NONPARAMETRIC TIME SERIES PREDICTION.
II. KERNAL ESTIMATORS BASED ON LOCAL MEDIANS

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Abstract. The local and global asymptotic properties of nonparametric estimators of conditional medians constructed based on local medians are obtained for stationary time series which satisfies weak mixing conditions. For this class of nonparametric estimators, the results presented in this paper constitute a generalization of the corresponding ones (based on random samples) of Truong and Stone (1987, Asymptotic properties of kernel estimators based on local medians. *Preprint*) to time series.

1. INTRODUCTION

One of the most important problems in univariate time series analysis is that of *predicting* (or *forcasting*) a future value of a discrete time stationary random process from its past values. This prediction problem may be described as follows: Let n, m and d be positive integers and suppose that X_t, $t = 0, \pm 1, \pm 2, \ldots$ is a discrete time, strictly stationary random process. It is desired to predict the value of X_{n+m} from $X_{n-d+1}, \ldots, X_n$.

An m-th step prediction rule is a function $\theta(x_{n-d+1}, \ldots, x_n)$ of the past; that is, $\theta(\cdot)$ is a (Borel measureable) real-valued function defined on $\mathbf{R}^d$. Construction of such a rule can have two purposes: (i) to predict the future X_{n+m} as accurately as possible; (ii) to understand the structural relationship between the future X_{n+m} and the past $(X_{n-d+1}, \ldots, X_n)$.

In bivariate time series analysis, one of the main concerns is to investigate the relationship between the input series $\{X_t; t = 0, \pm 1, \ldots\}$ and the output series $\{Z_t; t = 0, \pm 1, \ldots\}$. Here it is useful to consider Z_n as a function of $(X_{n-d+1}, \ldots, X_n)$.

Note that by letting $\mathbf{X}_n = (X_{n-d+1}, \ldots, X_n)$ and $Y_n = X_{n+m}$ in the univariate case, or $\mathbf{X}_n = (X_{n-d+1}, \ldots, X_n)$ and $Y_n = Z_n$ in the bivariate case; then the above set ups are special cases of the following situation: Let $(\mathbf{X}, Y)$ be a pair of random variables that are respectively d and 1 dimensional; the random variable Y is called the response and the random vector $\mathbf{X}$ is refered to as the predictor variable. The main objective is to construct a function $\theta(\cdot)$ so as to (i) study the relationship between the response and the explanatory variable or (ii) obtain the predictor $\theta(\mathbf{X})$ of Y based on $\mathbf{X}$.

The simplest and most widely used measure of accuracy of $\theta(\mathbf{X})$ as a predictor of Y is the *Mean Square Error*, $E|Y - \theta(\mathbf{X})|^2$. The function $\theta(\cdot)$ which minimizes this measure of accuracy is the regression function of Y on $\mathbf{X}$, defined by $\theta(\mathbf{X}) = E(Y|\mathbf{X})$.

Recently, there has been an increasing interest in adopting the *Mean Absolute Deviation* $E|Y - \theta(\mathbf{X})|$ as a measure of accuracy, especially when outliers may be present (Bloomfield and Steiger, 1983). The optimal function $\theta(\cdot)$ is now defined so that $\theta(\mathbf{X})$ is the conditional median, Median$(Y|\mathbf{X})$, of Y given $\mathbf{X}$. Note that this function is not

necessary uniquely defined.

In practice, it is necessary to construct estimators of these functions based on a set of observations. Time series prediction is the generic term revolving around the construction of estimators of these predictors based on a realization $(\mathbf{X}_1, Y_1), \ldots, (\mathbf{X}_n, Y_n)$ from the stationary process $(\mathbf{X}_t, Y_t), t = 0, \pm 1, \ldots$.

Parametric Approach vs Nonparametric Approach

To estimate these predictors, the *parametric* approach starts with specific assumptions about the relationship between the response (or future) and the explanatory variables (or past) and about the variation in the response (future) that may or may not be accounted for by the explanatory variables. For instance, the standard regression method (or autoregressive method in time series) starts with an a priori model for the regression function $\theta(\cdot)$ which, by assumption or prior knowledge, is a linear function that contains finitely many unknown parameters. Under the assumption that the joint distribution is Gaussian, it is an optimal prediction rule; if the distribution is non-Gaussian, it is not generally posible to determine the funtion $\theta(\cdot)$; so one might settle for the *best* linear predctor. By constrast, in the *nonparametric* approach, the regression function will be estimated directly without assuming such an a priori model for $\theta(\cdot)$. As pointed out in Stone (1985), the nonparametric approach is more *flexible* than the standard regression method; *flexibility* means the ability of the model to provide accurate fits in a wide variety of realistic situations, inaccuracy here leading to *bias* in estimation. In recent years, nonparametric estimation has become an active area in time series analysis because of its flexibility in fitting data (Bierens, 1983; Collomb, 1984; Collomb and Hardle, 1984; Robinson, 1983).

The present approach deals with the asymptotic properties (in terms of rates of convergence) of a class of nonparametric estimators constructed by kernel methods based on local medians. It is hoped that the results obtained here serve as a starting point for further development and understanding of the sampling properties of more complicated nonparametric procedures involving robustification, local polynomial fits, additive regression, and spline approximation.

Some previous work on nonparametric estimation in time series will be surveyed in the next section.

2. DEVELOPMENTS IN TIME SERIES PREDICTION

The theory and practice of linear model fitting has now attained a refined state; see, for example, Brillinger (1981), Hannan (1970), Hannan (1973), Dunsmuir and Hannan (1976) and Priestley, 1979). While the study of non-linear models in time series is still in its early stages, what has been learned so far is sufficient to indicate that this is a very rich and potentially rewarding field. Analysis of particular series have shown that non-linear models can provide better fits to the data (as one would expect) and, more importantly, that the structure underlying the data can not be captured by linear models.

