - \Phi(\underset{\sim}{w};\underset{\sim}{0},\underset{\sim}{\Sigma}_N)\right)\right|$$

$$= \sum_{\phi\subseteq J\subseteq A_0} \left|\int_{E^{pq}} 1\left\{Q(J) \le x\right\}a^*(J)d\left(P_{H_0}\left\{\underset{\sim}{T}_N \le \underset{\sim}{w}\right\} - \Phi(\underset{\sim}{w};\underset{\sim}{0},\underset{\sim}{\Sigma}_N)\right)\right|$$

$$\leq \sum_{\phi \subseteq J \subseteq A_0} \left\{ d_1 \sup_{\underset{\sim}{w} \in E^{pq}} \left| g^J(\underset{\sim}{w}) \right| \sum_{i=1}^{N} \sum_{j=1}^{p} \sum_{\ell=1}^{q} |c^*_{Nij\ell}|^{3+\delta} N^{\delta} + \right.$$

$$\left. + \, d_2 \, \tilde{S}_{g^J}\left(d_3 \sum_{i=1}^{N} \sum_{j=1}^{p} \sum_{\ell=1}^{q} |c^*_{Nij\ell}|^{3+\delta} N^{\delta}, \Phi(\cdot\,; \underset{\sim}{0}, \underset{\sim}{\Sigma}_N) \right) \right\} \tag{4.9}$$

where

$$g^J(\underset{\sim}{w}) = 1\{Q(J) \leq x\} \;\; 1\{\underset{\sim}{Z}_{J:J'} > \underset{\sim}{0}, \underset{\sim}{\Delta}^{-1}_{J'J'} \underset{\sim}{Z}_{J'} \leq \underset{\sim}{0}\}. \tag{4.10}$$

By noting that (under H_0 as well as $\{K_N\}$)

$$1\{\underset{\sim}{Z}_{J:J'} > \underset{\sim}{0}, \underset{\sim}{\Delta}^{-1}_{J'J'} \underset{\sim}{Z}_{J'} \leqq \underset{\sim}{0}\} = 1\{\underset{\sim}{Z}_{J:J'} > \underset{\sim}{0}\} \; 1\{\underset{\sim}{\Delta}^{-1}_{J'J'} \underset{\sim}{Z}_{J'} \leqq \underset{\sim}{0}\}, \tag{4.11}$$

we have

(i) $$\sum_{\phi \subseteq J \subseteq A_0} \sup_{\underset{\sim}{w} \in E^{pq}} \left| g^J(\underset{\sim}{w}) \right| = \sum_{\phi \subseteq J \subseteq A_0} \sup_{\underset{\sim}{w} \in E^{pq}} \left| 1\{Q(J) \leq x\} 1\{\underset{\sim}{Z}_{J:J'} > \underset{\sim}{0}\} 1\{\underset{\sim}{\Delta}^{-1}_{J'J'} \underset{\sim}{Z}_{J'} \leq \underset{\sim}{0}\} \right|$$

$$\leq \sum_{\phi \subseteq J \subseteq A_0} \sup_{\underset{\sim}{w} \in E^{pq}} \left| 1\{\underset{\sim}{Z}_{J:J'} > \underset{\sim}{0}, \underset{\sim}{\Delta}^{-1}_{J'J'} \underset{\sim}{Z}_{J'} \leq \underset{\sim}{0}\} \right| = 1 \, . \tag{4.12}$$

and

(ii) $$\sum_{\phi \subseteq J \subseteq A_0} \tilde{S}_{g^J}(\varepsilon, \Phi) = \sum_{\phi \subseteq J \subseteq A_0} \sup_{\underset{\sim}{y} \in R(J)} \int_{E^{pq}} S_{g^J}(\underset{\sim}{w} + \underset{\sim}{y}; \varepsilon) d\Phi(\underset{\sim}{w}; \underset{\sim}{0}, \underset{\sim}{\Sigma}_N)$$

$$= \sum_{\phi \subseteq J \subseteq A_0} \left\{ \sup_{\underset{\sim}{y} \in R(J)} \int_{E^{pq}} \sup_{\underset{\sim}{w} \in R(J)} \left\{ \left| g^J(\underset{\sim}{w} + \underset{\sim}{y}) - g^J(\underset{\sim}{v} + \underset{\sim}{y}) \right|, \| \underset{\sim}{w} - \underset{\sim}{v} \| < \varepsilon, \; \underset{\sim}{v} \in R(J) \right\} d\Phi(\underset{\sim}{w}; \underset{\sim}{0}, \underset{\sim}{\Sigma}_N) \right.$$