So far, the study of non-linear models has been restricted to a few specific forms. For example, Priestley (1980), Tong and Lim (1980), Nicholls and Quinn (1980), and Haggan and Ozaki (1980, 1981) consider various non-linear filters of, possibly independent, identically distributed Gaussian random variables. In practice it may be difficult to decide a priori, which, if any, of these models is best suited to a given set of data.

Asymptotic results for the conditional expectation has been established by Doukhan and Ghindes (1980), Collomb (1984), Bierens (1983) and Robinson (1983) under various mixing conditions. In Robinson (1983), pointwise consistency and a central limit theorem was obtained for kernel estimators based on local averages under the α-mixing condition. Collomb (1984) and Bierens (1983) considered the uniform consistency and rate of convergence for kernel estimators based on local averages under the ϕ-mixing condition, which is considerably stronger than the α-mixing condition. Collomb and Härdle (1984) considered the uniform rate of convergence (also under ϕ-mixing) for a class of robust nonparametric estimators that did not include local medians. Truong and Stone (1987b) considered the local and global properties of kernel estimators based on local averages and local medians under α-mixing condition; the L^∞ rate of convergence remains unsolved. Under another type of mixing condition defined in terms of cumulants, Truong (1987) modified the method in Truong and Stone (1987a, 1987b) to yield uniform rate of convergence for

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kernel estimators based on local averages.

The present approach deals with the local and global asymptotic properties of the kernel estimators based on local medians. Results are given in Section 3.

For this class of nonparametric estimators, the results presented there constitute an answer and an extension to one of the open questions of Stone (1982). In the random sample case, Härdle and Luckhaus (1984) considered the L^∞ rate of convergence for a class of robust nonparametric estimators including an estimator of the conditional median. But the problem of L^q ($1 \le q < \infty$) rates of convergence was still unsolved. Proofs of these results are given in Section 5.

3. NONPARAMETRIC TIME SERIES PREDICTION

Results on the local and global rates of convergence of nonparametric estimators of conditional expectations and conditional medians based on a realization of a discrete time stationary time series will be treated in this section. Recall that d is the dimensionality of the explanatory variable $\mathbf{X}$ and let U denote a nonempty bounded open neighborhood of the origin of $\mathbf{R}^d$. Let $\{(\mathbf{X}_i, Y_i), i = 0, \pm 1, \ldots\}$ be an $(d + 1)$ vector-valued strictly stationary series and set $\theta(\mathbf{x}) = \text{Median}(Y_0|\mathbf{X}_0 = \mathbf{x})$. Let $(\mathbf{X}_1, Y_1), \ldots, (\mathbf{X}_n, Y_n)$ denote a realization of this process.

Assumption 1. *There is a positve constant M_0 such that*

$$|\theta(\mathbf{x}) - \theta(\mathbf{x}')| \le M_0 \|\mathbf{x} - \mathbf{x}'\| \qquad \text{for } \mathbf{x}, \mathbf{x}' \in U,$$

where $\|\mathbf{x}\| = (x_1^2 + \cdots + x_d^2)^{1/2}$ for $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbf{R}^d$.

Assumption 2. *The distribution of $\mathbf{X}_0$ is absolutely continuous and its density $f(\cdot)$ is bounded away from zero and infinity. That is, there is a positive constant M_1 such that $M_1^{-1} \le f(\mathbf{x}) \le M_1$ for $\mathbf{x} \in U$.*

The following technical condition is required for bounding the variance of various terms in the proof.

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Assumption 3. *The conditional distribution of* $\mathbf{X}_1$ *given* $\mathbf{X}_0$ *is absolutely continuous and its density* $h(\cdot\,|\mathbf{x})$ *is bounded away from zero and infinity. That is* $M_1^{-1} \leq h(\mathbf{y}|\mathbf{x}) \leq M_1$ *for* $\mathbf{x}$ *and* $\mathbf{y} \in U$.

Let $(Z_1, \ldots, Z_k)$ denote a vector of k random variables with $E|Z_j|^k < \infty$, for $j = 1, \ldots, k$. The rth order joint cumulant, $\text{cum}(Z_1, \ldots, Z_k)$, of $(Z_1, \ldots, Z_k)$ is given by

$$\text{cum}(Z_1, \ldots, Z_k) = \sum (-1)^{p-1}(p-1)! E\left(\prod_{j \in \nu_1} Z_j\right) \cdots E\left(\prod_{j \in \nu_p} Z_j\right)$$

where the sum extends over all partitions $(\nu_1, \ldots, \nu_p)$, $p = 1, 2, \ldots, k$, of $(1, \ldots, k)$. Equivalently (Brillinger, 1981), $\text{cum}(Z_1, \ldots, Z_k)$ is given by the coefficient of $i^k t_1 \cdots t_k$ in the Taylor series expansion of $\log E[\exp(i \sum_1^k Z_j t_j)]$ about the origin. An important special case of this definition occurs when $Z_j = Z, j = 1, \ldots, k$. This then gives the cumulant of order k of a univriate random variable. See Brillinger (1981) for methods on computing cumulants.