$$= \sum_{\phi \subseteq J \subseteq A_0} \sup_{\underset{\sim}{y} \in R(J)} \Phi_{\underset{\sim}{0}, \underset{\sim}{\Sigma}_N}\left((\partial A_J(\underset{\sim}{w}))^{\varepsilon} + \underset{\sim}{y} \right) \tag{4.13}$$

where

$$A_J(\underset{\sim}{w}) = \left\{ \underset{\sim}{w} \in E^{pq}; \; Q(J) \leq x \; , \; \underset{\sim}{Z}_{J:J'} > \underset{\sim}{0}, \; \underset{\sim}{\Delta}^{-1}_{J'J'} \underset{\sim}{Z}_{J'} \leq 0 \right\} \tag{4.14}$$

and $(\partial A_J(\underset{\sim}{w}))^{\varepsilon}$ is the set of all points whose distances from the boundary of $A_J(\underset{\sim}{w})$ less than ε. Let C_0 be the class of all convex Borel subsets of E^{pq}, then for every J $(\phi \subseteq J \subseteq A_0)$, $A_J(\underset{\sim}{w}) \in C_0$. Hence, using Corollary 3.2 of Bhattacharya and Rao (1976) and some manipulations, we get

$$\sum_{\phi \subseteq J \subseteq A_0} \tilde{S}_{g^J}(\varepsilon, \Phi) \leq |\underset{\sim}{\Sigma}_N|^{\frac{1}{2}} 2^{\frac{5}{2}} \, \Gamma\left(\tfrac{pq+1}{2}\right) \left(\Gamma\left(\tfrac{pq}{2}\right)\right)^{-1} \varepsilon \, . \tag{4.15}$$

Taking $\varepsilon = d_3 \sum_{i=1}^{N} \sum_{j=1}^{p} \sum_{\ell=1}^{q} |c^*_{Nij\ell}|^{3+\delta} N^{\delta}$ and combining (4.9), (4.12) and (4.15), we obtain that (4.7) holds. Q.E.D.

Next we investigate the rates of convergence of distribution function of Q_N^2 to its limiting mixture central χ^2 distribution under a sequence of restricted contiguous alternative $\{K_N\}$. Under $\{K_N\}$, the distribution function depends on unknown parameter $\xi_{Ni} = ((\gamma_{j\ell}\, c^*_{Nij\ell})) = ((\theta_{Nij\ell}))$, so we make some extra conditions on the distribution function and the unknown parameters $\theta_{Nij\ell}$ $\forall$ $i=1,\dots,N$, $j=1,\dots,p$ and $\ell=1,\dots,q$.

[B2] For every $j=1,\dots,p$ and $\ell=1,\dots,q$

$$\Sigma_{i=1}^{N}\ \theta_{Nij\ell} = 0 \quad\text{and}\quad \Sigma_{i=1}^{N}\ \theta^2_{Nij\ell} = 1\ . \tag{4.16}$$

[B3] For the j^{th} marginal density function $f_{[j]}(x_{ij};\theta_j)$, we assume that there exist constants d_5 and d_6 such that for $\max_{1\le j\le p}\ \max_{1\le \ell\le q}\ \max_{1\le i\le N} |\theta_{Nij\ell}| \le d_5$,

$$\int |\nabla f_{[j]\ell}(x;\theta_j)|dx \le d_6 \quad \forall\ j=1,\dots,p;\ \ell=1,\dots,q \tag{4.17}$$

LEMMA 4.3. Let h be a bounded Borel measurable function defined on E^{pq}, then under the assumptions [A1]-[A4] and [B2]-[B3], there exist constants d_7, d_8 and d_9 such that

$$\Big|\int_{E^{pq}} h(w)\,d(P_{K_N}\{T_N \le w\} - \Phi_{pq}(w;\Sigma_N\lambda,\Sigma_N)\Big|$$

$$\le d_7 \sup_{w\in E^{pq}} |h(w)| \sum_{i=1}^{N}\sum_{j=1}^{p}\sum_{\ell=1}^{q} |c^*_{Nij\ell}|^{3+\delta}N^{\delta}(1+|\gamma_{j\ell}|^{3+\delta})$$

$$+ d_9\,\tilde{S}_h\Big(d_8 \sum_{i=1}^{N}\sum_{j=1}^{p}\sum_{\ell=1}^{q} |c^*_{Nij\ell}|^{3+\delta}N^{\delta}(1+|\gamma_{j\ell}|^{3+\delta}),\Phi(\cdot;\Sigma_N\lambda,\Sigma_N)\Big) \tag{4.18}$$

Proof. It follows from Lemma 4.1 and (7) in remarks of Hušková (1980).

THEOREM 4.4. Let X_i, $1\le i\le N$, be N independent p-dimensional stochastic vectors having continuous c.d.f. $F(x,\xi_{Ni})$, $\xi_{Ni} = ((\gamma_{J\ell}\, c^*_{Nij\ell}))$. Then under the assumptions [A1]-[a4] and [B2]-[B3], there exists constant d_{10} (not depending on N) such that for every real x ,

$$\sup_{x\in E} \Big|P_{K_N}\Big\{Q_N^2 \le x\Big\} - P_{K_N}\Big\{Q^{*2} \le x\Big\}\Big| \le d_{10} \sum_{i=1}^{N}\sum_{j=1}^{p}\sum_{\ell=1}^{q} |c^*_{Nij\ell}|^{3+\delta}N^{\delta}(1+|\gamma_{j\ell}|^{3+\delta})\ .$$

The proof follows directly from Theorem 4.2 and Lemma 4.3, and hence is omitted.

5. ASYMPTOTIC POWER COMPARISON OF Q_N^2 and R_N^2

While testing against a restricted alternative, in the parametric case, one may compare the classical LRT with its restricted alternative version , so as to gather information on the gain in the sensitivity of the test. In the nonparametric case, we intend to compare the asymptotic power functions of Q_N^2 and R_N^2 with the same objective. The power superiority of the restricted tests over the unrestricted ones in some specific nonparametric problems has been studied by Chatterjee and De (1974) and Chinchilli and Sen (1981), among others. De (1976) has shown that for a randomized block design problem, his Q_N^2 and the traditional R_N^2 both have the same approximate Bahadur-slope for any alternative in the restricted parameter space. We may observe that this feature is generally true for the LRT. Towards this, we assume that $\underset{\sim}{X}_i$, $i=1,\ldots,N$, are i.i.d.r.v. with the d.f. $\Phi_{pq}(\cdot;\ \underset{\sim}{\beta},\ \underset{\sim}{\Sigma}^0)$, where $\underset{\sim}{\Sigma}^0$ is assumed to be given. Consider the usual LRT and define

$$K_N(\underset{\sim}{X};\underset{\sim}{\beta},\underset{\sim}{0}) = N^{-1}\{\log[L(\underset{\sim}{X};\underset{\sim}{\beta},\underset{\sim}{\Sigma}^0)/L(\underset{\sim}{X};\underset{\sim}{0},\underset{\sim}{\Sigma}^0)]\};\ \underset{\sim}{X}= (\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_N), \tag{5.1}$$

where $L(\underset{\sim}{X};\underset{\sim}{\beta},\underset{\sim}{\Sigma}^0)$ stands for the likelihood function. By the classical Law of Large Numbers, we obtain that

$$\lim_{N\to\infty} K_N(\underset{\sim}{X};\underset{\sim}{\beta},\underset{\sim}{0}) = I(\underset{\sim}{\beta};\underset{\sim}{0}) \text{ a.s. when } \Phi_{pq}(\cdot;\underset{\sim}{\beta},\underset{\sim}{\Sigma}^0) \text{ holds}, \tag{5.2}$$

where $I(\underset{\sim}{\beta};\underset{\sim}{0})$ stands for the Kullback-Liebler Information; it is defined as

$$I(\underset{\sim}{\beta};\underset{\sim}{0}) = E_{\underset{\sim}{\beta}}\{\log[f(\underset{\sim}{X}_1;\underset{\sim}{\beta},\underset{\sim}{\Sigma}^0)/f(\underset{\sim}{X}_1;\underset{\sim}{0},\underset{\sim}{\Sigma}^0)]\}\ ,\ \text{for every } \underset{\sim}{\beta} \neq \underset{\sim}{0}\ . \tag{5.3}$$