A weak dependence condition on the stationary sequence will now be described. Let $f(\cdot)$ denote a real-valued, measurable functions on $\mathbf{R}^{d+1}$. Set

$$c(i, j) = \sup_{|f| \leq 1} \text{cum}\{f(\mathbf{X}_i, Y_i), f(\mathbf{X}_j, Y_j)\}.$$

and

$$c(t_1, t_2, \ldots, t_k) = \sup_{|f_i| \leq 1} \text{cum}\{f(\mathbf{X}_{t_1}, Y_{t_1}), f(\mathbf{X}_{t_2}, Y_{t_2}), \ldots, f(\mathbf{X}_{t_k}, Y_{t_k})\}.$$

If the series $(\mathbf{X}_t, Y_t)$, $t = 0, \pm 1, \ldots$ is strictly stationary, then

$$c(t_1, t_2, \ldots, t_k) = c(t_1 + u, t_2 + u, \ldots, t_k + u).$$

for $t_1, \ldots, t_k, u = 0, \pm 1, \ldots$. In this case, we sometimes use the asymmetric notation

$$c(t_1, t_2, \ldots, t_{k-1}) = c(t_1, t_2, \ldots, t_{k-1}, 0).$$

The stationary sequence is said to be mixing if $c(t_1, \ldots, t_k) \to 0$ as $t_1, \ldots, t_k \to \infty$. See Brillinger (1981).

Assumption 4. $\sum_{j \geq N} c(j) = O(N^{-1})$.

The following condition is similar to Assumption 2.6.3 of Brillinger (1981).

Assumption 5. $\sum_k C_k z^k / k! < \infty$ *for z in a neighborhood of zero, where*

$$C_k = \sum_{v_1 = -\infty}^{\infty} \cdots \sum_{v_{k-1} = -\infty}^{\infty} |c(v_1, \ldots, v_{k-1})|.$$

A condition on the conditional distribution of Y given $\mathbf{X}$ is required to guarantee the uniqueness of the conditional median (uniqueness will ensure consistency) and also the achievability of the desired rate of conververgence. If the conditional density is not bounded away from zero around the median the desired rate of convergence will not be achievable. (The same condition is required in order to obtain the usual asymptotic result about the sample median in the univariate case.)

Assumption 6. *The conditional distribution of Y_0 given $\mathbf{X}_0 = \mathbf{x}$ is absolutely continuous and its density $g(y|\mathbf{x}, \theta)$ is bounded away from zero and infinity over a neighborhood of the median θ. That is, there is a positive constant ϵ_0 such that $M_1^{-1} \leq g(y|\mathbf{x}, \theta) \leq M_1$ for $\mathbf{x} \in U$ and $y \in (\theta(\mathbf{x}) - \epsilon_0, \theta(\mathbf{x}) + \epsilon_0)$.*

The kernel estimators of conditional medians will now be described. For each $n \geq 1$, let $(\mathbf{X}_1, Y_1), \ldots, (\mathbf{X}_n, Y_n)$ be a realization of the (strictly) stationary time series. Let δ_n, $n \geq 1$, be positive numbers that tend to zero as n tends to infinity. Set $I_n(\mathbf{x}) = \{i : 1 \leq i \leq n \text{ and } \|\mathbf{X}_i - \mathbf{x}\| \leq \delta_n\}$ and $N_n(\mathbf{x}) = \#(I_n(\mathbf{x}))$. Also set $\hat{\theta}_n(\mathbf{x}) = \text{Median}\{Y_i : i \in I_n(\mathbf{x})\}$ (use the average of the two middle ordered statistics if $N_n(\mathbf{x})$ is even).

Given positive numbers a_n and b_n, $n \geq 1$, let $a_n \sim b_n$ mean that a_n / b_n is bounded away from zero and infinity. Given random variables V_n, $n \geq 1$, let $V_n = O_{pr}(b_n)$ mean that the random variables $b_n^{-1} V_n$, $n \geq 1$ are bounded in probability or, equivalently, that

$$\lim_{c \to \infty} \limsup_n P(|V_n| > cb_n) = 0.$$

Set $r = (2 + d)^{-1}$.

Theorem 1. *Suppose that Assumptions 1–4 and 6 hold and that $\delta_n \sim n^{-r}$. Then*

$$|\hat{\theta}_n(\mathbf{0}) - \theta(\mathbf{0})| = O_{pr}(n^{-r}).$$

The proof of this theorem, which will be given in Section 5, is basically a refinement of the corresponding one given in Truong and Stone (1987b), with additional arguments involving asymptotic independence.

Let C be a fixed compact subset of U having a nonempty interior and let $g(\cdot)$ be a real-valued function on $\mathbf{R}^d$. Set

$$\|g\|_q = \left\{ \int_C |g(\mathbf{x})|^q \, d\mathbf{x} \right\}^{\frac{1}{q}}, \qquad 1 \leq q < \infty;$$
$$\|g\|_\infty = \sup_{\mathbf{x} \in C} |g(\mathbf{x})|.$$

Theorem 2. *Suppose that Assumptions 1–6 hold and that $\delta_n \sim (n^{-1} \log n)^r$. Then there exists a $c > 0$ such that*

$$\lim_n P\left(\|\hat{\theta}_n - \theta\|_\infty \geq c(n^{-1} \log n)^r \right) = 0.$$

The proof of this theorem, which can be found in Section 5, is simpler and more intuitive than the corresponding proof given by Härdle and Luckhaus (1984) (only the calculation of binomial probability is required).

Theorem 3. *Suppose that Assumptions 1–6 hold and that $\delta_n \sim n^{-r}$. Then there exists a $c > 0$ such that*

$$P(\|\hat{\theta}_n - \theta\|_q \geq cn^{-r}) = 0 \qquad 1 \leq q < \infty.$$

The proof of this theorem, which will be given in Section 5, uses the result of Theorem 2 and the argument is a refinement of that in the i.i.d. case (Truong and Stone, 1987b).