For testing H_0: $\underset{\sim}{\beta} = \underset{\sim}{0}$ vs. K: $\underset{\sim}{\beta} \neq \underset{\sim}{0}$, it is easy to check that

$$I(\underset{\sim}{\beta};\underset{\sim}{0}) = \underset{\sim}{\beta}'(\underset{\sim}{\Sigma}^0)^{-1}\underset{\sim}{\beta}\ ,\ \text{for every } \underset{\sim}{\beta} \neq \underset{\sim}{0}\ . \tag{5.4}$$

Thus, the unrestricted LRT achieves the Kullback-Leibler Information in (5.4). For testing H_0: $\underset{\sim}{\beta}= \underset{\sim}{0}$ against K*: $\underset{\sim}{A}\underset{\sim}{\beta} \geq \underset{\sim}{0}$, where $\underset{\sim}{A}$ is defined by (2.1), it is easy to show that the restricted LRT also achieves the same Information. For this restricted alternative problem, the LRT statistic is given by

$$Q_N^{*2} = N\bar{\underset{\sim}{X}}_N'((\underset{\sim}{\Sigma}^0)^{-1}-\underset{\sim}{A}'(\underset{\sim}{\Delta}^0)^{-1}\underset{\sim}{A})\bar{\underset{\sim}{X}}_N + \sum_{\emptyset \subseteq J \subseteq A_0} \{\underset{\sim}{Z}^{0\,\prime}_{J:J'}\underset{\sim}{\Delta}^{0\ -1}_{JJ:J'}\underset{\sim}{Z}^0_{J:J'}\}1\{\underset{\sim}{Z}^0_{J:J'}>\underset{\sim}{0},\underset{\sim}{\Delta}^{0\ -1}_{J'J'}\underset{\sim}{Z}^0_{J'}\leq \underset{\sim}{0}\}, \tag{5.5}$$

where the partitioned vectors and matrices are defined as in (2.22)-(2.24) and

$$\Delta^0 = A\,\Sigma^0\,A' \quad \text{and} \quad Z^0 = N^{\frac{1}{2}} A \bar{X}_N = N^{-\frac{1}{2}} A (\Sigma_{i=1}^{N} X_i). \tag{5.6}$$

Thus

$$P_{H_0}\{ N^{-1} Q_N^{*2} \geq t\} = P_{H_0}\{ Q_N^{*2} \geq Nt\} = \Sigma_{k(J)=0}^{a_0} e^0_{k(J)} \, P\{\chi^2_{pq-a_0+k(J)} \geq Nt \} ,$$

where the $e^0_{k(J)}$ stand for the sums of the multinormal orthant probabilities corresponding to the $Z_{J:J'}$, for which the k(J) have the common value k (=0,..., a_0). Therefore, by some routine steps, we obtain that for evry t($\geq$0),

$$\lim_{N\to\infty} \{ -N^{-1} \log P_{H_0}\{ N^{-1} Q_N^{*2} \geq t\}\} = t/2 \quad \text{a.s.} \quad \Phi_{pq}(.;\beta,\Sigma^0). \tag{5.7}$$

Moreover, under the same setup,

$$N^{-1} Q_N^{*2} \xrightarrow{p} \beta'(\Sigma^{0^{-1}} - A'\Delta^{0^{-1}} A)\beta +$$

$$\sum_{\phi \subseteq J \subseteq A_0} \{\mu^{0'}_{J:J'} \Delta^{0-1}_{JJ:J'} \mu^0_{J:J'}\} 1\{\mu^0_{J:J'} > 0\} 1 \{\Delta^{0^{-1}}_{J'J'} \mu^0_{J'} \leq 0\} \quad \text{where } \mu^0 = A\,\beta . \tag{5.8}$$

It is easy to show that (5.8) equals to $\beta'\Sigma^{0\,-1}\beta$, so that the exact Bahadur slopes for both the unrestricted and restricted LRT's are the same. Thus, we need a finer asymptotic comparison to discriminate the two LRT's. In this context, the concept of approximate Bahadur-Cochran deficiency, developed by Chandra and Ghosh (1978), may be adapted to force this distinction.