With a simple modification of Assumption 6, Theorems 1–3 are easily extended to yield rates of convergence for nonparametric estimators of other conditional quantiles.

4. DISCUSSION

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For $n \geq 1$, let $(\mathbf{X}_1, Y_1), \ldots, (\mathbf{X}_n, Y_n)$ be a random sample of size n from the distribution of $(\mathbf{X}, Y)$ and let k denote a non-negative integer. Let $\theta(\cdot)$ be the regression function, $E(Y \mid \mathbf{X} = \cdot)$, of Y on $\mathbf{X}$ and suppose that $\theta(\cdot)$ has bounded $(k+1)$th derivative. Set $r = p/(2p + d)$ where $p = k + 1$. Stone (1980, 1982) showed that if $1 \leq q < \infty$, then n^{-r} is the optimal rate of convergence in both pointwise and L^q norms; while $(n^{-1} \log n)^{-r}$ is the optimal rate of convergence in L^∞ norm. To find an estimator of $\theta(\cdot)$ that achieves these optimal rates of concergence, given $\mathbf{x}$, let $\hat{P}_n(\cdot; \mathbf{x})$ be the polynomial on $\mathbf{R}^d$ of degree k that minimizes

$$\sum_{I_n(\mathbf{x})} [\, Y_i - \hat{P}_n(\mathbf{X}_i; \mathbf{x})\,]^2$$

and set $\hat{\theta}_n(\mathbf{x}) = \hat{P}_n(\mathbf{x}; \mathbf{x})$ (if $q = \infty$, define $\hat{\theta}_n$ as above over a finite subset of C and then extend it to all of C by suitable interpolation). Note that this estimator can be easily obtained by solving the corresponding normal equation.

Based on results presented in the previous sections, the following generalization to the case of conditional medians seems plausible. Suppose that the conditional median $\theta(\cdot)$ has bounded pth derivative. To find an estimator that achieves the above L^q ($1 \leq q \leq \infty$) rates of convergence, given $\mathbf{x}$, let $\hat{P}_n(\cdot; \mathbf{x})$ be a polynomial on $\mathbf{R}^d$ of degree k which minimizes

$$\sum_{I_n(\mathbf{x})} |Y_i - \hat{P}_n(\mathbf{X}_i; \mathbf{x})|$$

and set $\hat{\theta}_n(\mathbf{x}) = \hat{P}_n(\mathbf{x}; \mathbf{x})$. Though there may not be a unique solution, this numerical optimization problem is readily solved by the simplex method (see, for example, Bloomfield and Steiger (1983)). The corresponding generalization to time series is straightforward.

One drawback that the nonparametric approach has is the high *dimensionality*, which can be thought of in terms of the *variance* in estimation. In other words: A *huge* data set may be required for nonparametric estimation of a function of many variables; otherwise the variance of the estimator may be unacceptably large. This drawback is serious especially in time series analysis where the future usually depends on much of the past.

A possible solution would be to use *additivity* as in Stone (1985) to alleviate *curse of dimensionality*. More formally, let $\theta(\cdot)$ be the regression function defined on $\mathbf{R}^d$ and

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suppose that θ is additive; that is, that there is smooth functions $\theta_1(\cdot), \ldots, \theta_d(\cdot)$ defined on $\mathbf{R}^1$ such that

$$\theta(x_1, \ldots, x_d) = \mu + \theta_1(x_1) + \cdots + \theta_d(x_d),$$

where $\mu = E(Y)$. Using B-splines, an estimator of $\theta(\cdot)$ can be constructed to achieve the optimal rates of convergence n^{-r}, where r now is equal to $p/(2p + 1)$. The rates of convergence here do not depend on the dimensional parameter d. Another nice feature about this estimator is that it is smoother and is as flexible as ordinary nonparametric procedures constructed by the kernel method.

The corresponding methodology is generalized immediately to time series, and it is an interesting open problem to determine whether the asymptotic properties described above (with r independent of d) also hold in this context.

5. PROOF OF THEOREMS

Proof of Theorem 1. Let B_{ni} be the event that $\|\mathbf{X}_i\| \leq \delta_n$. According to Assumption 1, $\theta(\mathbf{X}_i) \leq \theta(0) + M_0 \delta_n$ whenever $\|\mathbf{X}_i\| \leq \delta_n$. Thus

$$\tfrac{1}{2} - P(Y_i \geq \theta(0) + c\delta_n \mid B_{ni}) \geq P(M_0 \delta_n \leq Y_i - \theta(0) \leq c\delta_n \mid B_{ni}).$$

Hence by Assumption 6

$$\tfrac{1}{2} - P(Y_i \geq \theta(0) + c\delta_n \mid B_{ni}) \geq (c - M_0) M_1^{-1} \delta_n \qquad \text{for } c_0 > M_0. \tag{5.1}$$

Set $K_i = 1_{\{\|\mathbf{X}_i\| \leq \delta_n\}}$ and $Z_i = 1_{\{Y_i \geq \theta(0) + c\delta_n\}} - P(Y_i \geq \theta(0) + c\delta_n \mid \mathbf{X}_i)$. Then $E[Z_i] = 0$ and, by the first argument in the proof of Lemma 7 (Truong, 1987),

$$\mathrm{Var}(\textstyle\sum_i K_i Z_i) = \mathrm{Var}(\textstyle\sum_{I_n} Z_i) = O(n\delta_n^d).$$

According to (5.1)

$$\tfrac{1}{2} - N_n^{-1} \textstyle\sum_i K_i P(Y_i \geq \theta(0) + c\delta_n \mid \mathbf{X}_i) = \tfrac{1}{2} - N_n^{-1} \textstyle\sum_i P(Y_i \geq \theta(0) + c\delta_n \mid B_{ni})$$

$$\geq (c - M_0) M_1^{-1} \delta_n \qquad \text{for } c > M_0.$$

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Consequently, by Lemma 6 (Truong, 1987) and Tchebychev's inequality

$$P(\hat{\theta}_n(\mathbf{0}) \geq \theta(\mathbf{0}) + c\delta_n) \leq P\left(N_n^{-1}\sum_{I_n} 1_{\{Y_i \geq \theta(\mathbf{o}) + c\delta_n\}} \geq \tfrac{1}{2}\right)$$

$$\leq P(N_n^{-1}\sum_{I_n} Z_i \geq \tfrac{1}{2} - N_n^{-1}\sum_i P(Y_i \geq \theta(\mathbf{0}) + c\delta_n \mid B_{ni}))$$

$$\leq P(N_n^{-1}\sum_{I_n} Z_i \geq (c - M_0)M_1^{-1}\delta_n)$$

$$\leq P(N_n^{-1}\sum_{I_n} Z_i \geq (c - M_0)M_1^{-1}\delta_n); N_n \geq \tfrac{1}{2}n\delta_n^d) + P(N_n < \tfrac{1}{2}n\delta_n^d)$$

$$= \frac{O(1)}{(c - M_0)^2}\frac{n\delta_n^d}{(n\delta_n^d\delta_n)^2} + o(1) = o(1) \qquad \text{as } n, c \to \infty,$$

since δ_n is chosen so that $n\delta_n^d\delta_n^2 = 1$, or equivalently, $\delta_n = n^{-r}$. This completes the proof of Theorem 1.

Proof of Theorem 2. Without loss of generality it can be assumed that $C = [-\tfrac{1}{2}, \tfrac{1}{2}]^d$. Set $L_n = [n^{2r}]$. Let W_n be the collection of $(2L_n + 1)^d$ points in C each of whose coordinates is of the form $j/(2L_n)$ for some integer j such that $|j| \leq L_n$. Then C can be written as the union of $(2L_n)^d$ subcubes, each having length $2\lambda_n = (2L_n)^{-1}$ and all of its vertices in W_n. For each $\mathbf{x} \in C$ there is a subcube Q_w with center w such that $\mathbf{x} \in Q_w$. Let C_n denote the collection of the centers of these subcubes. Then

$$P\left(\sup_{\mathbf{x} \in C} |\hat{\theta}_n(\mathbf{x}) - \theta(\mathbf{x})| \geq \delta_n\right) = P\left(\max_{C_n}\sup_{\mathbf{x} \in Q_w} |\hat{\theta}_n(\mathbf{x}) - \theta(\mathbf{x})| \geq \delta_n\right).$$

It follows from $\lambda_n \sim n^{-2r}$ and Assumption 1 that $|\theta(\mathbf{x}) - \theta(w)| \leq M_0\|\mathbf{x} - w\| \leq M_0\delta_n$ for $\mathbf{x} \in Q_w$, $w \in C_n$ (for n sufficiently large). Therefore, to prove the lemma, it is sufficient to show that there is a positive constant c such that

$$\lim_n P\left(\max_{w \in C_n}\sup_{\mathbf{x} \in Q_w} |\hat{\theta}_n(\mathbf{x}) - \theta(w)| \geq c\delta_n\right) = 0. \tag{5.2}$$

To prove (5.2), let $\eta \equiv \sqrt{d}$, $\mathbf{x} \in Q_w$ and $N_n' \equiv N_n'(w) = \#\{i : \|\mathbf{X}_i - w\| \leq \delta_n - \eta\lambda_n\}$. Now $N_n \equiv N_n(\mathbf{x}) = \#\{i : \|\mathbf{X}_i - \mathbf{x}\| \leq \delta_n\} \geq N_n'$ for $\mathbf{x} \in Q_w$, $\{\hat{\theta}_n(\mathbf{x}) - \theta(w) \geq c\delta_n\} \subseteq \{N_n^{-1}\sum_{I_n} 1_{\{Y_i \geq \theta(w) + c\delta_n\}} \geq \tfrac{1}{2}\} \subseteq \{\sum_{I_n^*} 1_{\{Y_i \geq \theta(w) + c\delta_n\}} \geq \tfrac{1}{2}N_n'\}$, where $I_n^* \equiv I_n^*(w) = \{i : 1 \leq i \leq n \text{ and } \|\mathbf{X}_i - w\| \leq \delta_n + \eta\lambda_n\}$. Thus

$$\cup_{Q_w}\left\{\hat{\theta}_n(\mathbf{x}) - \theta(w) \geq c\delta_n\right\} \subseteq \left\{\sum_{I_n^*} 1_{\{Y_i \geq \theta(w) + c\delta_n\}} \geq \tfrac{1}{2}N_n'\right\}. \tag{5.3}$$

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Set $N_n^* \equiv N_n^*(w) = \#I_n^*(w)$. By Assumptions 2-5 and Markov's inequality there are positive constants k_1 and k_2 such that

$$\lim_n P(\Psi_n) = 1, \tag{5.4}$$

where $\Psi_n = \{N_n^*(w) - N_n'(w) \leq k_1 n^r \text{ and } N_n^*(w) \geq k_2 n \delta_n^d \text{ for all } w \in C_n\}$.