Let k_N^0 (k_N) and α_{1N} (α_{2N}) be the critical value and size of the unrestricted (restricted) LRT sequence ($\{R_N^{*2}\}$, $\{Q_N^{*2}\}$) when the power at β is β^*. Then, using the results in Section 2 of Chandra and Ghosh (1982), we obtain that

$$K_N^0 = \beta'\Sigma^{0^{-1}}\beta + 2N^{-\frac{1}{2}}\Phi^{-1}(1-\beta^*)\sqrt{\beta'\Sigma^{0^{-1}}\beta} + [(\Phi^{-1}(1-\beta^*))^2 + pq-1]N^{-1} + o(N^{-1}), \tag{5.9}$$

$$\alpha_{1N} = P_{H_0}\left\{N^{-1} R_N^{*2} \geq k_N^0\right\} =$$

$$= (Nk_N^0)^{\frac{pq}{2}-1} e^{-\frac{1}{2}Nk_N^0}\, 2^{-\frac{pq}{2}} \left(\Gamma(\tfrac{pq}{2})\right)^{-1} \int_0^\infty e^{-\frac{1}{2}x}\left(1 + x(Nk_N^0)^{-1}\right)^{\frac{pq-1}{2}} dx$$

$$= \left(\frac{Nk_N^0}{2}\right)^{\frac{pq}{2}-1} \left(\Gamma(\tfrac{pq}{2})\right)^{-1} e^{-\frac{1}{2}Nk_N^0} \left(1 + [(\tfrac{pq}{2})-1](Nk_N^0)^{-1} + o(N^{-1})\right) \tag{5.10}$$

and further that $k_N^0 - k_N = o(N^{-1})$, so that we have

$$\alpha_{2N} = P_{H_0}\left\{N^{-1}Q_N^{*2} \geq k_N\right\} = \sum_{k(J)=0}^{a_0} (Nk_N/2)^{\frac{pq-a_0+k(J)}{2}-1} e^{-\frac{1}{2}Nk_N} e^0_{k(J)} \left\{\Gamma\left(\frac{pq-a_0+k(J)}{2}\right)\right\}^{-1}$$

$$(1 + [(pq-a_0+k(J))/2 - 1](Nk_N)^{-1} + o(N^{-1})) , \qquad (5.11)$$

so that $\alpha_{2N} = \alpha_{1N}\, e^0_{a_0}\left[1+\left[\frac{Nk_N^0}{2}\right]^{-\frac{1}{2}} \left(\frac{e^0_{a_0-1}}{e^0_{a_0}}\right) \frac{\Gamma[\frac{pq}{2}]}{\Gamma[\frac{pq-1}{2}]} \; [1-\frac{1}{2}(Nk_N^0)^{-1} + o(N^{-1})] + o(N^{-1})\right]$

and hence

$$\log \alpha_{2N} = \log \alpha_{1N} + \log e^0_{a_0} + o(1) . \qquad (5.12)$$

Consequently, by using Theorem 2.3.1 of Chandra and Ghosh (1978) we conclude that the approximate Bahadur-Cochran deficiency of the unrestricted LRT with respect to the restricted one is $-2(\underset{\sim}{\beta}'(\underset{\sim}{\Sigma}^0)^{-1}\underset{\sim}{\beta})^{-1}\log e^0_{a_0}$. When $\underset{\sim}{\Sigma}^0$ is not known, the LRT statistics, Q_N^{02}, for H_0: $\underset{\sim}{\beta} = \underset{\sim}{0}$ against K^*: $\underset{\sim}{A}\underset{\sim}{\beta} \geq \underset{\sim}{0}$, has the same form as in (5.5) with $\underset{\sim}{\Sigma}^0$ being replaced by $\underset{\sim}{S} = (N-1)^{-1}\Sigma_{i=1}^N(\underset{\sim}{X}_i-\underset{\sim}{\bar{X}}_N)(\underset{\sim}{X}_i-\underset{\sim}{\bar{X}}_N)'$. In this case, following Perlman (1969), we obtain that under H_0, for every $x \geq 0$,

$$P\{\, Q_N^{02} \leq x \,\} = \Sigma_{k(J)=0}^{a_0} e^0_{k(J)} \, P\{\chi^2_{pq-a_0+k(J)}/\chi^2_{N-pq} \leq x\}. \qquad (5.13)$$

By very similar steps, it follows that in this case, both the unrestricted and restricted LRT's have the same Bahadur slope, and the approximate Bahadur-Cochran deficiency of the former with respect to the latter is still positive.