Indeed, note that $N_n^* - N_n' = \#\{i : \delta_n - \eta \lambda_n \leq \|\mathbf{X}_i - w\| \leq \delta_n + \eta \lambda_n\}$ with $q_n = P(\delta_n - \eta \lambda_n \leq \|\mathbf{X}_i - w\| \leq \delta_n + \eta \lambda_n) \sim ((\delta_n + \eta \lambda_n)^d - (\delta_n - \eta \lambda_n)^d) \sim \delta_n^d(\lambda_n/\delta_n)$ as $n \to \infty$. It follows from $n\delta_n^{d+2} \sim \log n$ and $\lambda_n \sim n^{-2r}$ that $nq_n = O(n^r) \to \infty$ as $n \to \infty$. Thus by Assumption 5, there is a positive constant c_0 such that (see the proof of Lemma 9 of Truong (1987))

$$P\left(N_n^*(w) - N_n'(w) \geq 2nq_n \text{ for some } w \in C_n\right) = [n^{2r}]^d \max_{C_n} \exp[-c_0 n^r/(\log n)^{3r}] \to 0,$$

as $n \to \infty$. Similarly, $\lim_n P\left(N_n^*(w) \leq \frac{1}{2} n p_n(w) \text{ for some } w \in C_n\right) = 0$, where $p_n(w) = P(\|\mathbf{X}_i - w\| \leq \delta_n + \eta \lambda_n) \sim \delta_n^d$. Thus (5.4) is proven.

Note that $n^r N_n^{*-1} \leq n^r (k_2 n \delta_n^d)^{-1} < \delta_n$ on Ψ_n. According to (5.3) and (5.4), there is a positive constant k_3 such that

$$\begin{aligned}
&P\left(\max_{C_n} \sup_{x \in Q_w} [\hat{\theta}_n(\mathbf{x}) - \theta(w)] \geq c\delta_n\right) \\
&\leq P\left(\cup_{C_n} \cup_{Q_w} \{\hat{\theta}_n(\mathbf{x}) - \theta(w) \geq c\delta_n\}\right) \\
&\leq P\left(\cup_{C_n} \{\sum_{I_n^*} 1_{\{Y_i \geq \theta(w) + c\delta_n\}} \geq \frac{1}{2} N_n'\}\right) \\
&\leq P\left(\cup_{C_n} \{\sum_{I_n^*} 1_{\{Y_i \geq \theta(w) + c\delta_n\}} \geq \frac{1}{2} N_n^* - \frac{1}{2} k_1 n^r\} \cap \Psi_n\right) + P(\Psi_n^c) \\
&\leq P\left(\cup_{C_n} \{N_n^{*-1} \sum_{I_n^*} 1_{\{Y_i \geq \theta(w) + c\delta_n\}} \geq \frac{1}{2} - k_3 \delta_n\}\right) + P(\Psi_n^c). \tag{5.5}
\end{aligned}$$

Let $B_{ni} = B_{ni}(w)$ denote the event $\|\mathbf{X}_i - w\| \leq \delta_n + \eta \lambda_n$ for $i = 1, \ldots, n$. According to Assumption 1, $\theta(\mathbf{X}_i) \leq \theta(w) + M_0(\delta_n + \eta \lambda_n)$ whenever $\|\mathbf{X}_i - w\| \leq \delta_n + \eta \lambda_n$. Thus

$$\frac{1}{2} - P(Y_i \geq \theta(w) + c\delta_n \mid B_{ni}) \geq P(M_0(\delta_n + \eta \lambda_n) \leq Y_i - \theta(w) \leq c\delta_n \mid B_{ni}).$$

Hence by Assumption 6, there is a positive constant M_2 such that

$$\frac{1}{2} - P(Y_i \geq \theta(w) + c\delta_n \mid B_{ni}) \geq cM_2\delta_n \qquad \text{for } n \text{ sufficient large.} \tag{5.6}$$

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Set $K_i = 1_{\{\|\mathbf{X}_i - w\| \leq \delta_n + \eta\lambda_n\}}$ and $Z_i = 1_{\{Y_i \geq \theta(w) + c\delta_n\}} - P(Y_i \geq \theta(w) + c\delta_n \mid \mathbf{X}_i)$. Then $E(Z_i) = 0$. By (5.6)

$$\tfrac{1}{2} - N_n^{*-1}\sum_{I_n^*} P(Y_i \geq \theta(w) + c\delta_n \mid \mathbf{X}_i) \geq cM_2\delta_n. \qquad (5.7)$$

It now follows from (5.7) that there is a positive constant M_3 such that

$$P\left(\cup_{C_n}\left\{N_n^{*-1}\sum_{I_n^*} 1_{\{Y_i \geq \theta(w) + c\delta_n\}} \geq \tfrac{1}{2} - k_3\delta_n\right\}\right)$$

$$\leq P\left(\cup_{C_n}\left\{N_n^{*-1}\sum_{I_n^*} Z_i \geq \tfrac{1}{2} - N_n^{*-1}\sum_{I_n^*} P(Y_i \geq \theta(w) + c\delta_n \mid \mathbf{X}_i) - k_3\delta_n\right\}\right)$$

$$\leq [n^{2r}]^d \max_{C_n} P\left(N_n^{*-1}\sum_{I_n^*} Z_i \geq cM_2\delta_n - k_3\delta_n\right)$$

$$\leq [n^{2r}]^d \max_{C_n} P\left(N_n^{*-1}\sum_{I_n^*} Z_i \geq cM_3\delta_n\right) \quad \text{for } n \text{ sufficient large.} \qquad (5.8)$$