We may note that in the above discussion, we have allowed both α_{1N} and α_{2N} to converge to 0 , and for a fixed power β^* , we have drawn the relative performance picture. This conclusion may not necessarily apply to the conventional case where the level of significance is held fixed, and for contiguous alternatives [such as in (2.18)], the relative power pictures are studied. In situations where the Pitman-efficiency measure is adoptable, the two sequence of competing statistics have the same type of distributions [viz., non-central chi square or normal] differing in some noncentrality parameters, and the relative picture of these noncentralities convey the efficiency picture. However, in our case, Q_N^2 and R_N^2 have different asymptotic distributions and the Pitman-measure is not adoptable. Another plausible approach for this local comparison is to go through a second order local scheme. i.e., to compare the slopes of the asymptotic local power functions at the null point. For this, note that under $\{K_N\}$ in (2.18)

$$\beta_{R^2}(\alpha;\underset{\sim}{\gamma}) = \exp(-\{\underset{\sim}{\lambda}'\underset{\sim}{\Sigma}\underset{\sim}{\lambda}\}/2)\ \Sigma_{h=0}^{\infty}(\underset{\sim}{\lambda}'\underset{\sim}{\Sigma}\underset{\sim}{\lambda})^{h}2^{-h}(h!)^{-1}P\{\chi^2_{pq+2h} \geq x'_\alpha\}, \tag{5.14}$$

where x'_α stands for the upper $100\alpha\%$ point of the central χ^2 d.f. with pq DF. Also, by Theorem 3.2, for the UI-LMPR test Q_N^{*2}, under $\{K_N\}$ in (2.18), the asymptotic power is given by

$$\beta_{Q^{*2}}(\alpha;\underset{\sim}{\gamma}) = \Sigma_{h=0}^{\infty}\ d_h^*\ P\{\chi^2_{pq-a_0+h} \geq x_\alpha\}, \tag{5.15}$$

where x_α is the critical level for Q^{*2}. Since both (5.14) and (5.15) are smooth functions, by the Taylor expansion, we have for small $(\underset{\sim}{\lambda}'\underset{\sim}{\Sigma}\underset{\sim}{\lambda})$

$$\beta_{R^2}(\alpha;\underset{\sim}{\gamma}) = \alpha + O(\{(\underset{\sim}{\lambda}'\underset{\sim}{\Sigma}\underset{\sim}{\lambda})^{\frac12}\}^2); \tag{5.16}$$

$$\beta_{Q^{*2}}(\alpha;\underset{\sim}{\gamma}) = \alpha + b^*(\underset{\sim}{\lambda}'\underset{\sim}{\Sigma}\underset{\sim}{\lambda})^{\frac12} + O(\{(\underset{\sim}{\lambda}'\underset{\sim}{\Sigma}\underset{\sim}{\lambda})^{\frac12}\}^2), \tag{5.17}$$

where

$$\begin{aligned}
b^* = \frac{1}{\sqrt{\underset{\sim}{\lambda}^{*\prime}\underset{\sim}{\Sigma}\underset{\sim}{\lambda}^*}}\Bigg\{&\Big[\int_{\underset{\sim}{\Delta}^{-1}\underset{\sim}{Z}\leq\underset{\sim}{0}} \underset{\sim}{\eta}^{*\prime}\underset{\sim}{\Delta}^{-1}\underset{\sim}{Z}\, d\Phi_{a_0}(\underset{\sim}{Z};\underset{\sim}{0},\underset{\sim}{\Delta})\Big]P\{\chi^2_{pq-a_0}\geq x_\alpha\} + \sum_{k(J)=1}^{a_0-1}\frac{\Gamma(\frac{k(J)}{2})}{2(\sqrt{\pi})^{k(J)}}\\
&\times\Big[\sum_{k(J)}^{*}\int_{\underset{\sim}{\Delta}^{-1}_{J'J'}\underset{\sim}{Z}_{J'}\leq\underset{\sim}{0}}\underset{\sim}{\eta}^{*\prime}_{J'}\underset{\sim}{\Delta}^{-1}_{J'J'}\underset{\sim}{Z}_{J'}\,d\Phi_{k(J')}(\underset{\sim}{Z}_{J'};\underset{\sim}{0},\underset{\sim}{\Delta}_{J'J'})\int_{\underset{\sim}{\Delta}^{\frac12}_{JJ:J'}\underset{\sim}{\tau}_J\geq\underset{\sim}{0}}d\underset{\sim}{\tau}_J\Big]\\
&\times P\{\chi^2_{pq-a_0+k(J)}\geq x_\alpha\} + \frac{1}{\sqrt{2\pi}}\sum_{k(J)=1}^{*} e^*_{a_0-1}(\underset{\sim}{0})\underset{\sim}{\eta}^{*\prime}_{J:J'}\underset{\sim}{\Delta}^{-\frac12}_{JJ:J'}P\{\chi^2_{pq-a_0+2}\geq x_\alpha\}\\
&+\sum_{k(J)=2}^{a_0}\frac{\Gamma(\frac{1+k(J)}{2})}{\sqrt{2}(\sqrt{\pi})^{k(J)}}\Big[\sum_{k(J)}^{*}e^*_{k(J')}(\underset{\sim}{0})\int_{\underset{\sim}{\Delta}^{\frac12}_{JJ:J'}\underset{\sim}{\tau}_J\geq\underset{\sim}{0}}\underset{\sim}{\eta}^{*\prime}_{J:J'}\underset{\sim}{\Delta}^{-\frac12}_{JJ:J'}\underset{\sim}{\tau}_J\,d\underset{\sim}{\tau}_J\Big]\\
&\times P\{\chi^2_{pq-a_0+k(J)+1}\geq x_\alpha\}\Bigg\}
\end{aligned}\tag{5.18}$$

with $\underset{\sim}{\lambda}^* = \varepsilon_1^{-1}\underset{\sim}{\lambda}$ and $\underset{\sim}{\eta}^* = \varepsilon_1^{-1}\underset{\sim}{\eta}$, for $\varepsilon_1 > 0$. If $\underset{\sim}{\Delta} = \underset{\sim}{I}$, then (5.18) reduces to

$$\begin{aligned}
b^* = \frac{a_0\overline{\eta^*}}{(\sqrt{2\pi\underset{\sim}{\lambda}^{*\prime}\underset{\sim}{\Sigma}\underset{\sim}{\lambda}^*})2^{a_0-1}}\Bigg\{&\sum_{k(J)=0}^{a_0-1}\binom{a_0-1}{k(J)}\Big[P\{\chi^2_{pq-a_0+2+k(J)}\geq x_\alpha\}\\
&-P\{\chi^2_{pq-a_0+k(J)}\geq x_\alpha\}\Big]\Bigg\} > 0, \text{ where } \overline{\eta^*} = a_0^{-1}\sum_{i=1}^{a_0}\eta_i^*
\end{aligned}\tag{5.19}$$

In general, b* is not very simple in form. However, we may simplify this further as

$$\begin{aligned}
b^* = \frac{1}{\sqrt{\underset{\sim}{\lambda}^{*\prime}\underset{\sim}{\Sigma}\underset{\sim}{\lambda}^*}}\Bigg\{&\Big[\Big(\int_{\underset{\sim}{\Delta}^{-1}\underset{\sim}{Z}\leqq\underset{\sim}{0}}\underset{\sim}{\eta}^{*\prime}\underset{\sim}{\Delta}^{-1}\underset{\sim}{Z}\,d\Phi_{a_0}(\underset{\sim}{Z};\underset{\sim}{0},\underset{\sim}{\Delta})\Big)P\{\chi^2_{pq-a_0}\geq x_\alpha\}\\
&+\sum_{k(J)=1}^{a_0}\Big(\sum_{k(J)}^{*}\int_{R(J)}\underset{\sim}{\eta}^{*\prime}\underset{\sim}{\Delta}^{-1}\underset{\sim}{Z}\,d\Phi_{a_0}(\underset{\sim}{Z};\underset{\sim}{0},\underset{\sim}{\Delta})\Big)P\{\chi^2_{pq-a_0+k(J)+1}\geq x_\alpha\}\Big] +
\end{aligned}$$

$$+\sum_{k(J)=1}^{a_0}\Big(\sum_{k(J)}^{*}\int_{R(J)}\eta^{*\prime}_{J}\Delta^{-1}_{J\cdot J'}Z_{J'}\,d\Phi_{a_0}(Z;\,0,\,\Delta)\Big)\Big[\,P\{\chi^2_{pq-a_0+k(J)}\geq x_\alpha\}$$

$$-\,P\{\chi^2_{pq-a_0+k(J)+1}\geq x_\alpha\}\,\Big]\Big\},\text{ where } R(J) \text{ defined in (3.8).} \qquad (5.20)$$