Set $p_n = p_n(w) = P(\|\mathbf{X}_i - w\| \leq \delta_n + \eta\lambda_n)$ (by stationary, p_n does not depend on i). Then $p_n \sim \delta_n^d$. Note that $\sum_{I_n^*} Z_i = \sum_i K_i Z_i$, $E(K_i Z_i) = 0$ and $E|K_i Z_i| = O(\delta_n^d)$. By Lemma 8 and 9 (Truong, 1987), there are positive constants M_4 and M_5 such that

$$P\left(N_n^{*-1}\sum_{I_n^*} Z_i \geq cM_3\delta_n\right) \leq P\left(N_n^* < \tfrac{1}{2}np_n\right)$$

$$+ P\left(N_n^{*-1}\sum_{I_n^*} Z_i \geq cM_3\delta_n; N_n^* \geq \tfrac{1}{2}np_n\right)$$

$$\leq \exp(-M_4 n\delta_n^d) + \exp(-c^2 M_5 n\delta_n^{d+2}) \quad \text{for } w \in C_n.$$

Thus

$$[n^{2r}]^d \max_{C_n} P\left(N_n^{*-1}\sum_{I_n^*} Z_i \geq cM_3\delta_n\right) \leq [n^{2r}]^d[\exp(-M_4 n\delta_n^d) + \exp(-c^2 M_5 n\delta_n^{d+2})].$$

Note that δ_n is chosen so that $n\delta_n^{d+2} \sim \log n$. Thus there is a positive integer c such that

$$[n^{2r}]^d \max_{C_n} P\left(N_n^{*-1}\sum_{I_n^*} Z_i \geq cM_3\delta_n\right) \to 0 \quad \text{as } n \to \infty. \qquad (5.9)$$

Hence by (5.4), (5.5), (5.8) and (5.9)

$$\lim_n P\left(\max_{C_n} \sup_{\mathbf{x} \in Q_w} [\hat{\theta}_n(\mathbf{x}) - \theta(w)] \geq c\delta_n\right) = 0 \quad \text{for } c > 0. \qquad (5.10)$$

Similarly,

$$\lim_n P\left(\max_{C_n} \sup_{\mathbf{x} \in Q_w} [\hat{\theta}_n(\mathbf{x}) - \theta(w)] \geq -c\delta_n\right) = 0 \quad \text{for } c > 0. \qquad (5.11)$$

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It follows from (5.10) and (5.11) that (5.2) is valid. This completes the proof of Theorem 2.

Proof of Theorem 3. By Assumption 1, $\theta(\cdot)$ is bounded on C (compact). Thus it follows from Theorem 2 that there is a positive constant $T \geq 1$ such that

$$\lim_n P(\Phi_n) = 1 \tag{5.12}$$

where $\Phi_n \equiv \{\|\hat{\theta}_n\|_\infty \leq T\}$. For $i = 1, \ldots, n$, set

$$Y_i' = \begin{cases} -T & \text{if } Y_i \leq -T; \\ Y_i & \text{if } |Y_i| \leq T; \\ T & \text{if } Y_i \geq T. \end{cases}$$

Put $\bar{\theta}_n(\mathbf{x}) \equiv \text{Median}\{Y_i' : i \in I_n(\mathbf{x})\}$. Note that $\bar{\theta}_n(\mathbf{x}) = \hat{\theta}_n(\mathbf{x})$ except on Φ_n^c for $\mathbf{x} \in C$. Thus by (5.12), in order to prove the theorem, it is sufficient to show

$$\lim_n P\left(\|\bar{\theta}_n - \theta\|_q \geq cn^{-r}\right) = 0. \tag{5.13}$$

To prove (5.13), we may assume that C is contained in the interior of the cube $C_0 = [-\frac{1}{2}, \frac{1}{2}]^d \subset U$. By (5.4) or Lemma 9 of Truong (1987), there is a positive constant k_4 such that

$$\lim_n P(\Omega_n) = 1, \tag{5.14}$$

where $\Omega_n = \{N_n(\mathbf{x}) \geq k_4 n \delta_n^d \text{ for } \mathbf{x} \in C\}$.

Write $P_{\Omega_n}(\cdot) = P(\cdot \, ; \Omega_n) = P(\cdot \cap \Omega_n)$ and $E_{\Omega_n}(W) = \int w \, dP_{\Omega_n}$, where W is a real-valued random variable. By (5.14), there is a sequence of positive numbers $\epsilon_n \to 0$ such that

$$P\left(\int_C |\bar{\theta}_n(\mathbf{x}) - \theta(\mathbf{x})|^q \, d\mathbf{x} \geq (cn^{-r})^q\right) \leq P\left(\int_C |\bar{\theta}_n(\mathbf{x}) - \theta(\mathbf{x})|^q \, d\mathbf{x} \geq (cn^{-r})^q; \Omega_n\right) + \epsilon_n$$

$$\leq \frac{E_{\Omega_n}\left[\int_C |\bar{\theta}_n(\mathbf{x}) - \theta(\mathbf{x})|^q \, d\mathbf{x}\right]}{(cn^{-r})^q} + \epsilon_n. \tag{5.15}$$