By noting that

$$\Big(\int_{\Delta^{-1}Z\leq 0}\eta^{*\prime}\Delta^{-1}Z\,d\Phi_{a_0}(Z;\,0,\,\Delta)\Big)\,P\{\chi^2_{pq-a_0}\geq x_\alpha\}+\sum_{k(J)=1}^{a_0}\Big(\sum_{k(J)}^{*}\int_{R(J)}$$

$$\eta^{*\prime}\Delta^{-1}Z\,d\Phi_{a_0}(Z;\,0,\,\Delta)\Big)\,P\{\chi^2_{pq-a_0+k(J)+1}\geq x_\alpha\}$$

$$>\Big(\sum_{\emptyset\subsetneq J\subsetneq A_0}\int_{R(J)}\eta^{*\prime}\Delta^{-1}Z\,d\Phi_{a_0}(Z;\,0,\,\Delta)\Big)\,P\{\chi^2_{pq-a_0}\geq x_\alpha\}=0, \qquad (5.21)$$

we conclude that the slope of the asymptotic power function at the null point for the restricted UI-LMPR test is larger than that of the unrestricted one. This result gives us the asymptotic local power superiority of the restricted test Q_N^2 to that of the unrestricted one R_N^2, in the restricted alternative space $\Omega_o = \{\beta \in E^{pq}: \eta \geq 0\}$.

We may finally remark that for testing the homogeneity against alternatives which put constraints on the parameters in the linear form of lower dimensional hyperspaces i.e., $H_0: \beta = 0$ vs. $H_1: \beta \in \Gamma^* = \{\beta \in E^{pq}: A \text{ vec } \beta = 0\}$, under local (contiguous) alternatives, both Q_N^2 and R_N^2 have non-central chi square d.f.'s with pq-a and pq DF respectively, and with the same non-centrality parameter. Since, the power function of a non-central chi square variable , for a fixed non-centrality parameter, is a non-increasing function of its DF, we conclude that for all $\beta \in \Gamma^*$, Q_N^2 is asymptotically locally more powerful than R_N^2 .

References

Barlow, R.E., Bartholomew, D.J., Bremner, J.M. and Brunk, H.D. (1972). Statistical Inference under Order Restrictions, Wiley, New York.

Brown, L.D. (1971). Non-local asymptotic optimality of appropriate likelihood ratio tests, Ann. Math. Statist. 42, 1206-1240.

Bhattacharya, R.N. and Ranga Rao, R. (1976). Normal Approximation and Asymptotic Expansions. Wiley, New York.

Chandra, T.K. and Ghosh, J.K. (1978). Comparison of tests with same Bahadur efficiency. Sankhya Ser. A 40, 253-277.

Chandra, T. K. and Ghosh, J.K. (1982). Deficiency for multiparameter testing problems In Statistics and Probability : Essays in honor of C.R. Rao (eds:G. Kallianpur, P.R. Krishnaiah & J.K. Ghosh), North Holland, Amsterdam, pp. 179-191.

Chatterjee, S.K. and De, N.K. (1974). On the power superiority of certain bivariate location tests against restricted alternatives. Calcutta Statist. Assoc. Bull. 23, 73-84.

Chatterjee, S.K. and Sen, P.K. (1964). Nonparametric tests for the bivariate two-sample location problem. Calcutta Statist. Assoc. Bull. 13, 18-58.

Chinchilli, V.M. and Sen, (1981). Multivariate linear rank statistics and the union-intersection principle for the orthant restriction problem. Sankhya Ser. B 43, 152-171.

De, N.K. (1976). Rank tests for randomized blocks against ordered alternatives. Calcutta Statist. Assoc. Bull. 25, 1-27.

Hoeffding, W. (1951). Optimum nonparametric tests. Proc. Second Berkeley Symp. Math. Statist. Prob., Univ. of California Press (ed. J. Neyman), 83-92.

Hušková, M. (1980). Some asymptotic results on the multivariate rank statistics. In Multivariate Analysis, Vol. V (Ed. P.K. Krishnaiah). New York: Academic, pp. 223-237.

Kudo, A. (1963). A multivariate analogue of the one-sided test. Biometrika 50, 403-418.

Perlman, M.D. (1969). One-sided testing problems in multivariate analysis. Ann. Math. Statist. 40, 549-567.

Puri, M.L. and Sen, P.K. (1969). A class of rank order tests for a general linear hypothesis. Ann. Math. Statist. 40, 1325-1343.

Puri, M.L. and Sen, P.K. (1985). Nonparametric Methods in General Linear Models. John Wiley, New York.

Roy, S.N. (1953). On a heuristic method of test construction and its use in multivariate analysis. Ann. Math. Statist. 24, 220-238.

Tsai, M.T. and Sen, P.K. (1987). On the asymptotic optimality of UI-LMP rank tests for restricted alternatives.