Set $Z_n(\mathbf{x}) = |\bar{\theta}_n(\mathbf{x}) - \theta(\mathbf{x})|$. By Assumption 1, $Z_n(\mathbf{x})$ is bounded by T for $\mathbf{x} \in C$. Thus there is a positive constant k_5 such that

$$E_{\Omega_n}[Z_n^q(\mathbf{x})] = \int_0^T qt^{q-1} P_{\Omega_n}\left(Z_n(\mathbf{x}) > t\right) dt$$

$$= \int_0^{2M_0\delta_n} qt^{q-1} P_{\Omega_n}\left(Z_n(\mathbf{x}) > t\right) dt + \int_{2M_0\delta_n}^T qt^{q-1} P_{\Omega_n}\left(Z_n(\mathbf{x}) > t\right) dt$$

$$\leq k_5\delta_n^q + \int_{2M_0\delta_n}^T qt^{q-1} P_{\Omega_n}\left(Z_n(\mathbf{x}) > t\right) dt. \tag{5.16}$$

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By Assumption 1-6, there is a positive number k_6 such that

$$\int_{2M_0\delta_n}^T q t^{q-1} P_{\Omega_n}\left(Z_n(\mathbf{x}) > t\right) dt \le k_6 \delta_n^q \qquad \text{for } \mathbf{x} \in C. \tag{5.17}$$

(The proof of (5.17) will be given shortly.) It follows from (5.16) and (5.17) that there is a positive constant k_7 such that

$$E_{\Omega_n}[Z_n^q(\mathbf{x})] \le k_7 \delta_n^q.$$

Thus there is a positive constant k_8 such that

$$E_{\Omega_n}\left[\int_C Z_n^q(\mathbf{x})\, d\mathbf{x}\right] = \int_C E_{\Omega_n}[Z_n^q(\mathbf{x})]\, d\mathbf{x} \le k_7 \delta_n^q. \tag{5.18}$$

The conclusion of Theorem 3 follows from (5.15) and (5.18).

Finally, (5.17) will be proven. Set $D_i = 1_{\{Y_i \ge \theta(\mathbf{x}) + t\}} - P(Y_i \ge \theta(\mathbf{x}) + t \mid \mathbf{X}_i)$ and let B_{ni} be the event $\|\mathbf{X}_i - \mathbf{x}\| \le \delta_n$ for $i = 1, \ldots, n$. Put $R_i = \frac{1}{2} - P(Y_i \ge \theta(\mathbf{x}) + t \mid B_{ni})$. By Assumption 6, there is a positive constant k_9 such that

$$N_n^{-1}\sum_{I_n} R_i \ge k_9 T^{-1}(t - M_0\delta_n) \qquad \text{for } M_0 \le t \le T,\ T > 1.$$

Thus (since $\{Y_i' > \theta(\mathbf{x}) + t\} \subset \{Y_i > \theta(\mathbf{x}) + t\}$)

$$\begin{aligned}
P_{\Omega_n}\left(\bar{\theta}_n(\mathbf{x}) - \theta(\mathbf{x}) > t\right) &\le P_{\Omega_n}\left(N_n^{-1}\sum_{I_n} 1_{\{Y_i' > \theta(\mathbf{x}) + t\}} \ge \tfrac{1}{2}\right) \\
&\le P_{\Omega_n}\left(N_n^{-1}\sum_{I_n} 1_{\{Y_i > \theta(\mathbf{x}) + t\}} \ge \tfrac{1}{2}\right) \\
&\le P_{\Omega_n}\left(N_n^{-1}\sum_{I_n} D_i \ge N_n^{-1}\sum_{I_n} R_i\right) \\
&\le P\left(\sum_{I_n} D_i \ge k_9 T^{-1}(t - M_0\delta_n) n \delta_n^d\right).
\end{aligned} \tag{5.19}$$

Note that $\sum_{I_n} D_i = \sum_i K_i D_i$, $E(K_i D_i) = 0$ and $E|K_i D_i| = O(\delta_n^d)$. Since D_i is bounded, by the first part of Lemma 7 of Truong (1987),

$$\mathrm{Var}(\sum_{I_n} D_i) = O(n\delta_n^d).$$

Consequently, by Markov's inequality (see Lemma 8 of Truong (1987)), there are positive constants k_{10} and k_{11} such that

$$P\left(\sum_{I_n} D_i \ge k_9 T^{-1}(t - M_0\delta_n) n \delta_n^d\right) \le k_{10} \exp[-k_{11} T^{-2}(t - M_0\delta_n)^2 n \delta_n^d] \tag{5.20}$$

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By (5.19) and (5.20), (note that $n\delta_n^d \sim \delta_n^{-2}$)

$$P_{\Omega_n}\left(\bar{\theta}_n(\mathbf{x}) - \theta(\mathbf{x}) > t\right) \leq k_{10}\exp[-k_{12}T^{-2}(t - M_0\delta_n)^2\delta_n^{-2}], \qquad (5.21)$$

and, similarly,

$$P_{\Omega_n}\left(\bar{\theta}_n(\mathbf{x}) - \theta(\mathbf{x}) < -t\right) \leq k_{10}\exp[-k_{12}T^{-2}(t - M_0\delta_n)^2\delta_n^{-2}], \qquad (5.22)$$

It now follows from (5.21) and (5.22) that there is a positive constant k_0 such that for $\mathbf{x} \in C$

$$\int_{2M_0\delta_n}^T t^{q-1}P_{\Omega_n}(|\bar{\theta}_n(\mathbf{x}) - \theta(\mathbf{x})| > t)\,dt \leq k_{10}\int_{2M_0\delta_n}^T t^{q-1}\exp[-k_{12}T^{-2}(t - M_0\delta_n)^2\delta_n^{-2}]\,dt$$

$$\leq 2^{q-1}\int_0^\infty t^{q-1}\exp[-k_{12}T^{-2}(t - M_0\delta_n)^2\delta_n^{-2}]\,dt$$

$$= k_0 T^q\delta_n^q,$$

as desired. This completes the proof of (5.17).

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